Time-frequency signal representations

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Abstract

This invited paper – of a tutorial and review character – presents an overview of two classes of time-frequency signal representations. The first class, in which the signal arises linearly, deals with the windowed Fourier transform and its sampled version (also known as the Gabor transform) and the inverse of the latter: Gabor's signal expansion. We will show how Gabor's signal expansion and the windowed Fourier transform are related and how they can benefit from each other. The second class, in which the signal arises quadratically (or bilinearly, as it is often called), is based on the Wigner distribution. We will show some examples of the Wigner distribution and discuss some of its important properties. Being a bilinear signal representation, the Wigner distribution shows artifacts in the case of multi-component signals. To reduce these artifacts, a large class of bilinear signal representations has been constructed, known as the shift-covariant Cohen class. We will consider this class and we will see how all its members can be considered as properly averaged versions of the Wigner distribution.

1 Introduction

It is sometimes convenient to describe a time signal $\varphi(t)$, say, not in the time domain, but in the frequency domain by means of its frequency spectrum, i.e., the Fourier transform $\bar{\varphi}(\omega)$ of the function $\varphi(t)$, which is defined by

$$\bar{\varphi}(\omega) = \int \varphi(t)\, e^{-j\omega t}\, \mathrm{d}t; \tag{1}$$

a bar on top of a symbol will mean throughout that we are dealing with a function in the frequency domain. (Unless otherwise stated, all integrations and summations in this paper extend from $-\infty$ to $+\infty$.) The frequency spectrum shows us the global distribution of the energy of the signal as a function of frequency. However, one is often more interested in the momentary or local distribution of the energy as a function of frequency.

An obvious candidate for a local-frequency description of a signal is the windowed Fourier transform with a moving window, which is well known in speech processing [1]. A local frequency spectrum like the windowed Fourier transform describes the signal in time and frequency, simultaneously. It is thus a function of two variables, derived, however, from a function of one variable. Therefore, it must satisfy certain restrictions, or, to put it another way: not any function of two variables is a local frequency spectrum. In this paper we will show that the windowed Fourier transform is completely determined by its values on the points of a certain time-frequency lattice, which is exactly the lattice suggested by Gabor as early as 1946.

Related to the windowed Fourier transform is Gabor's signal expansion [2]. In 1946, Gabor suggested the expansion of a signal into a discrete set of properly shifted and modulated Gaussian elementary signals. We will show that there exists a strong relationship between Gabor's signal expansion and the sampling of the windowed Fourier transform, and that Gabor's signal expansion can be used to reconstruct the signal from its sampled windowed Fourier transform.

Another favorite candidate for a local frequency spectrum is the Wigner distribution [3], introduced in 1932 by Wigner in mechanics to describe mechanical phenomena in a so-called phase space. The Wigner distribution is a representative of a rather broad class of bilinear time-frequency functions [4], which are related to each other by linear transformations. Some well-known time-frequency representations – like Woodward's ambiguity function [5], Rihaczek's complex energy density function [6], and the spectrogram [7] – belong to this class.

Many review papers on time-frequency signal analysis have been published. We mention in particular [8] and [9].

2 Windowed Fourier transform

The windowed Fourier transform $s(\tau,\omega)$ of the signal $\varphi(t)$ is defined as

$$s(\tau,\omega) = \int \varphi(t)\, w^*(t-\tau)\, e^{-j\omega t}\, \mathrm{d}t, \tag{2}$$

where $w(t)$ is the window function. It can be considered as the Fourier transform of the product of the signal $\varphi(t)$ and a conjugated and shifted version of the window function $w(t)$. The window function may be chosen rather arbitrarily; mostly it will be a function that is more or less concentrated around the origin. The windowed Fourier transform can then be considered as a windowed or short-time Fourier transform of the signal, which, indeed, can be interpreted as a local frequency spectrum. If the window function is chosen a very narrow function, like a

Dirac delta function, the windowed Fourier transform reduces to a pure time representation of the signal; if, on the other hand, the window function is chosen constant, the windowed Fourier transform reduces to a pure frequency representation. In general, the windowed Fourier transform is an intermediate signal description between the pure time and the pure frequency representation. We recall that the squared modulus of the windowed Fourier transform, $|s(\tau,\omega)|^2$, is known as the spectrogram [7].

Since the definition (2) of the windowed Fourier transform $s(\tau,\omega)$ can be considered as a Fourier transformation of the product $\varphi(t)\, w^*(t-\tau)$ with respect to t, we can easily find a way to reconstruct the signal $\varphi(t)$ from its windowed Fourier transform by simply writing down the corresponding inverse Fourier transformation. There exists another, more attractive way of reconstructing the signal from its windowed Fourier transform, viz., by means of the inversion formula

$$\varphi(t) \int |w(t)|^2 \,\mathrm{d}t = \frac{1}{2\pi} \iint s(\tau,\omega)\, w(t-\tau)\, e^{j\omega t} \,\mathrm{d}\tau \,\mathrm{d}\omega, \tag{3}$$

which represents the signal as a linear combination of shifted and modulated window functions, with the windowed Fourier transform $s(\tau,\omega)$ as a weighting function. The function $s(\tau,\omega)$ thus acts as a distribution function – a local frequency spectrum – and shows how the signal $\varphi(t)$ can be synthesized from the (τ,ω)-parameterized set of shifted and modulated versions of $w(t)$.

Reconstruction of $\varphi(t)$ from $s(\tau,\omega)$ is possible in many ways, and should be possible if we have only partial knowledge of $s(\tau,\omega)$. In the next section we will show how the signal can be reconstructed if we only know the values $s(mT,k\Omega)$ of the windowed Fourier transform on the rectangular time-frequency lattice $\tau = mT, \omega = k\Omega$.

3 Gabor's signal expansion

In 1946, Gabor suggested the expansion of a signal into a discrete set of elementary signals (or synthesis windows, as we will call them) [2]. Although Gabor restricted himself to a synthesis window that had a Gaussian shape, his signal expansion holds for rather arbitrarily shaped window functions [10, 11]. With the help of Gabor's signal expansion, we can express the signal $\varphi(t)$ as a superposition of properly shifted and modulated versions of a synthesis window $g(t)$, say, yielding

$$\varphi(t) = \sum_{mk} a_{mk}\, g(t-mT)\, e^{jk\Omega t}, \tag{4}$$

where the time shift T and the frequency shift Ω satisfy the relation $\Omega T = 2\pi$. Unlike the inversion formula (3), which represents the signal as a continuum of window functions, Gabor's signal expansion (4) represents the signal as a discrete set of synthesis windows that are shifted over discrete distances mT and that are modulated with discrete frequencies $k\Omega$.

3.1 The bi-orthonormal analysis window

In general, the discrete set of shifted and modulated synthesis windows $g_{mk}(t) = g(t-mT)\exp(jk\Omega t)$ need not be orthonormal, which implies that Gabor's expansion coefficients a_{mk} cannot be determined in the usual way, i.e., by determining the inner product of $g_{mk}(t)$ with $\varphi(t)$. In this section, we show how we can find an analysis window $w(t)$, say, that is bi-orthonormal to the set of synthesis windows in the sense

$$\int g(t)\, w^*(t-mT)\, e^{-jk\Omega t} \,\mathrm{d}t = \delta_m \delta_k, \tag{5}$$

where δ_m is the Kronecker delta ($\delta_0 = 1, \delta_m = 0$ for $m \neq 0$). With the help of this bi-orthonormal analysis window $w(t)$, the expansion coefficients then follow readily via the analysis formula

$$a_{mk} = \int \varphi(t)\, w^*(t-mT)\, e^{-jk\Omega t} \,\mathrm{d}t. \tag{6}$$

The relationship between Gabor's signal expansion and the windowed Fourier transform becomes apparent by noting that the right-hand side of relation (6) can be interpreted as a sampled windowed Fourier transform with window function $w(t)$.

Instead of the bi-orthonormality condition (5), we can as well use the equivalent condition (as long as $\Omega T = 2\pi$)

$$\sum_{mk} g_{mk}(t_1)\, w^*_{mk}(t_2) = \delta(t_1 - t_2). \tag{7}$$

The first bi-orthonormality condition (5) guarantees that if we start with an array of coefficients a_{mk}, construct a signal $\varphi(t)$ via Eq. (4), and subsequently substitute this signal into Eq. (6), we end up with the original coefficients array; the second, more practical bi-orthonormality condition (7) guarantees that if we start with a certain signal $\varphi(t)$, construct its Gabor coefficients a_{mk} via Eq. (6), and substitute these coefficients into Eq. (4), we end up with the original signal. We thus conclude that the two equations (4) and (6) form a transform pair.

To derive the window function $w(t)$, we start with Eq. (6) and define the Fourier transform $\bar{a}(\xi,\eta)$ of the two-dimensional array a_{mk}:

$$\bar{a}(\xi,\eta) = \sum_{mk} a_{mk}\, e^{-j2\pi(m\eta - k\xi)}. \tag{8}$$

After a proper rearranging of the factors, we write

$$\bar{a}(\xi,\eta) = T\, \tilde{\varphi}_T\left(\xi T, \eta\frac{2\pi}{T}\right)\, \tilde{w}^*_T\left(\xi T, \eta\frac{2\pi}{T}\right), \tag{9}$$

where $\tilde{\varphi}_\tau(t,\omega)$ is defined as

$$\tilde{\varphi}_\tau(t,\omega) = \sum_n \varphi(t+n\tau)\, e^{-jn\tau\omega}; \tag{10}$$

$\tilde{\varphi}_\tau(t,\omega)$ is often called the Zak transform of $\varphi(t)$. Note that Eq. (9) is the product equivalent of Eq. (6), and that the procedure that we described is comparable to the way in which the Fourier transformation brings a convolution of two time functions into product form. We remark that the Zak transform is, in essence, a Fourier transform. Hence, if we translate everything to discrete-time signals, fast algorithms can be used like the FFT, resulting in a fast Zak transform, see Eq. (10), and a fast Gabor transform (i.e., a sampled windowed Fourier transform), see Eq. (6). This technique is similar to the fast convolution, well-known in digital signal processing.

To proceed in determining the analysis window $w(t)$, we also express the Gabor expansion (4) in product form,

$$\tilde{\varphi}_T\left(\xi T, \eta\frac{2\pi}{T}\right) = \bar{a}(\xi,\eta)\,\tilde{g}_T\left(\xi T, \eta\frac{2\pi}{T}\right), \qquad (11)$$

and we do the same for the bi-orthonormality conditions (5) and (7):

$$T\,\tilde{g}_T\left(\xi T, \eta\frac{2\pi}{T}\right)\,\tilde{w}_T^*\left(\xi T, \eta\frac{2\pi}{T}\right) = 1. \qquad (12)$$

Clearly, the analysis window $w(t)$ can now be found from the synthesis window $g(t)$ by using the condition (12) that holds for their Zak transforms. Figure 1 shows the analysis window $w(t)$ that corresponds to a Gaussian synthesis window $g(t)$.

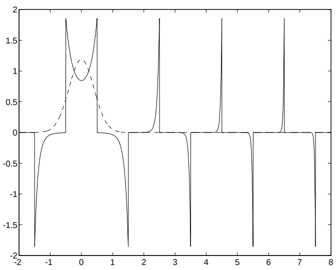
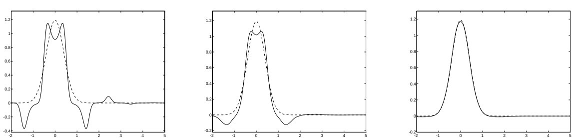


Figure 1: A Gaussian window $g(t) = 2^{1/4}\exp[-\pi(t/T)^2]$ (dashed line) and its corresponding optimum analysis window $T\,w(t)$ (solid line).

3.2 Oversampling

Without going into much detail, we remark that oversampling, i.e., $\Omega T < 2\pi$, allows us to impose additional conditions on the analysis window (which is no longer unique in the case of oversampling); we may thus construct analysis windows that have much nicer properties and resemble, for instance, the synthesis window, see Fig. 2. Indeed, in the case of rational oversampling,



(a) (b) (c)

Figure 2: A Gaussian window $g(t) = 2^{1/4}\exp[-\pi(t/\sigma_t)^2]$ (dashed line) and its corresponding optimum analysis window $(T/q)\,w(t)$ (solid line), for different values of rational oversampling $2\pi/\Omega T = p/q$, while maintaining the proportionality condition $\sigma_t/T = \sigma_\omega/\Omega = \sqrt{2\pi/\Omega T}$: (a) $2\pi/\Omega T = 7/6$, (b) $2\pi/\Omega T = 3/2$, and (c) $2\pi/\Omega T = 3$.

$2\pi/\Omega T = p/q$ (with $p \geq q \geq 1$), we can proceed in the same way as we did before, the only difference being that the product forms (9), (11), and (12) now take the forms of vector-matrix and matrix-matrix products,

$$\mathbf{a}(\xi,\eta) = \frac{pT}{q}\,\mathbf{W}^*(\xi,\eta)\,\boldsymbol{\phi}(\xi,\eta), \qquad (13)$$

$$\boldsymbol{\phi}(\xi,\eta) = \frac{1}{p}\,\mathbf{G}(\xi,\eta)\,\mathbf{a}(\xi,\eta), \qquad (14)$$

$$\mathbf{I}_q = \frac{T}{q}\,\mathbf{G}(\xi,\eta)\,\mathbf{W}^*(\xi,\eta), \qquad (15)$$

where $\boldsymbol{\phi}(\xi,\eta)$ is a q-dimensional column vector, $\mathbf{a}(\xi,\eta)$ is a p-dimensional column vector, $\mathbf{G}(\xi,\eta)$ and $\mathbf{W}(\xi,\eta)$ are $(q \times p)$-dimensional matrices, and $\mathbf{I}_q$ is the q-dimensional identity matrix; moreover $0 \leq \xi < 1$ and $0 \leq \eta < 1/p$. The optimum analysis window then follows from the generalized (Moore-Penrose) inverse:

$$\mathbf{W}^*_{opt}(\xi,\eta) = \frac{q}{T}\,\mathbf{G}^*(\xi,\eta)\,[\mathbf{G}(\xi,\eta)\,\mathbf{G}^*(\xi,\eta)]^{-1}. \quad (16)$$

3.3 Discrete-time case with oversampling

In the discrete-time case we write [12]

$$a_{mk} = \sum_n \varphi[n]\,w^*[n-mN]\,e^{-j2\pi kn/K}, \quad (17)$$

$$\varphi[n] = \sum_m \sum_{k=0}^{K-1} a_{mk}\,g[n-mN]\,e^{j2\pi kn/K}, \quad (18)$$

and we assume that $\varphi[n]$ and $w[n]$ have finite supports N_φ and N_w, respectively. The array a_{mk}, which is K-periodic in k, has a finite support M in m, where M satisfies the condition $MN \geq N_\varphi + N_w - 1$. The degree of oversampling is $K/N = p/q$, where $p \geq q \geq 1$ and p and q do not have common factors. If we introduce the integers J and L, and write $K = pJ$, $M = pL$, and $N = qJ$, the Gabor transform (17) and Gabor's signal expansion (18) can again be brought into product forms,

$$\mathbf{a}(\xi,\eta) = K\,\mathbf{W}^*(\xi,\eta)\,\boldsymbol{\phi}(\xi,\eta), \qquad (19)$$

$$\boldsymbol{\phi}(\xi,\eta) = \frac{1}{p}\,\mathbf{G}(\xi,\eta)\,\mathbf{a}(\xi,\eta), \qquad (20)$$

cf. Eqs. (13) and (14), and we can proceed as we did before to derive the relation between the analysis window $w[n]$ and the synthesis window $g[n]$. Note that the Gabor transform, which is based on finite-support functions $\varphi[n]$ and $w[n]$, can be calculated using fast algorithms. And if we are dealing with an infinitely long signal $\varphi[n]$, we simply divide it in parts that have a finite support and use the well-known overlap-add method.

3.4 Concluding remarks about Gabor

The Gabor transform (6), which has been introduced as the sampled Fourier transform of the windowed signal $\varphi(t)\,w^*(t-mT)$, can also be interpreted

- as the inner product of $\varphi(t)$ and the elements $w_{mk}(t)$ of an (m,k)-parameterized set of basis functions $w(t)$
- and as the sampled output signals of a k-parameterized filter bank with impulse responses $w_k'(t) = w^*(-t)\exp(jk\Omega t)$ and input signal $\varphi(t)$:

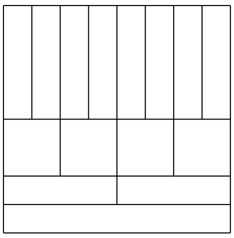
$$a_{mk} = e^{-jmk\Omega T}\int \varphi(t)\,w_k'(mT-t)\,\mathrm{d}t. \qquad (21)$$

Gabor's signal expansion (4) can be interpreted
• as a superposition of the elements $g_{mk}(t)$ of an (m,k)-parameterized set of basis functions $g(t)$ with weights a_{mk}
• and as a superposition of the outputs of a k-parameterized filter bank with impulse responses $g'_k(t) = g(t)\,\exp(jk\Omega t)$ and the k-parameterized input sequences a_{mk}:

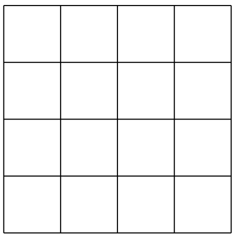
$$\varphi(t) = \sum_k \left(\sum_m a_{mk}\, g'_k(t - mT) \right). \qquad (22)$$

The second interpretations suggest a treatment in the frequency domain rather than in the time-domain, and we can benefit from the knowledge about filter banks.

There exists a relation between the Gabor expansion and a signal representation in terms of wavelets. The difference between these two representations is simply a different tiling of the time-frequency domain, resulting in a time-scale representation (for the wavelet tiling) rather than a time-frequency representation (for the Gabor tiling) of the signal.



Wavelet tiling
Constant-Q filter bank
multirate



Gabor tiling
Constant-bandwidth filter bank
constant rate

4 Wigner distribution

In 1932, Wigner introduced the distribution function [3]

$$W(t,\omega) = \int \varphi(t + \tfrac{1}{2}t')\, \varphi^*(t - \tfrac{1}{2}t')\, e^{-j\omega t'}\, \mathrm{d}t',$$
$$= \frac{1}{2\pi} \int \bar{\varphi}(\omega + \tfrac{1}{2}\omega')\, \bar{\varphi}^*(\omega - \tfrac{1}{2}\omega')\, e^{j\omega' t}\, \mathrm{d}\omega', \quad (23)$$

which again represents a signal in time and frequency, simultaneously, and which, for a broad class of signals, behaves like a local frequency spectrum. To show this, we give some easy examples in the next section.

Whereas the products $\varphi(t_1)\,\varphi^*(t_2)$ and $\bar{\varphi}(\omega_1)\,\bar{\varphi}^*(\omega_2)$ are related by means of a double Fourier transformation,

$$\bar{\varphi}(\omega_1)\,\bar{\varphi}^*(\omega_2)$$
$$= \iint \varphi(t_1)\,\varphi^*(t_2)\, e^{-j(\omega_1 t_1 - \omega_2 t_2)}\, \mathrm{d}t_1\, \mathrm{d}t_2, \quad (24)$$

the Wigner distribution $W(t,\omega)$ is somewhere 'in the middle' between the (t_1,t_2) and the (ω_1,ω_2) description, and arises when we apply the coordinate transformations $t_{1,2} = t \pm \frac{1}{2}t'$ and $\omega_{1,2} = \omega \pm \frac{1}{2}\omega'$, and subsequently integrate over t' only. A sole integration over t instead of t' also leads to an intermediate signal description, and the resulting function, now depending on the time lag t' and the Doppler shift ω', is known as the ambiguity function $A(t',\omega')$ and is used in the field of radar. It will be obvious that the Wigner distribution and the ambiguity function are related through a double Fourier transformation.

To calculate $W(t,\omega)$ in practice, using its time-domain definition, we have to restrict the integration interval for t'. We model this by using a window function $w(\frac{1}{2}t')$, so that the modified definition takes the form

$$W(t,\omega) \simeq P(t,\omega;w) = \int \varphi(t + \tfrac{1}{2}t')\, w(\tfrac{1}{2}t')$$
$$\times\, w^*(-\tfrac{1}{2}t')\, \varphi^*(t - \tfrac{1}{2}t')\, e^{-j\omega t'}\, \mathrm{d}t'$$
$$= \int S(t, \omega + \tfrac{1}{2}\theta; w)\, S^*(t, \omega - \tfrac{1}{2}\theta; w)\, \mathrm{d}\theta, \quad (25)$$

where $S(t,\omega;w) = s(t,\omega)\,\exp(j\omega t)$ is a slightly modified version of the windowed Fourier transform of $\varphi(t)$ with window function $w(t)$. The function $P(t,\omega;w)$ is known as the pseudo-Wigner distribution.

4.1 Examples of the Wigner distribution

The Wigner distribution of a Dirac delta function $\varphi(t) = \delta(t - t_\circ)$ reads $W(t,\omega) = \delta(t - t_\circ)$ and tells us exactly what we expect: at one moment $t = t_\circ$ (and one moment only!), all frequencies are present, whereas there is no contribution at other moments.

The Wigner distribution of a time-harmonic signal $\varphi(t) = \exp(j\omega_\circ t)$ reads $W(t,\omega) = 2\pi\,\delta(\omega - \omega_\circ)$ and tells us again what we expect: at all time moments, only one frequency $\omega = \omega_\circ$ manifests itself. We remark that a Dirac delta function and a harmonic signal are dual to each other, i.e., the Fourier transform of one function has the same form as the other function. Due to this duality, and in view of the fact that the definition of the Wigner distribution is similar in the time domain and the frequency domain, the Wigner distribution of a harmonic signal, with Fourier transform $\bar{\varphi}(\omega) = 2\pi\,\delta(\omega - \omega_\circ)$, is the same as the one of the Dirac delta function, but rotated in the time-frequency domain through 90°.

The Wigner distribution of a linear chirp signal $\varphi(t) = \exp(j\frac{1}{2}\alpha t^2)$ reads $W(t,\omega) = 2\pi\,\delta(\omega - \alpha t)$, and the linear chirp is apparent as a straight line in the time-frequency domain. The fact that a quadratic-phase function yields a Dirac delta function in the time-frequency domain is very attractive in optics [13, 14], especially in first-order optics, where many signals (like spherical waves) and systems (like lenses and sections of free space) are described by quadratic-phase functions.

For a smooth-phase signal $\varphi(t) = \exp[j\psi(t)]$, for instance an FM signal, where $\psi(t)$ is a sufficiently smooth function of time t, the Wigner distribution takes the form $W(t,\omega) \simeq 2\pi\,\delta(\omega - \mathrm{d}\psi/\mathrm{d}t)$, and we note that the Wigner distribution shows us the instantaneous frequency $\mathrm{d}\psi/\mathrm{d}t$.

4.2 Properties of the Wigner distribution

The Wigner distribution has many nice, conceptually attractive properties. We mention a few of them.
• The Wigner distribution is a real, bilinear signal representation, very well suited for energy considerations.

• The Wigner distribution is time-shift and frequency-shift covariant:

$$\begin{aligned} \text{if} \quad \varphi(t) &\rightarrow W(t,\omega), \\ \text{then} \quad \varphi(t-t_\circ)\, e^{j\omega_\circ t} &\rightarrow W(t-t_\circ, \omega-\omega_\circ). \end{aligned} \tag{26}$$

• A so-called canonical transformation of the signal,

$$\varphi_o(t_o) = (j2\pi b)^{-1/2} e^{j\phi} \times \int e^{jb^{-1}(\frac{1}{2}at_i^2 - t_o t_i + \frac{1}{2}dt_o^2)} \varphi_i(t_i)\, \mathrm{d}t_i \tag{27}$$

with $b \neq 0$, leads to a mere coordinate transformation of the Wigner distribution (with $ad - bc = 1$):

$$W_o(at + b\omega, ct + d\omega) = W_i(t, \omega). \tag{28}$$

For $a = d = \cos\gamma$ and $b = -c = \sin\gamma$, and choosing $\phi = \frac{1}{2}\gamma$, the canonical transformation $\varphi_i(t) \rightarrow \varphi_o(t)$ takes the form of a fractional Fourier transformation, $\varphi(t) \rightarrow (2\pi)^{-1/2}\, \bar{\varphi}_\gamma(t)$, with $\gamma = \frac{1}{2}\pi$ corresponding to the common Fourier transformation: $\bar{\varphi}_{\pi/2}(\omega) = \bar{\varphi}(\omega)$. The Wigner distribution then undergoes a mere rotation, $W_o(t,\omega) = W_i(t\cos\gamma - \omega\sin\gamma, t\sin\gamma + \omega\cos\gamma)$, with a rotation through 90° in the case of the common Fourier transformation.

• The signal intensity $|\varphi(t)|^2 = (2\pi)^{-1}\, |\bar{\varphi}_0(t)|^2$ follows as the integral over all frequencies,

$$\int W(t,\omega)\, \mathrm{d}\omega = |\bar{\varphi}_0(t)|^2 = 2\pi\, |\varphi(t)|^2; \tag{29}$$

for the fractional Fourier transform $\bar{\varphi}_\gamma(t)$ we have

$$\int W(t\cos\gamma - \omega\sin\gamma, t\sin\gamma + \omega\cos\gamma)\, \mathrm{d}\omega = |\bar{\varphi}_\gamma(t)|^2, \tag{30}$$

with the spectral intensity $|\bar{\varphi}(\omega)|^2 = \int W(t,\omega)\, \mathrm{d}t$ as a special case for $\gamma = \frac{1}{2}\pi$. The correctness of these marginals is an attractive property of the Wigner distribution.

• The Wigner distribution, although real, is not necessarily nonnegative; but we do have Moyal's formula

$$\frac{1}{2\pi} \iint W_{\varphi_1}(t,\omega)\, W_{\varphi_2}(t,\omega)\, \mathrm{d}t\, \mathrm{d}\omega = \left| \int \varphi_1(t)\, \varphi_2^*(t)\, \mathrm{d}t \right|^2 \geq 0, \tag{31}$$

which shows us that averaging one Wigner distribution with another one always yields a nonnegative result.

• Due to its bilinear signal dependence, the Wigner distribution is especially suited for non-stationary noise, if defined as

$$W(t,\omega) = \int \mathrm{E}\{\varphi(t + \tfrac{1}{2}t')\, \varphi^*(t - \tfrac{1}{2}t')\}\, e^{-j\omega t'}\, \mathrm{d}t', \tag{32}$$

using the expectation value $\mathrm{E}\{\varphi(t_1)\, \varphi^*(t_2)\}$. The Wigner distribution is thus a straightforward extension of the well-known power spectrum of stationary noise.

• An important property of the Wigner distribution $W(t,\omega)$ is that its normalized first-order moment with respect to ω yields the instantaneous frequency $\mathrm{d}\psi/\mathrm{d}t$ of the (analytic) signal $\varphi(t) = |\varphi(t)| \exp[j\psi(t)]$:

$$\frac{\mathrm{d}\psi(t)}{\mathrm{d}t} = \left(\int W(t,\omega)\, \mathrm{d}\omega \right)^{-1} \int \omega\, W(t,\omega)\, \mathrm{d}\omega. \tag{33}$$

It is this property that makes the Wigner distribution attractive for the study of FM signals.

4.3 The Cohen class

The Wigner distribution belongs to a broad class of time-frequency functions known as the Cohen class. Any function of this class is described by the general formula

$$C(t,\omega) = \frac{1}{2\pi} \iiint \varphi(\tau + \tfrac{1}{2}t')\, \varphi^*(\tau - \tfrac{1}{2}t') \times k(t,\omega,t',\omega')\, e^{-j(\omega t' - \omega' t + \omega'\tau)}\, \mathrm{d}\tau\, \mathrm{d}t'\, \mathrm{d}\omega' \tag{34}$$

and the choice of the kernel $k(t,\omega,t',\omega')$ selects one particular function of the Cohen class. The Wigner distribution, for instance, arises for $k(t,\omega,t',\omega') = 1$, whereas $k(t,\omega,t',\omega') = 2\pi\, \delta(\omega-\omega')\, \delta(t-t')$ yields the ambiguity function $A(t',\omega')$. In this section we will restrict ourselves to the case that $k(t,\omega,t',\omega')$ does not depend on the time variable t and the frequency variable ω, hence $k(t,\omega,t',\omega') = \bar{K}(t',\omega')$, in which case the resulting time-frequency distribution $C(t,\omega)$ is shift covariant, like the Wigner distribution $W(t,\omega)$, see Eq. (26).

4.4 Multi-component signals

The Wigner distribution is a bilinear signal representation; usually, however, we deal with a linear signal representation, and using a bilinear representation yields cross-terms if the signal consists of multiple components. The two-component signal $\varphi(t) = \varphi_1(t) + \varphi_2(t)$ yields the Wigner distribution

$$W_\varphi(t,\omega) = W_{\varphi_1}(t,\omega) + W_{\varphi_2}(t,\omega) + 2\,\mathrm{Re}\left\{ \int \varphi_1(t + \tfrac{1}{2}t')\, \varphi_2^*(t - \tfrac{1}{2}t')\, e^{-j\omega t'}\, \mathrm{d}t' \right\} \tag{35}$$

and we notice a cross-term in addition to the auto-terms $W_{\varphi_1}(t,\omega)$ and $W_{\varphi_2}(t,\omega)$. In the case of two time-harmonic components $\exp(j\omega_1 t)$ and $\exp(j\omega_2 t)$, for instance, the cross-term reads

$$4\pi\, \delta[\omega - \tfrac{1}{2}(\omega_1 + \omega_2)] \cos[(\omega_1 - \omega_2)t].$$

It appears at the frequency $\frac{1}{2}(\omega_1 + \omega_2)$, i.e., in the middle between the two auto-terms $W_{\varphi_1}(t,\omega) = 2\pi\, \delta(\omega - \omega_1)$ and $W_{\varphi_2}(t,\omega) = 2\pi\, \delta(\omega - \omega_2)$, and is modulated in the t direction with frequency $\omega_1 - \omega_2$. We can get rid of this cross-term by averaging the Wigner distribution with a kernel that is narrow in the ω direction and broad in the t direction. We thus remove the cross-term without seriously disturbing the auto-terms.

The requirement of removing cross-terms without seriously affecting the auto-terms has led to the shift-covariant Cohen class of bilinear signal representations. All members $C(t,\omega)$ of this class can be generated by a convolution (both for t and ω) of the Wigner distribution with an appropriate kernel $K(t,\omega)$:

$$C(t,\omega) = \iint K(t-t_\circ,\omega-\omega_\circ)\, W(t_\circ,\omega_\circ)\, \mathrm{d}t_\circ\, \mathrm{d}\omega_\circ. \tag{36}$$

Note that a convolution keeps the important property of shift covariance! After Fourier transforming the latter equation, we are led to an expression in the 'ambiguity domain,' and the convolution becomes a product:

$$\bar{C}(t',\omega') = \bar{K}(t',\omega')\, A(t',\omega'). \tag{37}$$

The product form (37) offers an easy way in the design of appropriate kernels.

4.5 Some basic Cohen kernels

Many kernels have been proposed in the past, and some already existing bilinear signal representations have been identified as belonging to the Cohen class with an appropriately chosen kernel. Table 1 mentions some of them. [4, 8, 9]

bilinear signal representation	$\bar{K}(t',\omega')$
Wigner $W(t,\omega)$	1
pseudo-Wigner $P(t,\omega;w)$	$w(\frac{1}{2}t')\, w^*(-\frac{1}{2}t')$
Kirkwood-Rihaczek	$\exp(-j\frac{1}{2}\omega' t')$
Levin	$\cos(\frac{1}{2}\omega' t')$
Born-Jordan	$\sin(\frac{1}{2}\alpha\omega' t')/\frac{1}{2}\alpha\omega' t'$
Choi-Williams	$\exp[-(\omega' t')^2/\sigma]$
spectrogram $\lvert S(t,\omega;w)\rvert^2$	$A_w(-t',-\omega')$

Table 1: Kernels $\bar{K}(t',\omega')$ of some basic Cohen-class bilinear signal representations.

In designing kernels, one may try to keep the interesting properties of the Wigner distribution; this reflects itself in conditions for the kernel. We recall that shift covariance is already maintained. To keep also the properties of realness, t marginal, and ω marginal, for instance, the kernel $\bar{K}(t',\omega')$ should satisfy the conditions $\bar{K}(t',\omega') = \bar{K}^*(-t',-\omega')$, $\bar{K}(0,\omega') = 1$, and $\bar{K}(t',0) = 1$, respectively. To keep the important property that for a signal $\varphi(t) = |\varphi(t)|\exp[j\psi(t)]$ the instantaneous frequency $\mathrm{d}\psi/\mathrm{d}t$ follows from the bilinear representation through its first-order moment, like it does for the Wigner distribution, see Eq. (33), the kernel should satisfy the condition $\bar{K}(0,\omega') =$ constant and $\partial\bar{K}/\partial t'\big|_{t'=0} = 0$. The Levin, Born-Jordan, and Choi-Williams representations clearly satisfy these conditions.

A less well-known, but very powerful kernel, results from modifying the definition of the pseudo-Wigner distribution, see Eq. (25), by performing an additional windowing in the ω direction [15]:

$$P(t,\omega;w,z) = \int S(t,\omega+\tfrac{1}{2}\theta;w)\, z(\theta) \times S^*(t,\omega-\tfrac{1}{2}\theta;w)\, \mathrm{d}\theta. \tag{38}$$

Note that $z(\theta)$ controls the behavior of $P(t,\omega;w,z)$: either more spectrogram-type $|S(t,\omega;w)|^2$ for a narrow window $z(\theta) \simeq \delta(\theta)$, with no cross-terms but blurred auto-terms, or more pseudo-Wigner-type $P(t,\omega;w)$ for a wide window $z(\theta) \simeq 1$, with highly localized auto-terms but annoying cross-terms. The corresponding kernel reads $K(t,\omega) = W_w(-t,-\omega)\, \bar{z}(-t\cos\gamma+\omega\sin\gamma)$, where we have added an additional angle γ with which we can align the kernel in the time-frequency domain, if necessary [16]; $\gamma = 0$ leads to the original definition (38).

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