# Wigner distribution function of a circular aperture

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## Abstract

The Wigner distribution function of a circular aperture is determined. Analytic expressions as well as numerical and graphical results are presented.

## 1 Introduction

In recent years the Wigner distribution function [1] has proven to be a valuable tool for the description of signals and systems in optics (see, for instance, [2, 3, 4, 5, 6] and the references cited there). The Wigner distribution function is a signal representation in space and (spatial) frequency, simultaneously, and thus forms an intermediate signal description between the pure space representation and the pure frequency representation. This simultaneous space-frequency description closely resembles the ray concept in geometrical optics, in which the position and direction of a ray are also given simultaneously, and its relationship to the generalized radiance is obvious [2, 7]. In a way, the Wigner distribution function can be considered as the amplitude of an optical ray passing through a certain position with a certain direction.

One of the important properties of the Wigner distribution function is that its propagation through Luneburg's first-order optical systems [8] (described by an $ABCD$-matrix [9]) can be expressed in a very simple form [3, 4]. A simple propagation law also arises when the light propagates through weakly inhomogeneous media, where diffraction effects can be neglected [10, 11]. In such media the amplitude of an optical ray remains invariant under propagation, and to find the propagation of the light we only have to determine the optical ray paths. Propagation of the Wigner distribution function in such media thus immediately leads to ray tracing. However, as soon as diffraction effects manifest themselves in optical systems, the method of ray tracing alone is no longer appropriate to determine the propagation of light. But, since the Wigner distribution function can take into account these diffraction effects as well, it may still be a valuable tool for the analysis of such more general systems.

In this paper we consider the effect of a circular aperture on the propagation of light. The important formula we need for that is equivalent to the formula that arises when we determine the Wigner distribution function of a time-harmonic, uniform, circular light source. We will therefore focus on this latter case.

## 2 Wigner distribution function

The Wigner distribution function $F(x, y, u, v)$ that corresponds to the complex function $\varphi(x, y)$ is defined as

$$F(x, y, u, v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi\left(x + \tfrac{1}{2}x', y + \tfrac{1}{2}y'\right) \varphi^*\left(x - \tfrac{1}{2}x', y - \tfrac{1}{2}y'\right) e^{-j(ux' + vy')} dx'dy'; \qquad (2.1)$$

the variables $x$ and $y$ denote space variables, the variables $u$ and $v$ denote spatial-frequency variables, the function $\varphi(x, y)$ represents the complex amplitude of the time-harmonic optical signal $\varphi(x, y)\exp(-j\omega t)$, and the asterisk $^*$ denotes complex conjugation. Obviously, with $x$ and $y$ considered as parameters, the Wigner distribution function is in fact the two-dimensional Fourier transform of the product

$\varphi\left(x+\frac{1}{2}x', y+\frac{1}{2}y'\right)\varphi^*\left(x-\frac{1}{2}x', y-\frac{1}{2}y'\right)$ with respect to the difference coordinates $x'$ and $y'$. Note that the Wigner distribution function $F(x, y, u, v)$ is real, and that the intensity $|\varphi(x, y)|^2$ can be determined from it by integrating over the frequency variables $u$ and $v$.

If the function $\varphi_i(x, y)$ is the complex amplitude of an optical signal just in front of a spatial modulator with modulation function $m(x, y)$, the complex amplitude just after the modulator reads $\varphi_o(x, y) = m(x, y)\varphi_i(x, y)$. It is not difficult to show that the Wigner distribution function $F_o(x, y, u, v)$ of the output signal $\varphi_o(x, y)$ is related to the Wigner distribution function $F_i(x, y, u, v)$ of the input signal $\varphi_i(x, y)$ through the relationship

$$F_o(x, y, u, v) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_m(x, y, u-u', v-v')F_i(x, y, u', v')du'dv', \tag{2.2}$$

where $F_m(x, y, u, v)$ is the Wigner distribution function of the modulation function $m(x, y)$. Note that Eq. (2.2) represents a two-dimensional convolution of the Wigner distribution functions $F_m(x, y, u, v)$ and $F_i(x, y, u, v)$ with respect to the frequency variables, and a mere multiplication with respect to the space variables.

To describe the effect of a spatial modulator, we thus need to determine the Wigner distribution function of its modulation function. This Wigner distribution function arises at the output of the modulator when the input signal takes the form $\varphi_i(x, y) = 1$, in which case the output signal is identical to the modulation function: $\varphi_o(x, y) = m(x, y)$. Therefore, in studying the effect of a spatial modulator, we might as well determine the Wigner distribution function of an optical signal whose complex amplitude reads $m(x, y)$. In this paper, in which we are interested in the effects of a circular aperture, we will therefore determine the Wigner distribution function of a circular light source, whose complex amplitude is uniform inside a circle and vanishes outside that circle.

## 3 Wigner distribution function of a circular light source

In the case of a uniform, circular light source with radius $a > 0$, the complex amplitude $\varphi(x, y)$ satisfies the conditions

$$\varphi(x, y) = \begin{cases} 1 & \text{for } 0 \le \sqrt{x^2+y^2} \le a \\ 0 & \text{elsewhere.} \end{cases} \tag{3.1}$$

We first remark that it can easily be seen from the definition (2.1) that the Wigner distribution function $F(x, y, u, v)$ vanishes outside the source, i.e. for $\sqrt{x^2+y^2} \ge a$. Moreover, since the signal $\varphi(x, y)$ is in fact a function of $\sqrt{x^2+y^2}$, we may, without loss of generality take $y = 0$ and $x \ge 0$; in that case the integrand in the definition of the Wigner distribution function [see Eq. (2.1)] is confined to the 'eye-shaped' interval (see Fig. 1)

$$\begin{array}{rlll} & -\sqrt{a^2-\left(x-\frac{1}{2}x'\right)^2} \le \frac{1}{2}y' \le \sqrt{a^2-\left(x-\frac{1}{2}x'\right)^2} & \text{while} & x-a \le \frac{1}{2}x' \le 0 \\ \text{and} & -\sqrt{a^2-\left(x+\frac{1}{2}x'\right)^2} \le \frac{1}{2}y' \le \sqrt{a^2-\left(x+\frac{1}{2}x'\right)^2} & \text{while} & 0 \le \frac{1}{2}x' \le a-x, \end{array} \tag{3.2}$$

with $x \le a$, and the integral in Eq. (2.1) can be represented in the form

$$\begin{aligned} F(x, 0, u, v) &= \int_{-2(a-x)}^{0} e^{-jux'}dx' \int_{-2\sqrt{a^2-(x-x'/2)^2}}^{2\sqrt{a^2-(x-x'/2)^2}} e^{-jvy'}dy' \\ &+ \int_{0}^{2(a-x)} e^{-jux'}dx' \int_{-2\sqrt{a^2-(x+x'/2)^2}}^{2\sqrt{a^2-(x+x'/2)^2}} e^{-jvy'}dy'. \end{aligned}$$

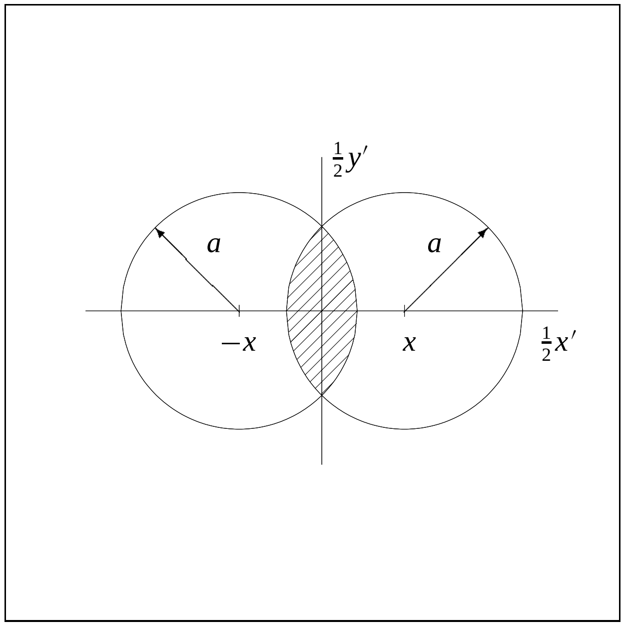


Figure 1: The ‘eye-shaped’ interval.

The integrations over $y'$ can easily be carried out, yielding

$$\begin{aligned} F(x,0,u,v) &= \int_{-2(a-x)}^{0} \frac{2}{v} \sin\left(2v\sqrt{a^2-\left(x-\tfrac{1}{2}x'\right)^2}\right) e^{-jux'} dx' \\ &+ \int_{0}^{2(a-x)} \frac{2}{v} \sin\left(2v\sqrt{a^2-\left(x+\tfrac{1}{2}x'\right)^2}\right) e^{-jux'} dx', \end{aligned}$$

where, for the time being, we assume that $v \neq 0$. When we substitute in the first integral $x - \frac{1}{2}x' = a\xi$ and in the second integral $x + \frac{1}{2}x' = a\xi$, we get

$$\begin{aligned} F(x,0,u,v) &= 2a \int_{x/a}^{1} \frac{2}{v} \sin\left(2av\sqrt{1-\xi^2}\right) e^{-j2u(x-a\xi)} d\xi \\ &+ 2a \int_{x/a}^{1} \frac{2}{v} \sin\left(2av\sqrt{1-\xi^2}\right) e^{j2u(x-a\xi)} d\xi \end{aligned}$$

and thus

$$F(x,0,u,v) = 16a^2\, \Re \left\{ e^{-j2ux} \int_{x/a}^{1} \frac{\sin\left(2av\sqrt{1-\xi^2}\right)}{2av}\, e^{j2au\xi} d\xi \right\}, \tag{3.3}$$

where $\Re$ denotes the real part. It is not difficult to show that for $v = 0$ we have

$$F(x,0,u,0) = 16a^2\, \Re \left\{ e^{-j2ux} \int_{x/a}^{1} \sqrt{1-\xi^2}\, e^{j2au\xi} d\xi \right\}. \tag{3.4}$$

Note that for $u = 0$ the integral in the latter equation can readily be determined, yielding

$$F(x,0,0,0) = \frac{16a^2}{2} \left[ \frac{\pi}{2} - \arcsin\frac{x}{a} - \frac{x}{a}\sqrt{1-\left(\frac{x}{a}\right)^2} \right], \tag{3.5}$$

but in general the integrals in Eqs. (3.3) and (3.4) do not seem to have known solutions.

It is the aim of this paper to present an (approximate) solution for the integral that arises in Eq. (3.3). Since we have the symmetry properties $F(x, 0, \pm u, \pm v) = F(x, 0, u, v)$, we may, without loss of generality, take $u \geq 0$ and $v \geq 0$ when we derive this solution. If we then use the substitutions $x = a \sin\alpha$ (with $0 \leq \alpha \leq \frac{1}{2}\pi$), $2au = R\sin\theta$ and $2av = R\cos\theta$ (with $R \geq 0$ and $0 \leq \theta \leq \frac{1}{2}\pi$), and introduce the function

$$K(\psi, R) = \int_0^{\psi} e^{jR\cos\beta}\cos\beta d\beta, \tag{3.6}$$

Eq. (3.3) can be represented in the form (see Appendix A)

$$F\left(a\sin\alpha, 0, \frac{R}{2a}\sin\theta, \frac{R}{2a}\cos\theta\right)$$

$$= \frac{16a^2}{2R}\Re\left\{je^{-jR\sin\alpha\sin\theta}[K(\alpha-\theta, R) - K^*(\alpha+\theta, R) - j\pi J_1(R)]\right\}, \tag{3.7}$$

where $J_1(z)$ is a first-order Bessel function (see, for instance, [12], Eq. 9.1.21). Note that for $R \to 0$, i.e. for $u \to 0$ and $v \to 0$, Eq. (3.7) reduces indeed to Eq. (3.5). Moreover, for $\alpha = 0$, i.e. for $x = 0$, and using the symmetry property $K(\theta, R) = -K(-\theta, R)$, Eq. (3.7) takes the form

$$F(0, 0, u, v) = 4\pi a^2 \frac{J_1(2a\sqrt{u^2+v^2})}{a\sqrt{u^2+v^2}}, \tag{3.8}$$

as should be expected; and for $\alpha = \frac{1}{2}\pi$, i.e. for $x = a$, and using the symmetry property $K(\frac{1}{2}\pi - \theta, R) - K^*(\frac{1}{2}\pi + \theta, R) - j\pi J_1(R) = 0$, the Wigner distribution function $F(a, 0, u, v)$ vanishes, as should also be expected.

The function $K(\psi, R)$ can be calculated numerically, of course, where, due to the symmetry properties of $K(\psi, R)$ around $\psi = 0$ and $\psi = \frac{1}{2}\pi$, the variable $\psi$ may be confined to the interval $0 \leq \psi \leq \frac{1}{2}\pi$. However, for large values of the parameter $R$, the integration may become difficult. Therefore, we have derived in Appendix B an approximation of $K(\psi, R)$ for large values of $R$ (and $0 \leq \psi \leq \frac{1}{2}\pi$), reading

$$\begin{aligned} K(\psi, R) &\simeq \sqrt{\frac{\pi}{R}}\left(1+\frac{3j}{8R}\right)\left[C\left(\sqrt{\frac{2R(1-\cos\psi)}{\pi}}\right) - jS\left(\sqrt{\frac{2R(1-\cos\psi)}{\pi}}\right)\right]e^{jR} \\ &+ \frac{j}{R}\frac{\sqrt{2}\cos\psi - \sqrt{1+\cos\psi}}{\sqrt{2}\sin\psi}e^{jR\cos\psi}, \end{aligned} \tag{3.9}$$

where $C(z)$ and $S(z)$ are the two Fresnel integrals (see, for instance, [12], Sec. 7.3). A plot of $|K(\psi, R)|$ has been presented in Fig. 2.

As an illustration we have depicted in Fig. 3 the Wigner distribution function $F(x, 0, u, v)$ as a function of $u$ and $v$ for several values of $x$ (with $0 \leq x \leq a$). We remark that the maximum value of $F(x, 0, u, v)$ always occurs for $u = 0$ and $v = 0$, where this maximum $F(x, 0, 0, 0)$ is given by Eq. (3.5); in the figures, we have adapted the scale of the vertical axis accordingly. We have also scaled the horizontal $u$ and $v$ axes inversely proportional to the widths $2(a-x)$ and $2\sqrt{a^2-x^2}$ of the 'eye-shaped' interval in the $\frac{1}{2}x'$ and $\frac{1}{2}y'$ directions, respectively [see Eq. (3.2)], which leads to figures that look more or less similar.

The similarity between the various figures is indeed striking. We are thus tempted to conclude that all functions $F(x, 0, u, v)$ are similar to $F(0, 0, u, v)$ [given by Eq. (3.8)], except for some scaling:

$$\begin{aligned} F(x, 0, u, v) &\simeq \frac{F(x, 0, 0, 0)}{F(0, 0, 0, 0)}F\left(0, 0, \frac{a-x}{a}u, \frac{\sqrt{a^2-x^2}}{a}v\right) \\ &= 4\pi a^2\left[1 - \frac{2}{\pi}\arcsin\frac{x}{a} - \frac{2}{\pi}\frac{x}{a}\sqrt{1-\left(\frac{x}{a}\right)^2}\right]\frac{J_1\left(2\sqrt{(a-x)^2u^2+(a^2-x^2)v^2}\right)}{\sqrt{(a-x)^2u^2+(a^2-x^2)v^2}}. \end{aligned} \tag{3.10}$$

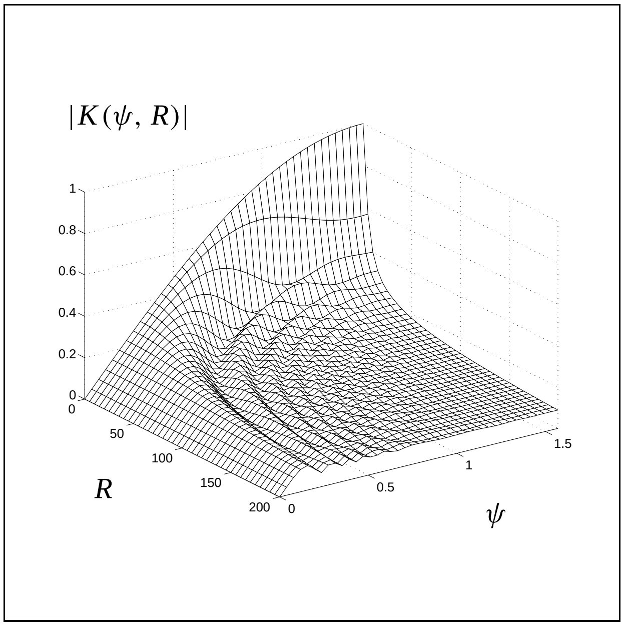


Figure 2: The function $K(\psi, R)$.

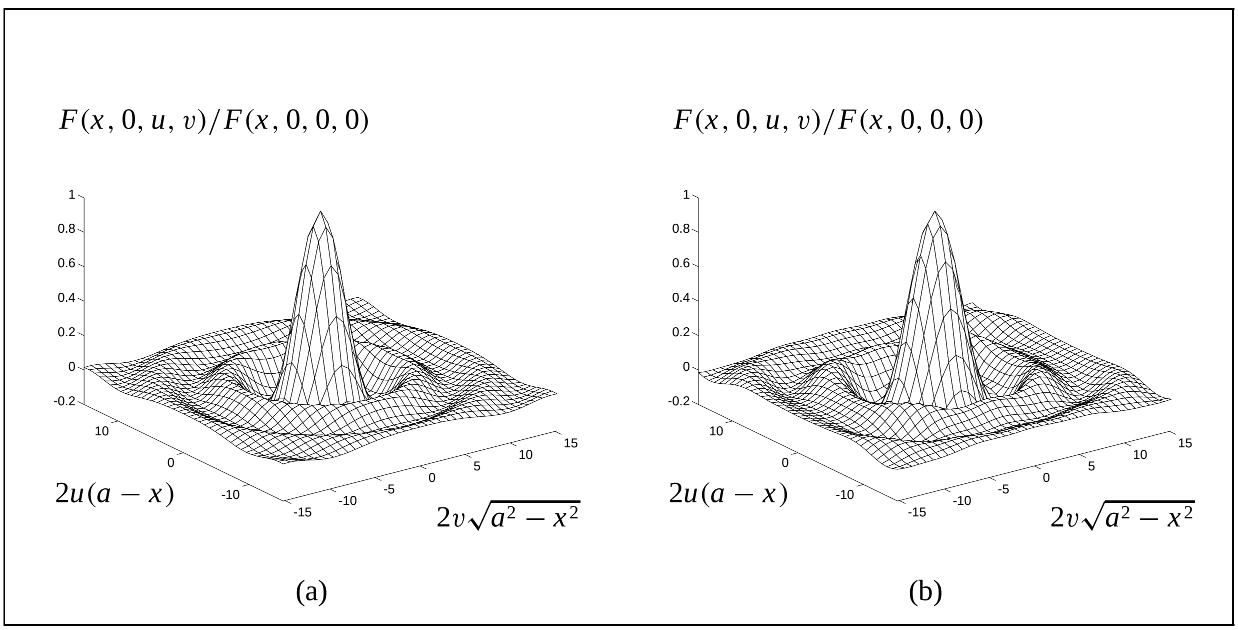


Figure 3: The Wigner distribution function $F(x, 0, u, v)$ of a circular source with radius $a$, for several values of the position variable $x$: (a) $x/a = 0$ and (b) $x/a = 0.95$.

It can easily be seen that the $\simeq$ sign in the latter equation becomes an $=$ sign for $x = 0$, for $x = a$, and for $u = v = 0$. Numerical investigation of the approximate expression (3.10) in relation to the exact expression (3.7) leads to the conclusion that the difference between these two expressions is less than 15% of the maximum value $F(x, 0, 0, 0)$.

The similarity between the Wigner distribution functions $F(x, 0, u, v)$ for different values of $x$ can also be deduced from examination of the 'eye-shaped' interval defined by Eq. (3.2). The boundary of this interval is given by the relationship $(\frac{1}{2}y')^2 + (x \mp \frac{1}{2}x')^2 = a^2$. If we scale the $x'$ and $y'$ variables according to $\frac{1}{2}x' = \xi(a - x)$ and $\frac{1}{2}y' = \eta\sqrt{a^2 - x^2}$, the boundary (expressed in the variables $\xi$ and $\eta$) is then given by the relationship

$$\eta^2 + \xi^2 - \frac{2x}{a + x}(\xi^2 \pm \xi) = 1. \tag{3.11}$$

We remark that for $x = 0$ the boundary is a circle, $\eta^2 + \xi^2 = 1$, while for $x = a$ the boundary is given by the intersection of two parabolas, $\eta^2 \mp \xi = 1$, see Fig. 4. However, for any value of $x$ with $0 \le x \le a$, the scaled 'eye-shaped' interval (3.11) is roughly the same and its two-dimensional Fourier transform will lead to roughly the same scaled Wigner distribution function.

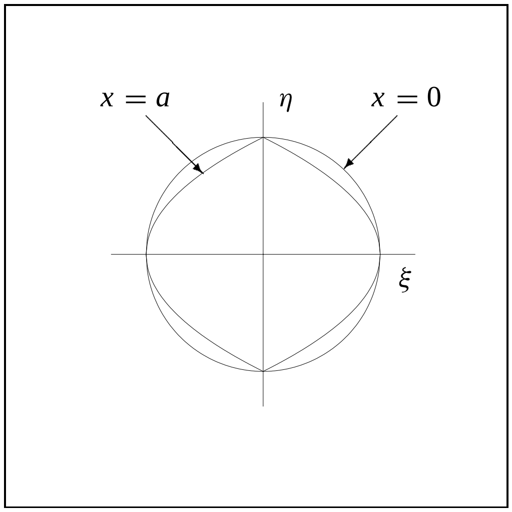


Figure 4: The scaled 'eye-shaped' interval for $x = 0$ (circle) and for $x = a$ (parabolas).

It might be interesting to compare the Wigner distribution function of a circular aperture with the Wigner distribution function of a rectangular aperture for which the signal $\varphi(x, y)$ satisfies the condition

$$\varphi(x, y) = \begin{cases} 1 & \text{for } 0 \le |x| \le a \text{ and } 0 \le |y| \le b \\ 0 & \text{elsewhere.} \end{cases} \tag{3.12}$$

In the latter case the Wigner distribution function takes the form

$$F(x, y, u, v) = 16(a - |x|)(b - |y|)\frac{\sin 2u(a - |x|)}{2u(a - |x|)}\frac{\sin 2v(b - |y|)}{2v(b - |y|)}, \tag{3.13}$$

in the interval $(0 \le |x| \le a, 0 \le |y| \le b)$ and vanishes outside that interval. For $u = v = 0$ we have $F(x, y, 0, 0) = 16(a - |x|)(b - |y|)$, which expression should be compared with Eq. (3.5), and for $x = y = 0$ we have $F(0, 0, u, v) = 16ab(\sin 2ua/2ua)(\sin 2vb/2vb)$, which expression should be compared with Eq. (3.8). We remark that the counterpart of the approximate expression (3.10) in the rectangular case reads

$$F(x, y, u, v) = \frac{F(x, y, 0, 0)}{F(0, 0, 0, 0)}F\left(0, 0, \frac{a - |x|}{a}u, \frac{b - |y|}{b}v\right), \tag{3.14}$$

and that this expression is exact.

## 4 Conclusion

We have determined the Wigner distribution function of a circular aperture and we have presented analytic expressions as well as numerical and graphical results. Knowledge of the Wigner distribution function of a circular aperture may be valuable in determining the effect of such an aperture in optical systems that are being analyzed by ray tracing.

## 5 Acknowledgement

The authors wish to express their gratitude to J. Boersma and J.J.A.M. Brands from the Mathematics Department, Eindhoven University of Technology, Netherlands, for helping them with the mathematics as presented in Appendix A and particularly in Appendix B.

## Appendix A. Derivation of Eq. (3.7)

The integral that occurs in Eq. (3.3) reads

$$I = \int_{x/a}^{1} \sin\left(2av\sqrt{1-\xi^2}\right)\, e^{j2au\xi}\, d\xi.$$

With the substitutions $2au = R\sin\theta$ and $2av = R\cos\theta$ (with $R \geq 0$ and $0 \leq \theta \leq \frac{1}{2}\pi$), $\xi = \sin\phi$ (with $0 \leq \phi \leq \frac{1}{2}\pi$), and $x/a = \sin\alpha$ (with $0 \leq \alpha \leq \frac{1}{2}\pi$), this integral takes the form

$$I = \int_{\alpha}^{\pi/2} \sin(R\cos\theta\cos\phi)\, e^{jR\sin\theta\sin\phi} \cos\phi d\phi.$$

Using the property $2j\sin z = e^{jz} - e^{-jz}$, we have

$$2jI = \int_{\alpha}^{\pi/2} e^{jR\cos(\phi-\theta)} \cos\phi d\phi - \int_{\alpha}^{\pi/2} e^{-jR\cos(\phi+\theta)} \cos\phi d\phi.$$

The problem thus reduces to determining the integral $I_0(\theta)$ defined by

$$I_0(\theta) = \int_{\alpha}^{\pi/2} e^{jR\cos(\phi-\theta)} \cos\phi d\phi,$$

from which the integral $I$ follows through the relation $2jI = I_0(\theta) - I_0^*(-\theta)$.

We now substitute $\phi - \theta = \beta$, yielding

$$\begin{aligned} I_0(\theta) &= \int_{\alpha-\theta}^{\pi/2-\theta} e^{jR\cos\beta} (\cos\theta\cos\beta - \sin\theta\sin\beta) d\beta \\ &= \cos\theta \int_{\alpha-\theta}^{\pi/2-\theta} e^{jR\cos\beta} \cos\beta d\beta - \sin\theta \int_{\alpha-\theta}^{\pi/2-\theta} e^{jR\cos\beta} \sin\beta d\beta = I_1(\theta) - I_2(\theta). \end{aligned}$$

The second integral $I_2(\theta)$ can easily be evaluated and yields

$$I_2(\theta) = \sin\theta \int_{\alpha-\theta}^{\pi/2-\theta} e^{jR\cos\beta} \sin\beta d\beta = \frac{\sin\theta}{jR} \left[ e^{jR\cos(\alpha-\theta)} - e^{jR\sin\theta} \right],$$

and the problem thus reduces to determining the integral

$$I_1(\theta) = \cos\theta \int_{\alpha-\theta}^{\pi/2-\theta} e^{jR\cos\beta} \cos\beta d\beta.$$

We now introduce the integral

$$K(\psi, R) = \int_0^{\psi} e^{jR\cos\beta} \cos\beta d\beta$$

from which the integral $I_1(\theta)$ follows through the relation $I_1(\theta) = \left[K\left(\frac{1}{2}\pi - \theta, R\right) - K(\alpha - \theta, R)\right]\cos\theta$. Note that $K(0, R) = 0$ and $K(\pi, R) = j\pi J_1(R)$ with $J_1(z)$ a first-order Bessel function (see, for instance, [12], Eq. 9.1.21). The relevant values of $\psi$ are $\psi = \frac{1}{2}\pi \pm \theta$ and $\psi = \alpha \pm \theta$ [where the plus signs arise from the fact that we will also need $I_0(-\theta)$], which values are confined to the interval $-\frac{1}{2}\pi \le \psi \le \pi$. However, due to the symmetry properties of the integrand, we have $K(\psi, R) = -K(-\psi, R)$ and $K\left(\frac{1}{2}\pi + \psi, R\right) - K^*\left(\frac{1}{2}\pi - \psi, R\right) = K\left(\frac{1}{2}\pi, R\right) - K^*\left(\frac{1}{2}\pi, R\right) = K(\pi, R) - K^*(0, R) = j\pi J_1(R)$, and we can restrict ourselves to nonnegative values of $\psi$, where $\psi$ may be confined to the interval $0 \le \psi \le \frac{1}{2}\pi$.

When we combine everything, the integral $I$ that we are looking for can be expressed in the form

$$I = \tfrac{1}{2}j[K(\alpha - \theta, R) - K^*(\alpha + \theta, R) - j\pi J_1(R)]\cos\theta + j\sin\theta \, \frac{\sin(R\cos\alpha\cos\theta)}{R} \, e^{jR\sin\alpha\sin\theta}.$$

We remark, however, that we can restrict ourselves to the first term of the latter expression, since we need only the real part of $Ie^{-jR\sin\alpha\sin\theta}$ to find the solution of Eq. (3.3). Note that for small values of $R$, i.e. for small values of $u$ and $v$, the integral $I$ can be approximated by $I \simeq \left(\frac{1}{4}\pi - \frac{1}{2}\alpha - \frac{1}{2}\sin\alpha\cos\alpha\right) R\cos\theta$, which immediately leads to Eq. (3.5).

## Appendix B. Derivation of Eq. (3.9)

To calculate $K(\psi, R)$ (with $0 \le \psi \le \frac{1}{2}\pi$) for large values of $R$, we substitute $\cos\beta = 1 - t$ (with $0 \le t \le 1$) and $\cos\psi = 1 - s$ (with $0 \le s \le 1$), yielding

$$K(\arccos[1-s], R) = \int_0^s e^{jR(1-t)} \frac{1-t}{\sqrt{1-(1-t)^2}} \, dt = e^{jR} \int_0^s e^{-jRt} \frac{1-t}{\sqrt{t(2-t)}} dt.$$

The relevant integral then reads

$$L(s, R) = \int_0^s e^{-jRt} g_0(t) \frac{dt}{\sqrt{t}} = L(s, R; g_0(s)),$$

with $g_0(t) = (1-t)/\sqrt{2-t}$ a function that is analytic for $|t| < \rho$ with $\rho > 1$. From

$$R \int_0^s e^{jR(s-t)} \frac{g_0(t) - g_0(0)}{\sqrt{t}} dt$$

$$= j \frac{g_0(t) - g_0(0)}{\sqrt{t}} e^{jR(s-t)} \Big|_0^s \quad - \quad j \int_0^s e^{jR(s-t)} \left( \frac{g_0(t) - g_0(0)}{\sqrt{t}} \right)' dt$$

$$= j \frac{g_0(s) - g_0(0)}{\sqrt{s}} \quad - \quad j e^{jRs} \int_0^s e^{-jRt} \frac{g_1(t)}{\sqrt{t}} dt$$

with $g_1(t) = g_0'(t) - \dfrac{g_0(t) - g_0(0)}{2t} = \dfrac{dg_0(t)}{dt} - \frac{1}{2} \sum_{m=1}^{\infty} \dfrac{d^m g_0(t)}{dt^m} \Big|_{t=0} \dfrac{t^{m-1}}{m!}$ an analytic function again, we get

$$L(s, R; g_0(s)) = g_0(0) \int_0^s e^{-jRt} \frac{dt}{\sqrt{t}} + [g_0(s) - g_0(0)] \frac{j}{R\sqrt{s}} e^{-jRs} - \frac{j}{R} L(s, R; g_1(s)).$$

If necessary, this procedure can be repeated to get a higher-order approximation of $L(s,\,R)$, leading to

$$L(s,\,R) = \sum_{n=0}^{\infty} \left(\frac{-j}{R}\right)^n \left[ g_n(0) \int_0^s e^{-jRt} \frac{dt}{\sqrt{t}} + [g_n(s) - g_n(0)] \frac{j}{R\sqrt{s}} e^{-jRs} \right],$$

with $g_0(s) = (1-s)/\sqrt{2-s}$ and $g_{n+1}(s) = g'_n(s) - [g_n(s) - g_n(0)]/2s$. Note that for small values of $s$ the expression $[g_n(s) - g_n(0)]/\sqrt{s}$ can be approximated by $g'_n(0)\sqrt{s}$, from which we conclude that $s = 0$ is not a singular point of this expression.

The remaining integral in the expression for $L(s,\,R)$ leads to

$$\int_0^s e^{-jRt} \frac{dt}{\sqrt{t}} = \sqrt{\frac{2\pi}{R}} \left[ C\left(\sqrt{\frac{2Rs}{\pi}}\right) - jS\left(\sqrt{\frac{2Rs}{\pi}}\right) \right],$$

where $C(z)$ and $S(z)$ are the two Fresnel integrals (see, for instance, [12], Sec. 7.3), and for large values of $R$ – and restricting ourselves to the terms that are proportional to $1/\sqrt{R}$, $1/R\sqrt{R}$, and $1/R$ – the function $K(\psi,\,R)$ can thus be approximated by

$$\begin{aligned} K(\psi,\,R) \;\simeq\; & \sqrt{\frac{\pi}{R}} \left(1 + \frac{3j}{8R}\right) \left[ C\left(\sqrt{\frac{2R(1-\cos\psi)}{\pi}}\right) - jS\left(\sqrt{\frac{2R(1-\cos\psi)}{\pi}}\right) \right] e^{jR} \\ & + \frac{j}{R} \frac{\sqrt{2}\cos\psi - \sqrt{1+\cos\psi}}{\sqrt{2}\sin\psi}\, e^{jR\cos\psi}. \end{aligned}$$

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