OPTIMUM OVERSAMPLING IN THE RECTANGULAR GABOR SCHEME

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ABSTRACT

The windowed Fourier transform of a time signal is considered, as well as a way to reconstruct the signal from a sufficiently densely sampled version of its windowed Fourier transform using a Gabor representation; following Gabor, sampling occurs on a two-dimensional time-frequency lattice with equidistant time intervals and equidistant frequency intervals. For sufficiently dense sampling, the synthesis window (which appears in Gabor's reconstruction formula) may be constructed such that it resembles a rather arbitrarily given function; this function may or may not be proportional to the analysis window (which is used in the windowed Fourier transform). It is shown that the resemblance can already be reached for a rather small degree of oversampling, if the sampling distances in the time and frequency directions are properly chosen. A procedure is presented with which the optimum ratio of the sampling intervals can be determined.

1 INTRODUCTION

In recent years, the sampled version of the windowed Fourier transform of an arbitrary signal has received considerable interest. The sampling values that result from this windowed Fourier transform (with a given analysis window) can be used to reconstruct the signal by means of the Gabor expansion [1] (with a proper synthesis window, to be determined from the analysis window). In sampling the windowed Fourier transform, we have some freedom in choosing the sampling distances in the time and the frequency direction, as long as the density of the sampling lattice, which is determined by the *product* of the two sampling distances, is sufficiently high. But the *ratio* between the two sampling distances appears to be important, as well. Indeed, if we require that the synthesis window shall become proportional to a given function – the analysis window, for instance – for a sufficiently small value of the product of the two sampling distances – i.e. a very large degree of oversampling – this limit will be reached for a much smaller degree of oversampling if the ratio of the sampling distances is properly chosen. The present paper presents a procedure with which the optimum value of this ratio can be determined.

2 WINDOWED FOURIER TRANSFORM

Let the *windowed Fourier transform* $S_\varphi(t,\omega)$ of a signal $\varphi(t)$ be defined as

$$S_\varphi(t,\omega) = \int \varphi(t')w^*(t'-t)e^{-j\omega t'}dt', \qquad (1)$$

where $w(t)$ is the window function. (Unless otherwise stated, all integrations and summations in this paper extend from $-\infty$ to $+\infty$.) We note that the windowed Fourier transform can be considered as the Fourier transform of the product of the signal $\varphi(t)$ and a shifted and complex conjugated version of the *analysis window* $w(t)$. The window function may be chosen rather arbitrarily; mostly it will be a function that is more or less concentrated around the origin. In this paper we will use the Gaussian window function

$$w(t) = 2^{\frac{1}{4}}e^{-\pi(t/T)^2} \qquad (2)$$

and the exponential window function

$$w(t) = \sqrt{a}e^{-a|t|/T} \quad \text{with} \quad a > 0 \qquad (3)$$

as examples. Note that we have normalized the window functions such that their L_2 norms $\int |w(t)|^2 dt$ equal T.

The signal $\varphi(t)$ can be reconstructed from its windowed Fourier transform (1) by an inverse Fourier transformation, of course. A different and more interesting way to reconstruct the signal from its windowed Fourier transform is by means of the *inversion formula*

$$\varphi(t')\int |w(t)|^2 dt = \frac{1}{2\pi}\iint S_\varphi(t,\omega)w(t'-t)e^{j\omega t'}dt d\omega; \qquad (4)$$

this relationship can easily be proved by substituting from the definition (1) of the windowed Fourier transform. However, in order to reconstruct the signal we need not know the entire windowed Fourier transform; it suffices to know its values at the points of the rectangular lattice ($t = m\alpha T, \omega = k\beta\Omega$) in the time-frequency domain, where $\Omega T = 2\pi$ and $\alpha\beta \leq 1$, and where m and k take all integer values. Note that the rectangular cells of this lattice occupy an area of $2\pi\alpha\beta$ in the time-frequency domain. With the sampling values defined as

$$a_{mk} = S_\varphi(m\alpha T, k\beta\Omega), \qquad (5)$$

the signal $\varphi(t)$ can then be reconstructed by considering these sampling values as the *coefficients* in *Gabor's signal expansion*, with a *synthesis window* $g(t)$ that still has to be determined; thus

$$\varphi(t) = \sum_m \sum_k a_{mk} g(t - m\alpha T) e^{jk\beta\Omega t}. \qquad (6)$$

In the case of *critical sampling* (i.e., $\alpha\beta = 1$), the synthesis window $g(t)$ is uniquely determined by the analysis window $w(t)$ [2, 3], whereas in the case of *oversampling* (i.e., $\alpha\beta < 1$), the synthesis window is no longer unique. In the oversampled case, the synthesis window is very often chosen such that it has minimum L_2 norm. This so-called *optimum* synthesis window $g_{opt}(t)$ not only has minimum L_2 norm, but it also resembles best (in a minimum L_2 norm sense, again) the analysis window $w(t)$. It is easy to see that for *infinite* oversampling ($\alpha T \downarrow 0$, $\beta\Omega \downarrow 0$), Gabor's signal expansion (6) becomes equivalent to the inversion formula (4), and the (optimum) synthesis window is indeed proportional to the analysis window: $g_{opt}(t) \int |w(t)|^2 dt \simeq \alpha\beta \; w(t)$.

As an illustration we have depicted in Fig. 1 the optimum synthesis windows $g_{opt}(t)$ that correspond to the Gaussian analysis window (2) for different values of α and β (with $\alpha = \beta$); note that the resemblance between the synthesis and the analysis window increases with increasing degree of oversampling. Several ways are described in the literature to determine the optimum synthesis window, see, for instance, [4] and the references cited there.

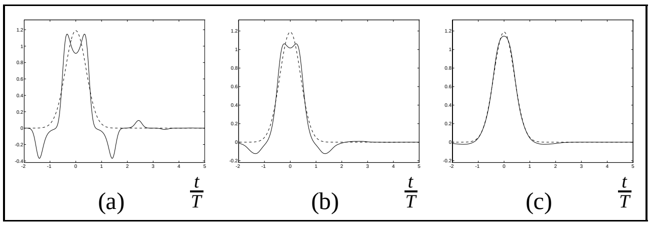

Figure 1: A Gaussian analysis window $w(t)$ (dashed line) and its corresponding optimum synthesis window $g_{opt}(t)$ (solid line) for different values of oversampling: (a) $\alpha = \beta = \sqrt{6/7}$, (b) $\alpha = \beta = \sqrt{2/3}$, and (c) $\alpha = \beta = \sqrt{2/5}$.

3 OPTIMUM SYNTHESIS WINDOW

One of the methods to determine the synthesis window – which method can be applied to the rather general case of *rational* oversampling, i.e., $\alpha\beta = q/p$, with p and q positive integers and $p \geq q \geq 1$ – is as follows [5, 6]. Let $\tilde{w}(t, \omega; \tau)$ be the *Zak transform* [7, 8] of the analysis window $w(t)$, defined by

$$\tilde{w}(t, \omega; \tau) = \sum_m w(t + m\tau) e^{-jm\omega\tau}, \qquad (7)$$

and let $\tilde{g}(t, \omega; \tau)$ be the Zak transform of the synthesis window $g(t)$, defined in a similar way. Let $\mathbf{W}(x, y)$, for any value of x and y between 0 and 1, be a $(q \times p)$-dimensional matrix whose *sr*-th entry $w_{sr}(x, y)$ reads

$$w_{sr}(x, y) = \tilde{w}\left((x + s)\frac{\alpha p T}{q}, \left[y + \frac{r}{p} \right] \frac{\Omega}{\alpha}; \alpha T \right), \qquad (8)$$

and let the $(q \times p)$-dimensional matrix $\mathbf{G}(x, y)$ be defined analogously. The relationship between the analysis and the synthesis window can then be expressed by the elegant matrix relationship

$$\frac{\alpha T}{q} \mathbf{W}\mathbf{G}^* = \mathbf{I}_q, \qquad (9)$$

where $\mathbf{I}_q$ is the $(q \times q)$-dimensional identity matrix and where the asterisk in connection with matrices and vectors denotes complex conjugation *and* transposition.

In the general case of oversampling we have $q < p$, which implies that (i) $\mathbf{W}$ is not a square matrix and does not have a normal inverse $\mathbf{W}^{-1}$, and that (ii) Eq. (9) represents an underdetermined set of linear equations and does not have a unique solution $\mathbf{G}^*$. It is well known that, under the condition that rank($\mathbf{W}$)=q, the *optimum solution* for $\mathbf{G}^*$ in the sense of *minimum L_2 norm* can be found with the help of the so-called *generalized (Moore-Penrose) inverse* [9] $\mathbf{W}^\dagger$, and the optimum solution $\mathbf{G}^*_{opt}$ then reads

$$\mathbf{G}^*_{opt} = \frac{q}{\alpha T} \mathbf{W}^\dagger = \frac{q}{\alpha T} \mathbf{W}^*(\mathbf{W}\mathbf{W}^*)^{-1}. \qquad (10)$$

It is not difficult to show that the minimum L_2 norm of $\mathbf{G}$ corresponds to the minimum L_2 norm of the Zak transform $\tilde{g}(t, \omega; \alpha T)$ and thus to the minimum L_2 norm of the window function $g(t)$.

Instead of looking for the optimum solution $\mathbf{G}^*_{opt}$ in the sense of minimum L_2 norm of $\mathbf{G}$, we could as well look for the optimum solution $\mathbf{G}^*_{\lambda F}$ in the sense of minimum L_2 norm of the difference $\mathbf{G} - \lambda\mathbf{F}$; in this way we would find the matrix $\mathbf{G}$ that resembles best the matrix $\lambda\mathbf{F}$. To find $\mathbf{G}^*_{\lambda F}$ we rewrite Eq. (9)

$$\frac{\alpha T}{q} \mathbf{W}(\mathbf{G} - \lambda\mathbf{F})^* = \mathbf{I}_q - \frac{\alpha T}{q} \mathbf{W}\lambda^*\mathbf{F}^*$$

and we conclude

$$\mathbf{G}^*_{\lambda F} = \mathbf{G}^*_{opt} + (\mathbf{I}_p - \mathbf{W}^\dagger\mathbf{W})\lambda^*\mathbf{F}^*. \qquad (11)$$

Note that the second term in the right-hand side of Eq. (11) vanishes if $\mathbf{F}$ is chosen proportional to $\mathbf{W}$, and that, for minimum L_2-norm of $\mathbf{G}_{\lambda F} - \lambda\mathbf{F}$, the optimum value λ_o of the proportionality factor λ should read

$$\lambda_o = \frac{\alpha\beta}{\int f(t) w^*(t) dt}. \qquad (12)$$

4 OPTIMUM SAMPLING LATTICE

The optimum synthesis window depends, of course, on the choice of the sampling parameters α and β. In Fig. 1 these sampling parameters were chosen identical ($\alpha = \beta$), since this choice yields the best result in the case that the synthesis and the analysis window should have the same Gaussian shape, as we will show. As an illustration of non-identical sampling parameters we have depicted in Fig. 2 the optimum synthesis windows that correspond to the Gaussian analysis window (2) for several values of α, while maintaining the relation $\alpha\beta = 1/3$. It is the aim of this paper to derive, for a fixed degree of oversampling $1/\alpha\beta$, the optimum values of the sampling parameters α and β in the case that we require that the synthesis window $g(t)$ shall resemble a function $f(t)$ that may be *different* from the analysis window $w(t)$.

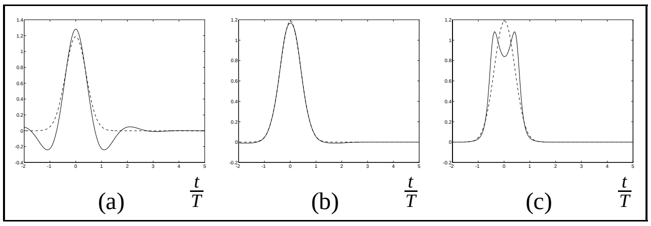

Figure 2: A Gaussian analysis window $w(t)$ (dashed line) and its corresponding optimum synthesis window $g_{opt}(t)$ (solid line) in the case of oversampling by a factor 3, for different values of α and β: (a) $\alpha = 1/3$, $\beta = 1$, (b) $\alpha = \beta = \sqrt{1/3}$, and (c) $\alpha = 1$, $\beta = 1/3$.

To find a procedure for determining the optimum parameter values we proceed as follows. Let us express the required function $f(t)$ by means of its Gabor expansion [cf. Eq. (6)]:

$$f(t) = \sum_m \sum_k c_{mk}\, g(t - m\alpha T) e^{jk\beta\Omega t}. \qquad (13)$$

The Gabor coefficients c_{mk} follow, of course, as the sampling values of the windowed Fourier transform $S_f(t,\omega)$: $c_{mk} = S_f(m\alpha T, k\beta\Omega)$. If the degree of oversampling $1/\alpha\beta$ is sufficiently high, the optimum synthesis window $g_{\lambda_o f}(t)$ resembles the function $\lambda_o f(t)$ and the Gabor expansion (13) takes the form

$$f(t) \simeq \frac{\alpha\beta}{\int f(t) w^*(t) dt} \sum_m \sum_k c_{mk}\, f(t - m\alpha T) e^{jk\beta\Omega t}. \qquad (14)$$

From Eq. (14) it is obvious that we would like the array of Gabor coefficients c_{mk} to be as concentrated around the origin as possible. We therefore look for those sampling parameters α and β – with their product $\alpha\beta$ = constant – for which the L_2 norm $\sum_m \sum_k |c_{mk}|^2$ of the array c_{mk} takes its minimum value.

We first illustrate the method by finding the optimum parameters α and β in the case that the analysis window is Gaussian according to Eq. (2), while we require that the synthesis window shall resemble a Gaussian function $f(t)$ that has a *different* width:

$$f(t) = \frac{\alpha\beta}{T} \sqrt{\frac{1+\gamma^2}{2}}\, 2^{\frac{1}{4}} e^{-\pi(\gamma t/T)^2}. \qquad (15)$$

Note that we have normalized the required function $f(t)$ such that $\int f(t) w^*(t) dt = \alpha\beta$ and, hence, $\lambda_o = 1$. The windowed Fourier transform of $f(t)$ then takes the form

$$\begin{aligned} S_f(t,\omega) &= e^{-j\omega t/(1+\gamma^2)} \qquad (16) \\ &\times \alpha\beta\, e^{-\pi[(\gamma t/T)^2 + (\omega/\Omega)^2]/(1+\gamma^2)}. \end{aligned}$$

We thus conclude that

$$|c_{mk}|^2 = |\alpha\beta|^2\, e^{-2\pi[\gamma^2 m^2 \alpha^2 + k^2 \beta^2]/(1+\gamma^2)}$$

and that $|c_{mk}|^2$ is symmetrical for $\beta = \gamma\alpha$. Hence, the L_2 norm of c_{mk} is minimum if the ratio between β and α is equal to γ. As an illustration we have depicted in Fig. 3 the optimum synthesis window $g_f(t)$ that resembles best a Gaussian-shaped function $f(t)$ whose width is double the width of the analysis window $w(t)$ [cf. Eqs. (2) and (15) with $\gamma = 1/2$] for several values of α, while maintaining the relation $\alpha\beta = 1/4$. The optimum synthesis window corresponds indeed to the case $\beta = \gamma\alpha$.

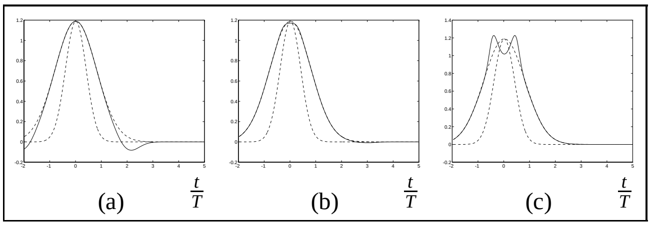

Figure 3: A Gaussian analysis window $w(t)$ (dashed line), a required Gaussian-shaped function $f(t)$ with double width (dashed line), and the corresponding optimum synthesis window $g_f(t)$ (solid line) in the case of oversampling by a factor 4, for different values of α and β: (a) $\alpha = \beta = 1/2$, (b) $\alpha = \sqrt{2}/2$, $\beta = \sqrt{2}/4$, and (c) $\alpha = 1$, $\beta = 1/4$.

In general, optimal choices for the sampling parameters α and β cannot be found analytically but have to be determined numerically by means of a computer. A good starting point for the numerical process may be found by considering the widths $2d_t$ and $2d_\omega$ of the function $S_f(t,\omega)$ in the t-direction and the ω-direction, respectively, and by taking the ratio of the sampling distance αT in the time direction and the sampling distance $\beta\Omega$ in the frequency direction equal to d_t/d_ω. As measures of d_t^2 and d_ω^2 we might choose, for instance, the normalized second-order moments of the one-dimensional functions $|S_f(t,0)|^2$ and $|S_f(0,\omega)|^2$, respectively:

$$d_t^2 = \frac{\int t^2 |S_f(t,0)|^2 dt}{\int |S_f(t,0)|^2 dt},\; d_\omega^2 = \frac{\int \omega^2 |S_f(0,\omega)|^2 d\omega}{\int |S_f(0,\omega)|^2 d\omega}. \qquad (17)$$

As an example again, let us determine the widths $2d_t$ and $2d_\omega$ in the case of Gaussian-shaped window functions, see Eqs. (2) and (15). For the widths $2d_t$ and $2d_\omega$ we find $\gamma 2d_t/T = 2d_\omega/\Omega = \sqrt{(1+\gamma^2)/\pi}$, and the optimum ratio $\beta/\alpha = (2d_\omega/\Omega)/(2d_t/T)$ takes indeed the value γ, as we already concluded before.

As a second example, we consider the exponential analysis window (3) and we require that the synthesis window shall have the same exponential shape $f(t) = (\alpha\beta/T) w(t)$. From the windowed Fourier transform

$$S_f(t,\omega) = e^{-j\frac{1}{2}\omega t}\, \frac{\alpha\beta}{1 + (\pi/a)^2 (\omega/\Omega)^2}\, e^{-a|t/T|}$$

$$\times \left(a \left| \frac{t}{T} \right| \frac{\sin[\pi(\omega/\Omega)(t/T)]}{\pi(\omega/\Omega)(t/T)} + \cos[\pi(\omega/\Omega)(t/T)] \right) \quad (18)$$

we then find the widths $2d_t/T = 2\sqrt{7/5}/a$ and $2d_\omega/\Omega = 2a/\pi$; hence, the ratio $\alpha/\beta = (2d_t/T)/(2d_\omega/\Omega)$ should take the value $\alpha/\beta = \pi\sqrt{7/5}/a^2$. For the special value $a = \sqrt{\pi}(7/5)^{1/4} \simeq 1.928$, the ratio α/β takes the value 1. As an illustration we have depicted in Fig. 4 the optimum synthesis windows that correspond to the exponential window (3) with $a = \sqrt{\pi}(7/5)^{1/4}$ for several values of α, while maintaining the relation $\alpha\beta = 1/9$.

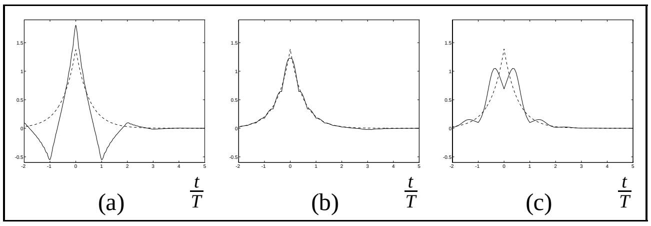

Figure 4: An exponential analysis window $w(t)$ (dashed line) and its corresponding optimum synthesis window $g_{opt}(t)$ (solid line) in the case of oversampling by a factor 9, for different values of α and β: (a) $\alpha = 1/9$, $\beta = 1$, (b) $\alpha = \beta = 1/3$, and (c) $\alpha = 1$, $\beta = 1/9$.

5 CONCLUSION

In this paper we have dealt with the windowed Fourier transform $S_\varphi(t, \omega)$ of a continuous-time signal $\varphi(t)$ using an analysis window $w(t)$, and Gabor's representation of such a time signal as a set of shifted and modulated versions of a synthesis window $g(t)$. It is well known that the coefficients that arise in the Gabor expansion can be identified as the values of the windowed Fourier transform, sampled on a rectangular grid in the time-frequency domain.

Whereas the relationship between the synthesis window $g(t)$ – which acts as a building block in Gabor's signal expansion – and the analysis window $w(t)$ – which is used in the windowed Fourier transform – has been investigated extensively in the past, in particular for finding ways to determine the synthesis window when the analysis window is given, less attention has been paid to the influence of the sampling geometry, i.e., the ratio between the sampling distances in the time and the frequency direction. However, if the analysis window $w(t)$ is given and we want the synthesis window $g(t)$ to resemble a required function $f(t)$ as closely as possible, the sampling geometry appears to be rather important.

In this paper we have shown that, although the synthesis window will automatically resemble the required function for a sufficiently large degree of oversampling, the resemblance between the synthesis window and the required function can be reached for a much smaller degree of oversampling if the sampling geometry is chosen correctly. In particular, we have presented a method, based on the windowed Fourier transform $S_f(t, \omega)$ of the required function $f(t)$, with which the optimum sampling geometry for a given degree of oversampling can be found. The method has been illustrated for the case that both the analysis window $w(t)$ and the required function $f(t)$ have Gaussian shapes with different widths, and for the case that the analysis window and the synthesis window have the same exponential shape.

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