GABOR TRANSFORM AND ZAK TRANSFORM WITH RATIONAL OVERSAMPLING

Martin J. Bastiaans
Technische Universiteit Eindhoven, Faculteit Elektrotechniek, EH 5.33
Postbus 513, 5600 MB Eindhoven, Netherlands
tel: +31 40 2473319; fax: +31 40 2448375; e-mail: M.J.Bastiaans@ele.tue.nl

ABSTRACT

Gabor's expansion of a signal into a set of shifted and modulated versions of an elementary signal is introduced, along with the inverse operation, i.e. the Gabor transform, which uses a window function that is related to the elementary signal and with the help of which Gabor's expansion coefficients can be determined. The Zak transform – with its intimate relationship to Gabor's signal expansion – is introduced. It is shown how the Zak transform can be helpful in determining Gabor's expansion coefficients and how it can be used in finding window functions that correspond to a given elementary signal. In particular, a simple proof is presented of the fact that the window function with minimum L_2 norm is identical to the window function whose difference from the elementary signal has minimum L_2 norm, and thus resembles best this elementary signal, and that this window function yields the Gabor coefficients with minimum L_2 norm.

1 INTRODUCTION

It is sometimes convenient to describe a time signal not in the time domain, but in the frequency domain by means of its *frequency spectrum*. The frequency spectrum shows us the *global* distribution of the energy of the signal as a function of frequency. However, one is often more interested in the *local frequency spectrum*, which shows us the momentary or *local* distribution of the energy as a function of frequency.

A candidate for a local frequency spectrum is *Gabor's signal expansion*. In 1946 Gabor [1] suggested the expansion of a signal into a discrete set of properly shifted and modulated Gaussian elementary signals [1, 2, 3, 4]. Although Gabor restricted himself to an elementary signal that had a Gaussian shape, his signal expansion holds for rather arbitrarily shaped elementary signals [2, 3, 4]; we will use Gabor's choice of a Gaussian-shaped elementary signal as an example only.

In his original paper, Gabor restricted himself to a *critical* sampling of the time-frequency domain, where the expansion coefficients can be interpreted as independent data, i.e., degrees of freedom of a signal. It is the aim of the present paper to extend Gabor's concepts to the case of *oversampling by a rational factor*, in which case the expansion coefficients are no longer independent. In particular, we will show how we can construct a window function that corresponds to a given elementary signal.

2 GABOR'S SIGNAL EXPANSION

Let us consider an *elementary signal* $g(t)$ and let us construct a discrete set of *shifted and modulated versions* of this elementary signal $g(t-m\alpha T)\exp(jk\beta\Omega t)$, where the time shift αT and the frequency shift $\beta\Omega$ satisfy the relationships $\Omega T = 2\pi$ and $\alpha\beta \leq 1$, and where m and k may take all integer values. Gabor stated in 1946 that any reasonably well-behaved signal $\varphi(t)$ can be expressed as a superposition of shifted and modulated versions of the elementary signal,

$$\varphi(t) = \sum_m \sum_k a_{mk} g(t-m\alpha T) e^{jk\beta\Omega t}, \qquad (1)$$

with properly chosen coefficients a_{mk} and with $\alpha\beta = 1$. Note that, unless otherwise stated, all summations and integrations in this paper extend from $-\infty$ to $+\infty$.

Gabor's original signal expansion was restricted to the special case $\alpha\beta = 1$, in which case the expansion coefficients a_{mk} can be identified as *degrees of freedom* of the signal. For $\alpha\beta > 1$, the set of shifted and modulated versions of the elementary signal is not complete and thus cannot represent any arbitrary signal, while for $\alpha\beta < 1$, the set is overcomplete which implies that Gabor's expansion coefficients become dependent and can no longer be identified as degrees of freedom. In the special case $\alpha\beta = 1$, it has been shown [2, 3, 4] how a *window function* $w(t)$ can be found such that the expansion coefficients can be determined via the so-called *Gabor transform*

$$a_{mk} = \int \varphi(t) w^*(t-m\alpha T) e^{-jk\beta\Omega t} dt. \qquad (2)$$

It is the aim of this paper to show how a window function can be found when the parameters α and β satisfy the relation $1/\alpha\beta = p/q \geq 1$, where p and q are positive *integers*.

In the case of *oversampling*, the relationship between the window function $w(t)$ and the elementary signal $g(t)$ follows from substituting from the Gabor transform (2) into Gabor's signal expansion (1). After some manipulation we get the condition

$$\frac{T}{\beta} \sum_m w^*\left(t + k\frac{T}{\beta} - m\alpha T\right) g(t-m\alpha T) = \delta[k], \qquad (3)$$

where $\delta[k]$ is a Kronecker delta, with $\delta[0] = 1$ and $\delta[k] = 0$ for $k \neq 0$. In principle, a window function can be derived from a given elementary signal with the help of Eq. (3); the way in which we should proceed, however, is not clear.

3 FOURIER TRANSFORM AND ZAK TRANSFORM

We now introduce the Fourier transform of a two-dimensional array [a_{mk}, for instance] and the Zak transform of a one-dimensional time signal [$\varphi(t)$, $g(t)$, and $w(t)$, for instance].

The *Fourier transform* $\bar{a}(x, y)$ of an array a_{mk} is defined according to

$$\bar{a}(x, y) = \sum_m \sum_k a_{mk} e^{-j2\pi(my-kx)}; \qquad (4)$$

we will throughout denote the Fourier transform of an array by the same symbol as the array itself, but marked by a bar on top of it. Note that the Fourier transform $\bar{a}(x, y)$ is *periodic* in the (time) variable x and the (frequency) variable y with period 1: $\bar{a}(x+m, y+k) = \bar{a}(x, y)$. Hence, in considering the Fourier transform $\bar{a}(x, y)$ we can restrict ourselves to the *fundamental Fourier interval* $(-\frac{1}{2} < x \leq \frac{1}{2}, -\frac{1}{2} < y \leq \frac{1}{2})$.

The *Zak transform* $\tilde{\varphi}(t, \omega; \tau)$ [5, 6, 7, 8] of a signal $\varphi(t)$ is defined as a one-dimensional Fourier transformation of the sequence $\varphi(t + m\tau)$ (with m taking on all integer values and t being a mere parameter), hence

$$\tilde{\varphi}(t, \omega; \tau) = \sum_m \varphi(t + m\tau) e^{-jm\omega\tau}; \qquad (5)$$

we will throughout denote the Zak transform of a signal by the same symbol as the signal itself, but marked by a tilde on top of it. We remark that the Zak transform $\tilde{\varphi}(t, \omega; \tau)$ is *periodic* in the frequency variable ω with period $2\pi/\tau$ and *quasi-periodic* in the time variable t with quasi-period τ: $\tilde{\varphi}(t + m\tau, \omega + k2\pi/\tau; \tau) = \tilde{\varphi}(t, \omega; \tau) \exp(jm\omega\tau)$. Hence, in considering the Zak transform we can restrict ourselves to the *fundamental Zak interval* $(-\frac{1}{2}\tau < t \leq \frac{1}{2}\tau, -\pi/\tau < \omega \leq \pi/\tau)$.

As an example we have depicted the Zak transform $\tilde{g}(t, \omega; \alpha T)$ of the Gaussian signal $g(t) = 2^{1/4} \exp[-\pi(t/T)^2]$ in Fig. 1 for several values of the parameter $\tau = \alpha T$, where we have restricted ourselves to the fundamental Zak interval; note that for small α, the Zak transform becomes almost independent of t, while for large α, the Zak transform becomes almost independent of ω.

4 TRANSFORMATION OF GABOR'S SIGNAL EXPANSION AND THE GABOR TRANSFORM

Using the Fourier transform and the Zak transform defined in the previous section, it can be shown [9] that Gabor's signal expansion (1) and the Gabor transform (2) can be transformed into the *sum-of-products forms*

$$\tilde{\varphi}\left((x+s)\frac{\alpha pT}{q}, y\frac{\Omega}{\alpha}; \alpha pT\right) = \frac{1}{p} \sum_{r=<p>} \bar{a}\left(x, y + \frac{r}{p}\right) \times \tilde{g}\left((x+s)\frac{\alpha pT}{q}, \left[y + \frac{r}{p}\right]\frac{\Omega}{\alpha}; \alpha T\right) \qquad (6)$$

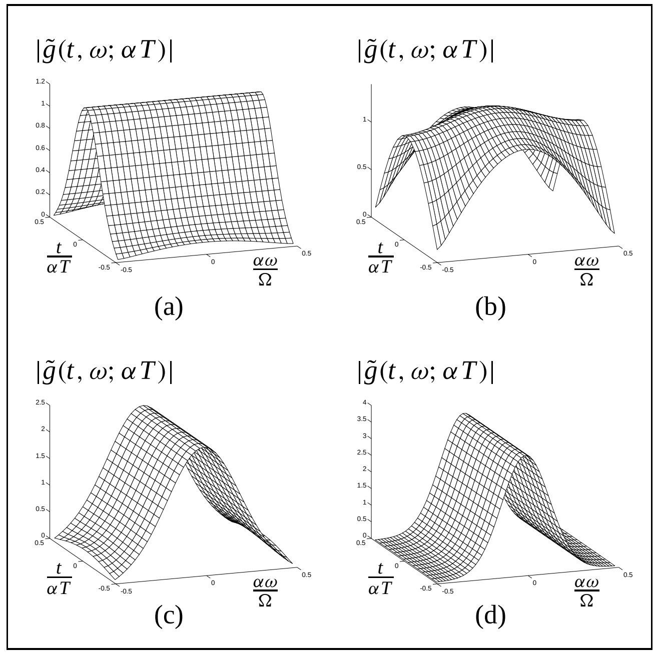

Figure 1: The Zak transform $\tilde{g}(t, \omega; \alpha T)$ in the case of a Gaussian elementary signal $g(t) = 2^{1/4} \exp[-\pi(t/T)^2]$ for different values of α: (a) $\alpha = 2$, (b) $\alpha = 1$, (c) $\alpha = \frac{1}{2}$, and (d) $\alpha = \frac{1}{3}$.

and

$$\bar{a}\left(x, y + \frac{r}{p}\right) = \frac{\alpha pT}{q} \sum_{s=<q>} \tilde{\varphi}\left((x+s)\frac{\alpha pT}{q}, y\frac{\Omega}{\alpha}; \alpha pT\right) \times \tilde{w}^*\left((x+s)\frac{\alpha pT}{q}, \left[y + \frac{r}{p}\right]\frac{\Omega}{\alpha}; \alpha T\right), \qquad (7)$$

respectively, where the expressions $r = <p>$ and $s = <q>$ are used throughout as short-hand notations for intervals of p and q successive integers.

We remark that, with $r = <p>$, the Fourier transform $\bar{a}(x, y)$ is completely determined by the p functions $a_r(x, y) = \bar{a}(x, y + r/p)$, where the variable y extends over an interval of length $1/p$. Likewise, with $s = <q>$, the Zak transform $\tilde{\varphi}(x\alpha pT/q, y\Omega/\alpha; \alpha pT)$ is completely determined by the q functions $\varphi_s(x, y) = \tilde{\varphi}([x+s]\alpha pT/q, y\Omega/alpha; \alpha pT)$, where the variable x extends over an interval of length 1. Moreover, with $r = <p>$ and $s = <q>$, the Zak transforms $\tilde{g}(x\alpha pT/q, y\Omega/\alpha; \alpha T)$ and $\tilde{w}(x\alpha pT/q, y\Omega/\alpha; \alpha T)$ are completely determined by the $q \times p$ functions $g_{sr}(x, y) = \tilde{g}([x+s]\alpha pT/q, [y + r/p]\Omega/\alpha; \alpha T)$ and $w_{sr}(x, y) = \tilde{w}([x+s]\alpha pT/q, [y + r/p]\Omega/\alpha; \alpha T)$, respectively, where x extends over an interval of length 1 and y over an interval of length $1/p$.

The p functions $a_r(x, y)$ can be combined into a p-dimensional column vector **a** and the q functions $\varphi_s(x, y)$ into a q-dimensional column vector $\boldsymbol{\phi}$. Moreover, the $q \times p$ functions $g_{sr}(x, y)$ and $w_{sr}(x, y)$ can be combined into the $(q \times p)$-dimensional matrices **G** and **W**, respectively. With the help of these vectors and matrices, Eqs. (6) and (7) can now be expressed in the elegant matrix-vector products

$$\phi = \frac{1}{p}\mathbf{G}\mathbf{a} \tag{8}$$

and

$$\mathbf{a} = \frac{\alpha p T}{q}\mathbf{W}^*\phi, \tag{9}$$

respectively, where, as usual, the asterisk in connection with vectors and matrices denotes complex conjugation *and* transposition. Note that Eq. (8) represents q equations in p unknowns, whereas Eq. (9) represents p equations in q unknowns. In the case of oversampling ($p > q \geq 1$) the former set of equations is thus *underdetermined*.

5 DETERMINATION OF THE WINDOW FUNCTION

We will now prove that Gabor's signal expansion (1) and the Gabor transform (2) form a transform pair, by showing that for any elementary signal $g(t)$ a window function $w(t)$ can be constructed. Instead of doing this by combining Gabor's signal expansion and the Gabor transform, which lead to Eq. (3), we will use the results (8) and (9) derived in the previous section.

If we substitute from Eq. (9) into Eq. (8) we get $\phi = (\alpha T/q)\,\mathbf{G}\mathbf{W}^*\phi$, which relation should hold for any arbitrary vector ϕ [i.e., for any arbitrary signal $\varphi(t)$]. This condition immediately leads to the relationship

$$\frac{\alpha T}{q}\mathbf{G}\mathbf{W}^* = \mathbf{I}_q, \tag{10}$$

where $\mathbf{I}_q$ is the $(q \times q)$-dimensional identity matrix.

We remark that in Gabor's original case of critical sampling ($p = q = 1$), the matrices $\mathbf{G}$ and $\mathbf{W}$ reduce to scalars and Eq. (10) takes a simple product form. In this case of critical sampling, the Zak transform of the window function could thus be easily found, in principle, as the inverse of the Zak transform of the elementary signal, and the resulting window function would be *unique*. However, the occurrence of zeros in the Zak transform of the elementary signal (see Fig. 1) prohibits such an easy procedure. The problems caused by these zeros can be overcome by oversampling.

In the case of oversampling, the window function that corresponds to a given elementary signal is *not unique*. This is in accordance with the fact that in the case of oversampling the set of shifted and modulated versions of the elementary signal is overcomplete, and that Gabor's expansion coefficients are dependent and can no longer be considered as degrees of freedom. In the case of oversampling, the general condition (10) enables us to construct a window function $w(t)$ for a given elementary signal $g(t)$, by solving a set of $q \times q$ equations in $q \times p$ unknowns. Since $q < p$, this set of equations is again underdetermined.

Let us now consider Eqs. (8) and (10) in the general case of oversampling. In that case we have $q < p$, which implies that $\mathbf{G}$ is not a square matrix and does not have a normal inverse $\mathbf{G}^{-1}$, and that Eqs. (8) and (10) do not have unique solutions. It is well known that, under the condition that rank($\mathbf{G}$) = q, the *optimum solutions* in the sense of the *minimum L_2 norm* can now be found with the help of the so-called *generalized (Moore-Penrose) inverse* $\mathbf{G}^\dagger = \mathbf{G}^*(\mathbf{G}\mathbf{G}^*)^{-1}$. The optimum solution $\mathbf{W}_{opt}$ then reads

$$\mathbf{W}_{opt} = \frac{q}{\alpha T}(\mathbf{G}^\dagger)^* \tag{11}$$

and the optimum solution $\mathbf{a}_{opt}$ reads

$$\mathbf{a}_{opt} = p\,\mathbf{G}^\dagger\phi = \frac{\alpha p T}{q}\mathbf{W}^*_{opt}\phi. \tag{12}$$

Of course, if we proceed in this way, we will find, for any x and y, the minimum L_2 norm solutions for the matrix $\mathbf{W}$ and the vector $\mathbf{a}$. It is not difficult to show, however, that the minimum L_2 norms of $\mathbf{W}$ and $\mathbf{a}$ correspond to the minimum L_2 norms of the Zak transform $\tilde{w}(x\alpha pT/q, y\Omega/\alpha; \alpha T)$ and the Fourier transform $\bar{a}(x, y)$, respectively, and thus, with the help of Parseval's energy theorem, to the minimum L_2 norms of the window function $w(t)$ and the array of Gabor coefficients a_{mk}, respectively.

Instead of looking for the optimum solution $\mathbf{W}_{opt}$ in the sense of the minimum L_2 norm of $\mathbf{W}$, we could as well look for the optimum solution $\mathbf{W}_F$ in the sense of the minimum L_2 norm of the difference $\mathbf{W} - \mathbf{F}$; in this way we would find the matrix $\mathbf{W}_F$ that resembles best the matrix $\mathbf{F}$. As a result we find $\mathbf{W}^*_F = \mathbf{W}^*_{opt} + (\mathbf{I}_p - \mathbf{G}^\dagger\mathbf{G})\mathbf{F}^*$.

An obvious choice for the matrix $\mathbf{F}$ would be a matrix that is proportional to the matrix $\mathbf{G}$. We then get $\mathbf{W}^*_G = \mathbf{W}^*_{opt} + [\mathbf{I}_p - \mathbf{G}^*(\mathbf{G}\mathbf{G}^*)^{-1}\mathbf{G}]\mathbf{G}^*$, but the second term in the right-hand side of this relationship vanishes. We thus reach the important conclusion that $\mathbf{W}_G = \mathbf{W}_{opt}$; hence, the window function $w_{opt}(t)$ that has the minimum L_2 norm is the same as the window function $w_g(t)$ whose difference from the elementary signal $g(t)$ has minimum L_2 norm, and resembles best this elementary signal.

In Fig. 2 we have depicted the Zak transforms of some window functions that correspond to a Gaussian elementary signal for different values of α and β, resulting in different values of oversampling p/q, while in Fig. 3 we have depicted these window functions themselves. We remark that the resemblance between the window function and the elementary signal increases with decreasing $\alpha\beta$.

6 CONCLUSION

We have introduced Gabor's expansion of a signal into a set of shifted and modulated versions of an elementary signal. We have also described the inverse operation – the Gabor transform – with which Gabor's expansion coefficients can be determined. While Gabor restricted himself to critically sampling the time-frequency domain on a rectangular lattice $(m\alpha T, k\beta\Omega)$ where the time shift αT and the frequency shift $\beta\Omega$ satisfy the relation $\Omega T = 2\pi$ and $\alpha\beta = 1$, we have considered the case of oversampling $\alpha\beta \leq 1$, and in particular the case $\alpha\beta = q/p$ where p and q are positive integers. We have shown that in this case of rational oversampling, the Gabor

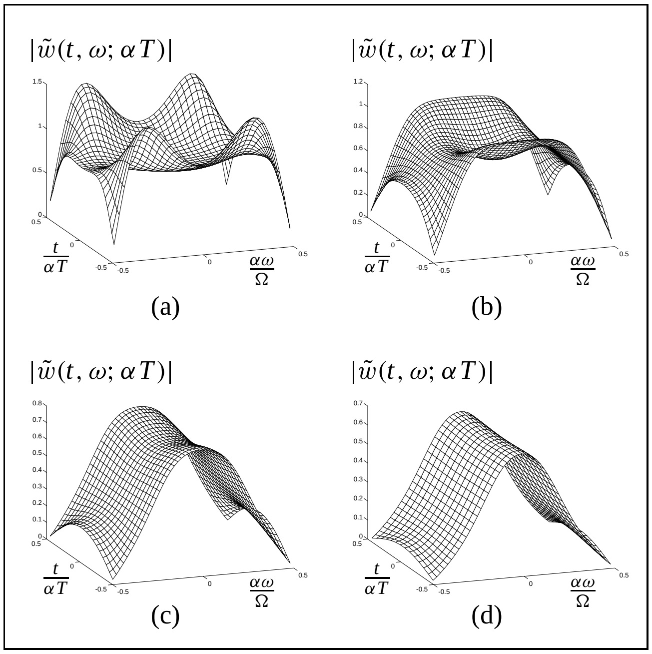

Figure 2: The Zak transform $\tilde{w}(t, \omega; \alpha T)$ that corresponds to a Gaussian elementary signal $g(t) = 2^{1/4}\exp[-\pi(t/T)^2]$ for different values of oversampling: (a) $\alpha = \beta = \sqrt{6/7}$, $p/q = 7/6$, (b) $\alpha = \beta = \sqrt{2/3}$, $p/q = 3/2$, (c) $\alpha = \beta = \sqrt{2/5}$, $p/q = 5/2$, and (d) $\alpha = \beta = \sqrt{2/7}$, $p/q = 7/2$.

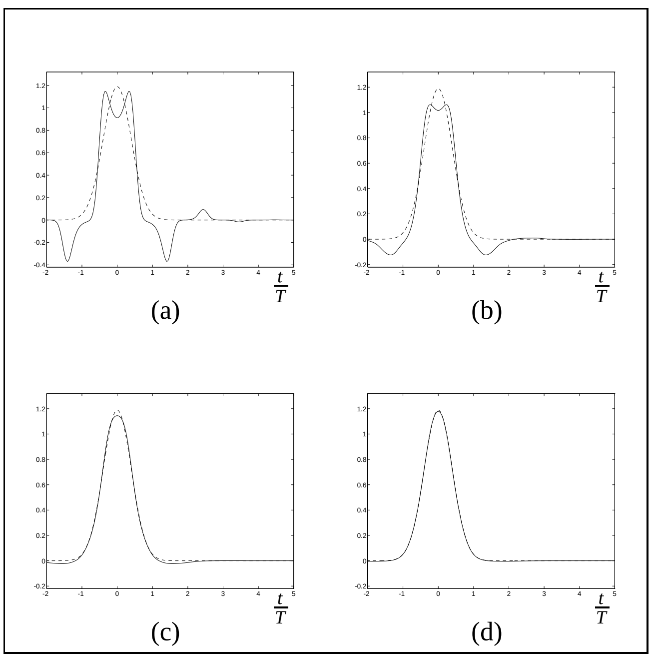

Figure 3: A Gaussian elementary signal $g(t/T) = 2^{1/4}\exp[-\pi(t/T)^2]$ (dashed line) and its corresponding window function $(\alpha T/q)w(t/T)$ (solid line) for different values of oversampling: (a) $\alpha = \beta = \sqrt{6/7}$, $p/q = 7/6$, (b) $\alpha = \beta = \sqrt{2/3}$, $p/q = 3/2$, (c) $\alpha = \beta = \sqrt{2/5}$, $p/q = 5/2$, and (d) $\alpha = \beta = \sqrt{2/7}$, $p/q = 7/2$.

transform and Gabor's signal expansion can be transformed into sum-of-products forms. Using these sum-of-products forms, it was possible to show that the Gabor transform and Gabor's signal expansion form a transform pair. In particular it was shown how the window function that appears in the Gabor transform, can be constructed from the elementary signal that is used in Gabor's signal expansion. We refer to a related paper [10], where the case of oversampling by a rational factor is considered, as well.

The process of oversampling introduces dependence between the Gabor coefficients; whereas these coefficients can be considered as degrees of freedom in the case of critical sampling, they can no longer be given such an interpretation in the case of oversampling. The additional freedom caused by oversampling, allowed us to construct the window function in such a way that it is mathematically well-behaved. In particular, we have shown that the window function with the minimum L_2 norm is identical to the window function whose difference from the elementary signal has the minimum L_2 norm, and thus resembles best this elementary signal, and that this window function yields the Gabor coefficients with the minimum L_2 norm.

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