Gabor's Expansion and the Zak Transform for Continuous-time and Discrete-time Signals: Critical Sampling and Rational Oversampling

by
Martin J. Bastiaans

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Gabor's expansion and the Zak transform for continuous-time and discrete-time signals: Critical sampling and rational oversampling

Martin J. Bastiaans

Abstract

Gabor's expansion of a signal into a discrete set of shifted and modulated versions of an elementary signal is introduced and its relation to sampling of the sliding-window spectrum is shown. It is shown how Gabor's expansion coefficients can be found as samples of the sliding-window spectrum, where – at least in the case of critical sampling – the window function is related to the elementary signal in such a way that the set of shifted and modulated elementary signals is bi-orthonormal to the corresponding set of window functions.

The Zak transform is introduced and its intimate relationship to Gabor's signal expansion is demonstrated. It is shown how the Zak transform can be helpful in determining the window function that corresponds to a given elementary signal and how it can be used to find Gabor's expansion coefficients.

The continuous-time as well as the discrete-time case are considered, and, by sampling the continuous frequency variable that still occurs in the discrete-time case, the discrete Zak transform and the discrete Gabor transform are introduced. It is shown how the discrete transforms enable us to determine Gabor's expansion coefficients via a fast computer algorithm, analogous to the well-known fast Fourier transform algorithm.

Not only Gabor's critical sampling is considered, but also the case of oversampling by a rational factor. An arrangement is described which is able to generate Gabor's expansion coefficients of a rastered, one-dimensional signal by coherent-optical means.

Keywords: signal representation, time-frequency analysis, Gabor's signal expansion, Gabor transform, Zak transform

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Contents

1 Introduction **1**

2 Gabor's signal expansion **3**

3 Zak transform **8**

4 Gabor transform and Zak transform for continuous-time signals **14**

5 Gabor transform and Zak transform for discrete-time signals **16**

6 Discrete Gabor transform and discrete Zak transform **19**

7 Oversampling by a rational factor **24**

8 Discrete Gabor and Zak transforms with oversampling **33**

9 Coherent-optical set-up **42**

10 Conclusion **45**

Appendix A. The second bi-orthonormality condition (2.8) **47**

Appendix B. Derivation of condition (7.4) **49**

Appendix C. The sum-of-products form (7.7) **50**

Appendix D. The sum-of-products form (7.8) **52**

Appendix E. The sum-of-products form (8.19) **55**

Appendix E. The sum-of-products form (8.20) **57**

References **60**

1 Introduction

It is sometimes convenient to describe a time signal $\varphi(t)$, say, not in the time domain, but in the frequency domain by means of its *frequency spectrum*, i.e., the *Fourier transform* $\bar{\varphi}(\omega)$ of the function $\varphi(t)$, which is defined by

$$\bar{\varphi}(\omega) = \int \varphi(t) e^{-j\omega t} dt; \tag{1.1}$$

a bar on top of a symbol will mean throughout that we are dealing with a function in the frequency domain. (Unless otherwise stated, all integrations and summations in this paper extend from $-\infty$ to $+\infty$.) The *inverse* Fourier transformation takes the form

$$\varphi(t) = \frac{1}{2\pi} \int \bar{\varphi}(\omega) e^{j\omega t} d\omega. \tag{1.2}$$

The frequency spectrum shows us the *global* distribution of the energy of the signal as a function of frequency. However, one is often more interested in the momentary or *local* distribution of the energy as a function of frequency.

The need for a *local frequency spectrum* arises in several disciplines. It arises in music, for instance, where a signal is usually described not by a time function nor by the Fourier transform of that function, but by its *musical score*; indeed, when a composer writes a score, he prescribes the frequencies of the tones that should be present at a certain moment. It arises in optics: geometrical optics is usually treated in terms of *rays*, and the signal is described by giving the directions (cf. frequencies) of the rays (cf. tones) that should be present at a certain position (cf. time moment). It arises also in mechanics, where the position and the momentum of a particle are given simultaneously, leading to a description of mechanical phenomena in a *phase space*.

A candidate for a local frequency spectrum is *Gabor's signal expansion*. In 1946 Gabor [15] suggested the expansion of a signal into a discrete set of properly shifted and modulated Gaussian elementary signals [5, 6, 7, 15, 16, 17]. A quotation from Gabor's original paper might be useful. Gabor writes in the summary:

> Hitherto communication theory was based on two alternative methods of signal analysis. One is the description of the signal as a function of time; the other is Fourier analysis. ... But our everyday experiences ... insist on a description in terms of *both* time and frequency. ... Signals are represented in two dimensions, with time and frequency as co-ordinates. Such two-dimensional representations can be called 'information diagrams,' as areas in them are proportional to the number of independent data which they can convey. ... There are certain 'elementary signals' which occupy the smallest possible area in the information diagram. They are harmonic oscillations modulated by a probability pulse. Each elementary signal can be considered as conveying exactly one datum, or one 'quantum of information.' Any signal can be expanded in terms of these by a process which includes time analysis and Fourier analysis as extreme cases.

Although Gabor restricted himself to an elementary signal that had a Gaussian shape, his signal expansion holds for rather arbitrarily shaped elementary signals [5, 6, 7].

We will restrict ourselves to one-dimensional time signals; the extension to two or more dimensions, however, is rather straightforward. Most of the results can be applied to continuous-time as well as discrete-time signals. We will treat continuous-time signals in Sections 2, 3, and 4 (and also in Sections 7 and 9), and we will transfer the concepts to the discrete-time case in Sections 5 and 6 (and also in Section 8). To distinguish continuous-time from discrete-time signals, we will denote the former with curved brackets and the latter with square brackets; thus $\varphi(t)$ is a continuous-time and $\varphi[n]$ a discrete-time signal. We will use the variables in a consistent manner: in the continuous-time case, the variables t, m and T have something to do with time, the variables ω, k and Ω have something to do with frequency, and the relation $\Omega T = 2\pi$ holds throughout; in the discrete-time case, the variables n, m and N have something to do with time, the variables θ, k and Θ have something to do with frequency, and the relation $\Theta N = 2\pi$ holds throughout.

In his original paper, Gabor restricted himself to a *critical* sampling of the time-frequency domain; this is the case that we consider in Sections 2-6. In Section 2 we introduce Gabor's signal expansion, we introduce a window function with the help of which the expansion coefficients can be found, and we show a way how, at least in principle, this window function can be determined. In Section 3 we introduce the Zak transform and we use this transform to determine the window function that corresponds to a given elementary signal in a mathematically more attractive way. A more general application of the Zak transform to Gabor's signal expansion is described in Section 4. We translate the concepts of Gabor's signal expansion and the Zak transform to discrete-time signals in Section 5. In Section 6, we introduce – on the analogy of the well-known discrete Fourier transform – a discrete version of the Zak transform; the discrete versions of the Fourier and the Zak transform enable us to determine Gabor's expansion coefficients by computer via a fast computer algorithm.

In Section 7 we extend Gabor's concepts to the case of *oversampling*, in particular oversampling by a *rational* factor. We use the Fourier transform and the Zak transform again to transform Gabor's signal expansion into a mathematically more attractive form and we show how a window function can be determined in this case of oversampling by a rational factor. In Section 8 the discrete Gabor and Zak transforms are considered again, but now with oversampling.

Finally, in Section 9, we will introduce an optical arrangement which is able to generate Gabor's expansion coefficients of a rastered, one-dimensional signal by coherent-optical means.

2 Gabor's signal expansion

Let us consider an *elementary signal* $g(t)$, which may or may not have a Gaussian shape. Gabor's original choice was a Gaussian [15],

$$g(t) = 2^{\frac{1}{4}} e^{-\pi (t/T)^2}, \tag{2.1}$$

where we have added the factor $2^{\frac{1}{4}}$ to normalize $(1/T)\int |g(t)|^2 dt$ to unity, but in this paper the elementary signal may have a rather arbitrary shape; we will use Gabor's choice of a Gaussian-shaped elementary signal as an example only. From the elementary signal $g(t)$, we construct a discrete set of *shifted and modulated versions* $g_{mk}(t)$ defined by

$$g_{mk}(t) = g(t - mT)e^{jk\Omega t}, \tag{2.2}$$

where the time shift T and the frequency shift Ω satisfy the relationship $\Omega T = 2\pi$, and where m and k may take all integer values. Gabor stated in 1946 that any reasonably well-behaved signal $\varphi(t)$ can be expressed in the form

$$\varphi(t) = \sum_m \sum_k a_{mk} g_{mk}(t), \tag{2.3}$$

with properly chosen coefficients a_{mk}. Thus *Gabor's signal expansion* represents a signal $\varphi(t)$ as a superposition of properly shifted (over discrete distances mT) and modulated (with discrete frequencies $k\Omega$) versions of an elementary signal $g(t)$. We note that there exists a completely *dual* expression in the frequency domain; in this paper, however, we will concentrate on the time-domain description.

Gabor's signal expansion is related to the *degrees of freedom* of a signal: each expansion coefficient a_{mk} represents one complex degree of freedom [8, 15]. If a signal is, roughly, limited to the space interval $|t| < \frac{1}{2}a$ and to the frequency interval $|\omega| < \frac{1}{2}b$, the number of complex degrees of freedom equals the number of Gabor coefficients in the time-frequency rectangle with area ab, this number being approximately equal to the *time-bandwidth product* $ab/2\pi$. The reason for Gabor to choose a Gaussian-shaped elementary signal was that for such a signal each shifted and modulated version, which conveys exactly one degree of freedom, occupies the smallest possible area in the time-frequency domain. Indeed, if we choose the elementary signal according to Eq. (2.1), the 'duration' of such a signal and the 'duration' of its Fourier transform – defined as the square roots of their normalized second-order moments (see [24], Sect. 8-2) – read $T/2\sqrt{\pi}$ and $\Omega/2\sqrt{\pi}$, respectively, and their product takes the minimum value $\frac{1}{2}$.

Two special choices of the elementary signal might be instructive. If we choose a *rectangular-shaped* elementary signal such that $g(t) = 1$ for $-\frac{1}{2}T < t \leq \frac{1}{2}T$ and $g(t) = 0$ outside that time interval, then Gabor's signal expansion has an easy interpretation: we simply consider the signal $\varphi(t)$ in successive time intervals of length T and describe the signal in each time interval by means of a Fourier series. In the case of a *sinc-shaped* elementary signal $g(t) = \sin(\pi t/T)/(\pi t/T)$ – and hence $\bar{g}(\omega) = T$ for $-\frac{1}{2}\Omega < \omega \leq \frac{1}{2}\Omega$ and $\bar{g}(\omega) = 0$ outside that frequency

interval – Gabor's signal expansion has again an easy interpretation: we simply consider the signal in successive frequency intervals of length Ω and describe the signal in each frequency interval by means of the well-known sampling theorem for band-limited signals.

For the rectangular- or sinc-shaped elementary signals considered in the previous paragraph, the discrete set of shifted and modulated versions of the elementary signal $g_{mk}(t)$ is *orthonormal*; in general, however, this need not be the case, which implies that Gabor's expansion coefficients a_{mk} cannot be determined in the usual way. Let us consider two elements $g_{mk}(t)$ and $g_{nl}(t)$ from the (possibly non-orthonormal) set of shifted and modulated versions of the elementary signal, and let their inner product be denoted by $d_{n-m,l-k}$; hence

$$\int g_{nl}^*(t) g_{mk}(t) dt = d_{n-m,l-k} = d_{m-n,k-l}^*. \tag{2.4}$$

It is easy to see that for Gabor's choice of a Gaussian elementary signal the array d_{mk} takes the form

$$d_{mk} = T(-1)^{mk} e^{-\frac{1}{2}\pi(m^2+k^2)}, \tag{2.5}$$

which does not have the form of a product of two Kronecker deltas $T\delta_m\delta_k$; therefore, the set of shifted and modulated versions of a *Gaussian* elementary signal is *not* orthonormal.

Gabor's expansion coefficients can easily be found, even in the case of a non-orthonormal set $g_{mk}(t)$, if we could find a *window function* $w(t)$ such that

$$a_{mk} = \int \varphi(t) w_{mk}^*(t) dt, \tag{2.6}$$

where we have used, again, the short-hand notation [cf. Eq. (2.2)]

$$w_{mk}(t) = w(t - mT) e^{jk\Omega t}.$$

Such a window function should satisfy the two *bi-orthonormality conditions* [5, 6, 7]

$$\int w_{nl}^*(t) g_{mk}(t) dt = \delta_{n-m}\delta_{l-k} \tag{2.7}$$

and

$$\sum_m \sum_k w_{mk}^*(t_1) g_{mk}(t_2) = \delta(t_1 - t_2); \tag{2.8}$$

we will show later that the first bi-orthonormality condition implies the second one, so we can concentrate on the first one. The first bi-orthonormality condition guarantees that if we start with an array of coefficients a_{mk}, construct a signal $\varphi(t)$ via Eq. (2.3) and subsequently substitute this signal into Eq. (2.6), we end up with the original coefficients array; the second bi-orthonormality condition guarantees that if we start with a certain signal $\varphi(t)$, construct its Gabor coefficients a_{mk} via Eq. (2.6) and subsequently substitute these coefficients into Eq. (2.3), we end up with the original signal. We thus conclude that the two equations (2.3) and (2.6) form a *transform pair*.

We remark that Eq. (2.6), with the help of which we can determine Gabor's expansion coefficients, is, in fact, a sampled version of the *sliding-window spectrum* [7, 10] (or complex spectrogram, or windowed Fourier transform, or short-time Fourier transform), where the sampling appears on the time-frequency lattice $(mT, k\Omega)$ with $\Omega T = 2\pi$. In quantum mechanics this lattice is known as the Von Neumann lattice [4, 21], but for obvious reasons we prefer to call it the *Gabor lattice* in the context of this paper. Hence, whereas sampling the sliding-window spectrum yields the Gabor coefficients, Gabor's signal expansion itself can be considered as a way to reconstruct a signal from its sampled sliding-window spectrum. The name *window function* for the function $w(t)$ that corresponds to a given elementary signal $g(t)$ will thus be clear.

It is easy to see that the window function $w(t)$ is proportional to the elementary signal $g(t)$ if the set $g_{mk}(t)$ is orthonormal. In the remainder of this section we show a first way in which a window function can be found if the set $g_{mk}(t)$ is non-orthonormal. We therefore express the window function $w(t)$ by means of its Gabor expansion [cf. Eq. (2.3)] with expansion coefficients c_{mk}, say [5],

$$w(t) = \sum_m \sum_k c_{mk} g_{mk}(t) \tag{2.9}$$

and try to find the array of coefficients c_{mk}. We therefore consider the first bi-orthonormality condition (2.7)

$$\delta_m \delta_k = \int w^*(t) g_{mk}(t) dt$$

and substitute from the Gabor expansion (2.9) for the window function, yielding

$$\delta_m \delta_k = \int \left[\sum_n \sum_l c^*_{nl} g^*_{nl}(t) \right] g_{mk}(t) dt.$$

We rearrange factors

$$\delta_m \delta_k = \sum_n \sum_l c^*_{nl} \int g^*_{nl}(t) g_{mk}(t) dt$$

and substitute from Eq. (2.4)

$$\delta_m \delta_k = \sum_n \sum_l c^*_{nl} d_{n-m,l-k} = \sum_n \sum_l c^*_{nl} d^*_{m-n,k-l}.$$

The first bi-orthonormality relation thus leads to the condition

$$\sum_n \sum_l c_{nl} d_{m-n,k-l} = \delta_m \delta_k, \tag{2.10}$$

in which the left-hand side has the form of a *convolution* of the given array d_{mk} with the array c_{mk} that we have to determine. Equation (2.10) can be solved, in principle, when we introduce the *Fourier transform* of the arrays according to

$$\bar{d}\left(\frac{t}{T}, \frac{\omega}{\Omega}\right) = \sum_m \sum_k d_{mk} e^{-j(m\omega T - k\Omega t)} \quad \text{(with } \Omega T = 2\pi\text{)}, \tag{2.11}$$

and a similar expression for $\bar{c}(t/T, \omega/\Omega)$. Note that these Fourier transforms are periodic in the time variable t and the frequency variable ω with periods T and Ω, respectively:

$$\bar{d}\left(\frac{t}{T}+m, \frac{\omega}{\Omega}+k\right) = \bar{d}\left(\frac{t}{T}, \frac{\omega}{\Omega}\right). \tag{2.12}$$

Hence, in considering such Fourier transforms we can restrict ourselves to the *fundamental Fourier interval* $(-\frac{1}{2}T < t \leq \frac{1}{2}T, -\frac{1}{2}\Omega < \omega \leq \frac{1}{2}\Omega)$. The *inverse* Fourier transformation reads

$$d_{mk} = \frac{1}{2\pi}\int_T \int_\Omega \bar{d}\left(\frac{t}{T}, \frac{\omega}{\Omega}\right) e^{j(m\omega T - k\Omega t)} dt d\omega \tag{2.13}$$

and a similar expression for c_{mk}; $\int_T \cdot dt$ and $\int_\Omega \cdot d\omega$ denote integrations over one period T and Ω, respectively. After Fourier transforming both sides of Eq. (2.10), the convolution transforms into a *product*, and Eq. (2.10) takes the form

$$\bar{c}\left(\frac{t}{T}, \frac{\omega}{\Omega}\right)\bar{d}\left(\frac{t}{T}, \frac{\omega}{\Omega}\right) = 1. \tag{2.14}$$

The function $\bar{c}(t/T, \omega/\Omega)$ can easily be found from the latter relationship, provided that the inverse of $\bar{d}(t/T, \omega/\Omega)$ exists, and inverse Fourier transforming $\bar{c}(t/T, \omega/\Omega)$ [cf. Eq. (2.13)] then results in the array c_{mk} that we are looking for.

Let us consider how things work out for Gabor's choice of a Gaussian elementary signal. After Fourier transforming the array (2.5) we get

$$\frac{1}{T}\bar{d}\left(\frac{t}{T}, \frac{\omega}{\Omega}\right) = \theta_3(\Omega t)\theta_3(\omega T) + \theta_3(\Omega t)\theta_2(\omega T) + \theta_2(\Omega t)\theta_3(\omega T) - \theta_2(\Omega t)\theta_2(\omega T), \tag{2.15}$$

with

$$\theta_3(z) = \theta_3(z; e^{-2\pi}) = \sum_m e^{-2\pi m^2} e^{j2mz} \tag{2.16}$$

and

$$\theta_2(z) = \theta_2(z; e^{-2\pi}) = \sum_m e^{-2\pi(m+\frac{1}{2})^2} e^{j(2m+1)z}. \tag{2.17}$$

The functions $\theta(z; e^{-2\pi})$ are known as *theta functions* [1, 31] with *nome* $e^{-2\pi}$. The Fourier transform $(1/T)\bar{d}(t/T, \omega/\Omega)$ has been depicted in Fig. 1, where we have restricted ourselves to the fundamental Fourier interval.

Note that the values of $(1/T)\bar{d}(t/T, \omega/\Omega)$ for $(t = \frac{1}{2}T + mT, \omega = \frac{1}{2}\Omega + k\Omega)$ read

$$\frac{1}{T}\bar{d}(\tfrac{1}{2}+m, \tfrac{1}{2}+k) = \theta_3^2(0)\left[1 - 2\frac{\theta_2(0)}{\theta_3(0)} - \frac{\theta_2^2(0)}{\theta_3^2(0)}\right].$$

Since the nome equals $e^{-2\pi}$, we have the relation $\theta_2(0)/\theta_3(0) = \sqrt{2}-1 = \tan(\pi/8)$ (see [31], p. 525), and we conclude that $\bar{d}(t/T, \omega/\Omega)$ has (double) zeros for $(t = \frac{1}{2}T + mT,\ \omega = \frac{1}{2}\Omega + k\Omega)$. Inversion of $\bar{d}(t/T, \omega/\Omega)$ in order to find $\bar{c}(t/T, \omega/\Omega)$ may thus be difficult.

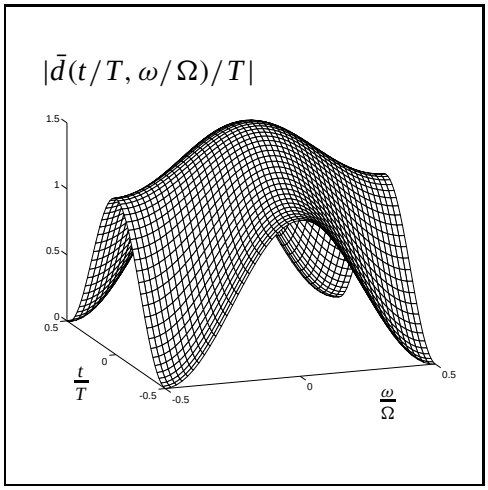

Figure 1: The Fourier transform $\bar{d}(t/T, \omega/\Omega)/T$ in the case of a Gaussian elementary signal $g(t) = 2^{\frac{1}{4}} e^{-\pi(t/T)^2}$.

Zeros of $\bar{d}(t/T, \omega/\Omega)$ not only prohibit an easy determination of the window function $w(t)$, but they lead to another unwanted property: they enable us to construct a (not identically zero) function $\bar{z}(t/T, \omega/\Omega)$ such that the product $\bar{z}(t/T, \omega/\Omega)\bar{d}(t/T, \omega/\Omega)$ [cf. Eq. (2.14)] vanishes. For a Gaussian elementary signal, with zeros for $(t = \frac{1}{2}T + mT, \omega = \frac{1}{2}\Omega + k\Omega)$, we might choose

$$\bar{z}\left(\frac{t}{T}, \frac{\omega}{\Omega}\right) = \sum_m \sum_k \delta\left(\frac{t}{T} - \tfrac{1}{2} - m\right) \delta\left(\frac{\omega}{\Omega} - \tfrac{1}{2} - k\right) z. \tag{2.18}$$

Inverse Fourier transforming this function yields the array

$$z_{mk} = (-1)^{m+k} z, \tag{2.19}$$

which is a *homogeneous solution* of Eq. (2.10). Hence, Gabor coefficients might *not* be *unique*: if c_{mk} are Gabor coefficients that determine the window function $w(t)$, then $c_{mk} + z_{mk}$ are valid Gabor coefficients, as well!

In the next section we will present a different and mathematically more attractive way to find the window function $w(t)$ that corresponds to a given elementary signal $g(t)$.

3 Zak transform

In this section we introduce the *Zak transform* [32, 33, 34] and we show its intimate relationship to Gabor's signal expansion. The Zak transform $\tilde{\varphi}(t, \omega; \tau)$ of a signal $\varphi(t)$ is defined as a Fourier transformation of the sequence $\varphi(t + m\tau)$ (with m taking on all integer values and t being a mere parameter), hence

$$\tilde{\varphi}(t, \omega; \tau) = \sum_m \varphi(t + m\tau) e^{-jm\omega\tau}; \tag{3.1}$$

we will throughout denote the Zak transform of a signal by the same symbol as the signal itself, but marked by a tilde on top of it. We remark that the function $\tilde{\varphi}(t, \omega; \tau)$ is *periodic* in the frequency variable ω with period $2\pi/\tau$ and *quasi-periodic* in the time variable t with quasi-period τ:

$$\tilde{\varphi}\left(t + m\tau, \omega + k\frac{2\pi}{\tau}; \tau\right) = \tilde{\varphi}(t, \omega; \tau) e^{jm\omega\tau}. \tag{3.2}$$

Hence, in considering the Zak transform we can restrict ourselves to the *fundamental Zak interval* $(-\frac{1}{2}\tau < t \leq \frac{1}{2}\tau, -\pi/\tau < \omega \leq \pi/\tau)$. The inverse relationship of the Zak transform (3.1) has the form

$$\varphi(t + m\tau) = \frac{\tau}{2\pi} \int_{2\pi/\tau} \tilde{\varphi}(t, \omega; \tau) e^{jm\omega\tau} d\omega; \tag{3.3}$$

it will be clear that the variable t in Eq. (3.3) can be restricted to an interval of length τ, with m taking on all integer values. From the properties of the Zak transform we mention Parseval's energy theorem, which leads to the relationship

$$\frac{1}{2\pi} \int_\tau \int_{2\pi/\tau} |\tilde{\varphi}(t, \omega; \tau)|^2 dt d\omega = \frac{1}{\tau} \int |\varphi(t)|^2 dt. \tag{3.4}$$

The Zak transform $\tilde{\varphi}(t, \omega; \tau)$ provides a means to represent an arbitrarily long one-dimensional time function (or one-dimensional frequency function) by a two-dimensional time-frequency function on a rectangle with *finite area* 2π. This two-dimensional function $\tilde{\varphi}(t, \omega; \tau)$ is known as the Zak transform, because Zak was the first who systematically studied this transformation in connection with solid state physics [32, 33, 34]. Some of its properties were known long before Zak's work, however. The same transform is called Weil-Brezin map and it is claimed that the transform was already known to Gauss [28]. It was also used by Gel'fand (see, for instance, [29], Chap. XIII); Zak seems, however, to have been the first to recognize it as the versatile tool it is. The Zak transform has many interesting properties and also interesting applications to signal analysis, for which we refer to [18, 19]. In this section we will show how the Zak transform can be applied to Gabor's signal expansion.

We want to make an observation to which we will return later on in this paper. Suppose that, for small τ for instance, we can approximate a function $g(t)$ by the piecewise constant function

$$g(t) = \sum_n g_n \text{rect}\left(\frac{t - n\tau}{\tau}\right), \tag{3.5}$$

where rect$(x) = 1$ for $-\frac{1}{2} < x \leq \frac{1}{2}$ and rect$(x) = 0$ outside that interval. In the time interval $-\frac{1}{2}\tau < t \leq \frac{1}{2}\tau$, the Zak transform $\tilde{g}(t, \omega; \tau)$ then takes the form

$$\tilde{g}(t, \omega; \tau) = \sum_n g_n e^{-jn\omega\tau} = \bar{g}(\omega\tau); \tag{3.6}$$

note that this Zak transform does not depend on the time variable t, and that the one-dimensional Fourier transform $\bar{g}(\omega\tau)$ of the sequence g_n arises. We remark that Parseval's energy theorem now leads to the relation

$$\frac{1}{2\pi}\int_\tau \int_{2\pi/\tau} |\tilde{g}(t, \omega; \tau)|^2 dt d\omega = \frac{\tau}{2\pi}\int_{2\pi/\tau} |\bar{g}(\omega\tau)|^2 d\omega = \sum_n |g_n|^2. \tag{3.7}$$

We still have to solve the problem of finding the window function $w(t)$ that corresponds to a given elementary signal $g(t)$ such that the two bi-orthonormality conditions [Eqs. (2.7) and (2.8)] are satisfied. We consider again the first bi-orthonormality condition (2.7)

$$\delta_m \delta_k = \int g(\tau) w^*_{mk}(\tau) d\tau$$

and apply a Fourier transformation [cf. Eq. (2.11)] to both sides of this condition, yielding

$$1 = \sum_m \sum_k \left[\int g(\tau) w^*(\tau - mT) e^{-jk\Omega\tau} d\tau\right] e^{-j(m\omega T - k\Omega t)}.$$

We rearrange factors

$$1 = \sum_m \left[\int g(\tau) w^*(\tau - mT) \left(\sum_k e^{-jk\Omega(\tau - t)}\right) d\tau\right] e^{-jm\omega T}$$

and replace the sum of exponentials by a sum of Dirac functions

$$1 = \sum_m \left[\int g(\tau) w^*(\tau - mT) \left(T\sum_n \delta(\tau - t - nT)\right) d\tau\right] e^{-jm\omega T}.$$

We rearrange factors again

$$1 = T\sum_m \sum_n \left[\int g(\tau) w^*(\tau - mT)\delta(\tau - t - nT) d\tau\right] e^{-jm\omega T}$$

and evaluate the integral

$$1 = T\sum_m \sum_n g(t + nT) w^*(t + [n - m]T) e^{-jm\omega T}.$$

After a final rearranging of factors we find

$$1 = T\sum_n g(t + nT) e^{-jn\omega T} \left[\sum_m w^*(t + [n - m]T) e^{j(n - m)\omega T}\right] =$$

$$= T\left[\sum_n g(t+nT)e^{-jn\omega T}\right]\left[\sum_m w(t+mT)e^{-jm\omega T}\right]^*,$$

in which expression we recognize [see Eq. (3.1)] the definitions for the Zak transforms $\tilde{g}(t,\omega;T)$ and $\tilde{w}(t,\omega;T)$ of the two functions $g(t)$ and $w(t)$, respectively; hence

$$T\tilde{g}(t,\omega;T)\tilde{w}^*(t,\omega;T) = 1. \tag{3.8}$$

The first bi-orthonormality condition (2.7) thus transforms into a *product*, enabling us to find the window function $w(t)$ that corresponds to a given elementary signal $g(t)$ in an easy way:

- from the elementary signal $g(t)$ we derive its Zak transform $\tilde{g}(t,\omega;T)$ via definition (3.1);
- under the assumption that division by $\tilde{g}(t,\omega;T)$ is allowed, the function $\tilde{w}(t,\omega;T)$ can be found with the help of relation (3.8);
- finally, the window function $w(t)$ follows from its Zak transform $\tilde{w}(t,\omega;T)$ by means of the inversion formula (3.3).

It is shown in Appendix A that the window function $w(t)$ found in this way also satisfies the second bi-orthonormality condition (2.8).

Let us consider Gabor's choice of a Gaussian elementary signal again. The Zak transform of a Gaussian reads

$$\tilde{g}(t,\omega;\alpha T) = 2^{\frac{1}{4}}e^{-\pi(t/T)^2}\theta_3\left(\alpha\pi\zeta^*;e^{-\pi\alpha^2}\right), \tag{3.9}$$

where

$$\theta_3(z;q) = \sum_m q^{m^2}e^{j2mz} \tag{3.10}$$

is a theta function again, in this case with nome $q = e^{-\pi\alpha^2}$, and where, for convenience, we have set $\zeta = \omega/\Omega + jt/T$. The Zak transform $\tilde{g}(t,\omega;\alpha T)$ has been depicted in Fig. 2 for several values of the parameter $\tau = \alpha T$, where we have restricted ourselves to the fundamental Zak interval. Note that for $\alpha \leq \frac{1}{3}$, the Zak transform $\tilde{g}(t,\omega;\alpha T)$ becomes almost independent of t, as we have mentioned before.

In the case of a Gaussian elementary signal and choosing $\alpha = 1$ (Gabor's original choice), the Zak transform of the window function takes the form

$$T\tilde{w}(t,\omega;T) = \frac{1}{\tilde{g}^*(t,\omega;T)} = 2^{-\frac{1}{4}}e^{\pi(t/T)^2}\frac{1}{\theta_3(\pi\zeta;e^{-\pi})}, \tag{3.11}$$

in which expression we have set again $\zeta = \omega/\Omega + jt/T$. In the fundamental interval the function $1/\theta_3(\pi\zeta;e^{-\pi})$ can be expressed as

$$\frac{1}{\theta_3(\pi\zeta;e^{-\pi})} = \left(\frac{K_o}{\pi}\right)^{-3/2}\left[c_0 + 2\sum_{m=1}^{\infty}(-1)^m c_m\cos(2\pi m\zeta)\right], \tag{3.12}$$

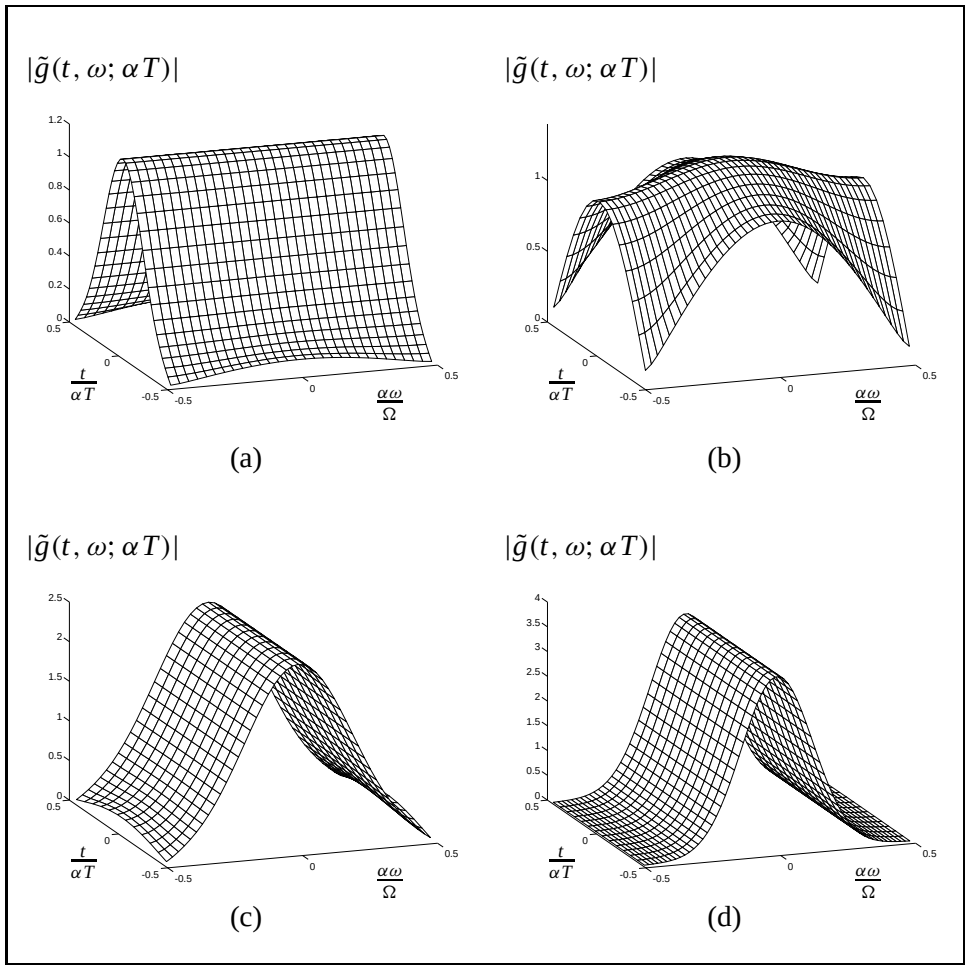

Figure 2: The Zak transform $\tilde{g}(t, \omega; \alpha T)$ in the case of a Gaussian elementary signal $g(t) = 2^{\frac{1}{4}} e^{-\pi (t/T)^2}$ for different values of $\tau = \alpha T$:
(a) $\alpha = 2$, (b) $\alpha = 1$, (c) $\alpha = \frac{1}{2}$, and (d) $\alpha = \frac{1}{3}$.

where the coefficients c_m are defined by

$$c_m = \sum_{n=0}^{\infty} (-1)^n e^{-\pi(n+\frac{1}{2})(2m+n+\frac{1}{2})} \tag{3.13}$$

(see, for instance, [31], p. 489, Example 14); the constant K_o is the complete elliptic integral for the modulus $\frac{1}{2}\sqrt{2}$: $K_o = 1.85407468$ (see, for instance, [31], p. 524). It is now easy to determine the window function $w(t)$ via the inversion formula (3.3), yielding

$$Tw(t+mT) = 2^{-\frac{1}{4}} e^{\pi(t/T)^2} \left(\frac{K_o}{\pi}\right)^{-3/2} (-1)^m c_m e^{2\pi m(t/T)} \tag{3.14}$$

with $-\frac{1}{2}T < t \leq \frac{1}{2}T$, and hence

$$Tw(t) = 2^{-\frac{1}{4}} e^{\pi(t/T)^2} \left(\frac{K_o}{\pi}\right)^{-3/2} \sum_{p+\frac{1}{2} \geq |t/T|}^{\infty} (-1)^p e^{-\pi(p+\frac{1}{2})^2}. \tag{3.15}$$

The Gaussian elementary signal $g(t)$ and its corresponding window function $Tw(t)$ have been depicted in Fig. 3.

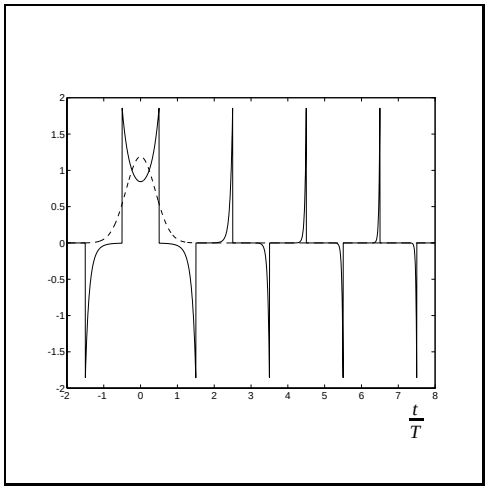

Figure 3: A Gaussian elementary signal $g(t) = 2^{\frac{1}{4}} e^{-\pi(t/T)^2}$ (dashed line) and its corresponding window function $Tw(t)$ (solid line).

A practical way to represent this particular window function is in the form [20]

$$Tw(t) = 2^{-\frac{1}{4}} \left(\frac{K_o}{\pi}\right)^{-3/2} (-1)^m e^{\pi[(t/T)^2 - (m+\frac{1}{2})^2]}$$

$$\times \sum_{p=m}^{\infty} (-1)^{p-m} e^{-\pi[(p+\frac{1}{2})^2 - (m+\frac{1}{2})^2]}, \tag{3.16}$$

where m is the nonnegative integer defined by $(m-\frac{1}{2})T < |t| \leq (m+\frac{1}{2})T$. Since the summation in the latter expression yields a result which is close to unity for any value of m (0.998133 for $m = 0$, 0.999997 for $m = 1$, ... , 1 for $m = \infty$), this representation leads to the approximation

$$Tw(t) \simeq 2^{-\frac{1}{4}} \left(\frac{K_o}{\pi}\right)^{-3/2} (-1)^m e^{\pi[(t/T)^2 - (m+\frac{1}{2})^2]} \tag{3.17}$$

with m defined by $(m-\frac{1}{2})T < |t| \leq (m+\frac{1}{2})T$.

We mention the property that in the case of a Gaussian elementary signal, whose Fourier transform has the same form as the elementary signal, the Fourier transform of the corresponding window function has the same form as the window function itself. Moreover, we note that for large positive r, the extrema of the window function read

$$Tw(\pm[r+\tfrac{1}{2}]T) \simeq 2^{-\frac{1}{4}} \left(\frac{K_o}{\pi}\right)^{-3/2} (-1)^r, \tag{3.18}$$

which implies that $|w(t)|$ does not decrease with increasing value of $|t|$. More properties of this particular window function can be found elsewhere [18].

Since the Zak transform $\tilde{g}(t,\omega;T)$ of the Gaussian elementary signal has simple zeros for $(t = \frac{1}{2}T + mT, \omega = \frac{1}{2}\Omega + k\Omega)$, we can again construct a (not identically zero) function

$$\tilde{z}(t,\omega;T) = \sum_m \sum_k \delta\left(\frac{t}{T} - \tfrac{1}{2} - m\right) \delta\left(\frac{\omega}{\Omega} - \tfrac{1}{2} - k\right) z \tag{3.19}$$

for which the product $T\tilde{g}(t,\omega;T)\tilde{z}^*(t,\omega;T)$ [cf. Eq. (3.8)] vanishes. Thus, with the help of the inversion formula (3.3), a function

$$z(t) = zT \sum_m (-1)^m \delta(t - \tfrac{1}{2}T - mT) \tag{3.20}$$

occurs, which is a *homogenous solution* of the bi-orthonormality relation (2.7). We conclude that if the Zak transform $\tilde{g}(t,\omega;T)$ of an elementary signal $g(t)$ has zeros, the corresponding window function may not be unique: if $w(t)$ is a window function, then $w(t) + z(t)$ is a proper window function, too.

Zeros in $\tilde{g}(t,\omega;T)$ may be even worse. When we apply Parseval's energy theorem (3.4) to $w(t)$ and substitute from relation (3.8) we get

$$\frac{1}{T}\int |w(t)|^2 dt = \frac{1}{2\pi}\int_T \int_\Omega |\tilde{w}(t,\omega;T)|^2 dt d\omega = \frac{1}{2\pi}\int_T \int_\Omega \frac{1}{|T\tilde{g}(t,\omega;T)|^2} dt d\omega. \tag{3.21}$$

From this relationship we conclude that in the case of zeros in $\tilde{g}(t,\omega;T)$, the corresponding window function $w(t)$ may not be quadratically integrable. This consequence of zeros in $\tilde{g}(t,\omega;T)$ is even worse than the fact that the window function is not unique; it may cause very bad convergence properties in the determination of Gabor's expansion coefficients.

4 Gabor transform and Zak transform for continuous-time signals

Now that we have shown that, at least in principle, a window function $w(t)$ can be found corresponding to a given elementary signal $g(t)$, we will focus again on the two relations (2.3) and (2.6). These two relations form a *transform pair*, as has been remarked before, and we will therefore associate more appropriate names for these relations. We define the *Gabor transform* by means of relation (2.6)

$$a_{mk} = \int \varphi(t) w^*_{mk}(t) dt;$$

the Gabor transform thus yields the array of Gabor coefficients a_{mk} that corresponds to a signal $\varphi(t)$. The *inverse* Gabor transform will then be defined by relation (2.3)

$$\varphi(t) = \sum_m \sum_k a_{mk} g_{mk}(t);$$

the inverse Gabor transform reconstructs the signal $\varphi(t)$ from its array of Gabor coefficients a_{mk}.

In Section 3 we have seen that the Zak transform can be helpful in determining the window function $w(t)$ for a given elementary signal $g(t)$. In this section we will study the possibility of applying the Zak transform to the Gabor transform and its inverse.

Let us first apply a Fourier transformation [cf. Eq. (2.11)] to the array of Gabor coefficients a_{mk},

$$\bar{a}\left(\frac{t}{T}, \frac{\omega}{\Omega}\right) = \sum_m \sum_k a_{mk} e^{-j(m\omega T - k\Omega t)},$$

and let us substitute from the Gabor transform (2.6):

$$\bar{a}\left(\frac{t}{T}, \frac{\omega}{\Omega}\right) = \sum_m \sum_k \left[\int \varphi(\tau) w^*(\tau - mT) e^{-jk\Omega\tau} d\tau\right] e^{-j(m\omega T - k\Omega t)}.$$

Along the same lines as the ones that we followed in deriving Eq. (3.8), we can now proceed to express the latter relationship in the form

$$\bar{a}\left(\frac{t}{T}, \frac{\omega}{\Omega}\right) = T\tilde{\varphi}(t, \omega; T)\tilde{w}^*(t, \omega; T). \tag{4.1}$$

Hence, the Gabor transform (2.6) can be transformed into the *product form* (4.1).

A product form can also be found for the inverse Gabor transform. If we apply a Zak transformation [see Eq. (3.1)] to both sides of Eq. (2.3), we get

$$\tilde{\varphi}(t, \omega; T) = \sum_n \left[\sum_m \sum_k a_{mk} g(t + nT - mT) e^{jk\Omega t}\right] e^{-jn\omega T}.$$

After rearranging factors

$$\tilde{\varphi}(t,\omega;T) =$$

$$= \sum_m \sum_k a_{mk} e^{-j(m\omega T - k\Omega t)} \left[\sum_n g(t + [n-m]T) e^{-j(n-m)\omega T} \right] =$$

$$= \left[\sum_m \sum_k a_{mk} e^{-j(m\omega T - k\Omega t)} \right] \left[\sum_n g(t+nT) e^{-jn\omega T} \right],$$

we immediately get the product relation

$$\tilde{\varphi}(t,\omega;T) = \bar{a}\left(\frac{t}{T}, \frac{\omega}{\Omega}\right) \tilde{g}(t,\omega;T). \tag{4.2}$$

Now that we have found product forms for the Gabor transform and its inverse, we have also found a different way of determining Gabor's expansion coefficients a_{mk}, without explicitly determining a window function $w(t)$:

- from the signal $\varphi(t)$ and the elementary signal $g(t)$ we derive their Zak transforms $\tilde{\varphi}(t,\omega;T)$ and $\tilde{g}(t,\omega;T)$, respectively, according to the definition (3.1);
- under the assumption that division by $\tilde{g}(t,\omega;T)$ is allowed, the function $\bar{a}(t/T, \omega/\Omega)$ can be found by means of the product relation (4.2);
- finally, the expansion coefficients a_{mk} follow from the function $\bar{a}(t/T, \omega/\Omega)$ with the help of the inverse Fourier transformation (2.13).

Again, we conclude that Gabor's expansion coefficients may be non-unique in the case that $\tilde{g}(t,\omega;T)$ has zeros. In that case homogeneous solutions z_{mk} may occur for which the Fourier transform $\bar{z}(t/T, \omega/\Omega)$ satisfies the relation

$$\bar{z}\left(\frac{t}{T}, \frac{\omega}{\Omega}\right) \tilde{g}(t,\omega;T) = 0. \tag{4.3}$$

Relation (4.3), which is similar to the product relation (4.2) with $\tilde{\varphi}(t,\omega;T) = 0$, can be transformed into the relation

$$\sum_m \sum_k z_{mk} g_{mk}(t) = 0, \tag{4.4}$$

which is similar to relation (2.3) with $\varphi(t) = 0$. Relation (4.4) shows that certain arrays of nonzero coefficients in Gabor's signal expansion may yield a zero result. We thus conclude that Gabor's signal expansion may be non-unique: if the array of coefficients a_{mk} yields the signal $\varphi(t)$, then the array $a_{mk} + z_{mk}$ yields the same signal.

5 Gabor transform and Zak transform for discrete-time signals

Until now we have only considered continuous-time signals. In this and the following section we will extend the concepts of the Gabor transform and the Zak transform to discrete-time signals [9].

Let us consider a discrete-time signal $\varphi[n]$; to distinguish the discrete-time case from the continuous-time case, we will use square brackets [ ] to denote a discrete-time signal, whereas we used curved brackets ( ) to denote a continuous-time signal. The Fourier transform of a discrete-time signal is defined by

$$\bar{\varphi}(\theta) = \sum_n \varphi[n] e^{-j\theta n}; \tag{5.1}$$

note that this Fourier transform is *periodic* in the frequency variable θ with period 2π. The inverse Fourier transformation reads

$$\varphi[n] = \frac{1}{2\pi} \int_{2\pi} \bar{\varphi}(\theta) e^{j\theta n} d\theta, \tag{5.2}$$

where the integration extends over one period 2π. As already expressed in the introductory Section 1, we will consistently use n, m and N as time variables, θ, k and Θ as frequency variables, and the relation $\Theta N = 2\pi$ holds throughout.

On the analogy of the Gabor transform (2.6) for continuous-time signals, we introduce the Gabor transform for discrete-time signals

$$a_{mk} = \sum_n \varphi[n] w^*_{mk}[n] \tag{5.3}$$

with the short-hand notation

$$w_{mk}[n] = w[n - mN] e^{jk\Theta n} \tag{5.4}$$

again [cf. Eq. (2.2)]; in the discrete-time case, the positive integer N and the parameter $\Theta = 2\pi/N$ are the respective counterparts of T and $\Omega = 2\pi/T$ in the continuous-time case. We remark that the Gabor transform a_{mk} for discrete-time signals is an array that is *periodic* in the frequency variable k with period N; this periodicity in the Gabor transform, like the periodicity in the Fourier transform, results from the discrete nature of the signal. On the analogy of the corresponding relation (2.3) for continuous-time signals, the inverse Gabor transform for discrete-time signals takes the form

$$\varphi[n] = \sum_m \sum_{k=<N>} a_{mk} g_{mk}[n], \tag{5.5}$$

where the summation over k extends over one period N of the periodic array a_{mk}.

On the analogy of the continuous-time case we need the Fourier transform of the array a_{mk}, which will be defined by [cf. Eq. (2.11)]

$$\bar{a}\left(\frac{n}{N}, \frac{\theta}{\Theta}\right) = \sum_m \sum_{k=<N>} a_{mk} e^{-j(m\theta N - k\Theta n)} \quad (\text{with } \Theta N = 2\pi), \tag{5.6}$$

where the summation over k extends again over one period N. We remark that this Fourier transform is periodic in the (discrete) time index n with period N and periodic in the (continuous) frequency variable θ with period Θ. The inverse Fourier transform reads [cf. Eq. (2.13)]

$$a_{mk} = \frac{1}{2\pi} \sum_{n=<N>} \int_{\Theta} \bar{a}\left(\frac{n}{N}, \frac{\theta}{\Theta}\right) e^{j(m\theta N - k\Theta n)} d\theta, \tag{5.7}$$

where the summation over n extends over one period N, and the integration over θ extends over one period Θ.

Furthermore we need the Zak transform for discrete-time signals, which will be defined by [cf. Eq. (3.1)]

$$\tilde{\varphi}(n, \theta; N) = \sum_{m} \varphi[n + mN] e^{-jm\theta N}. \tag{5.8}$$

The Zak transform in the discrete-time case is periodic in the (continuous) frequency variable θ with period Θ and quasi-periodic in the (discrete) time index n with quasi-period N [cf. Eq. (3.2)]:

$$\tilde{\varphi}\left(n + mN, \theta + k\frac{2\pi}{N}; N\right) = \tilde{\varphi}(n, \theta; N) e^{jm\theta N}. \tag{5.9}$$

The inverse Zak transform now reads [cf. Eq. (3.3)]

$$\varphi[n + mN] = \frac{N}{2\pi} \int_{2\pi/N} \tilde{\varphi}(n, \theta; N) e^{jm\theta N} d\theta, \tag{5.10}$$

where the time index n can be restricted to an interval of length N, with m taking on all integer values.

Using the discrete-time equivalents of the Gabor transform, the Fourier transform, and the Zak transform, it is not difficult to show that the Gabor transform (5.3) can be transformed into the product form [cf. Eq. (4.1)]

$$\bar{a}\left(\frac{n}{N}, \frac{\theta}{\Theta}\right) = N \tilde{\varphi}(n, \theta; N) \tilde{w}^*(n, \theta; N) \tag{5.11}$$

and the inverse Gabor transform (5.5) into the product form [cf. Eq. (4.2)]

$$\tilde{\varphi}(n, \theta; N) = \bar{a}\left(\frac{n}{N}, \frac{\theta}{\Theta}\right) \tilde{g}(n, \theta; N). \tag{5.12}$$

Let us now consider Gabor's choice of a Gaussian elementary signal $g(t)$ again, and let us consider a discrete-time version $g[n]$ of the elementary signal, *symmetrically* positioned with respect to the sampling grid on the t-axis with a sampling distance T/N:

$$g[n] = g\left(n\frac{T}{N}\right) = 2^{\frac{1}{4}} e^{-\pi(n/N)^2} \quad (N \text{ odd}), \tag{5.13}$$

$$g[n] = g\left([n+\tfrac{1}{2}]\frac{T}{N}\right) = 2^{\frac{1}{4}}e^{-\pi([n+\frac{1}{2}]/N)^2} \quad (N \text{ even}). \tag{5.14}$$

The corresponding Zak transforms $\tilde{g}(n,\theta;\,N)$ follow from substituting from Eqs. (5.13) and (5.14) into Eq. (5.8) and read [cf. Eq. (3.9)]

$$\tilde{g}(n,\theta;\,N) = 2^{\frac{1}{4}}e^{-\pi(n/N)^2}\theta_3(\pi\zeta^*;\,e^{-\pi}) \quad (\text{with } \zeta = \theta/\Theta + jn/N) \tag{5.15}$$

$$\tilde{g}(n,\theta;\,N) = 2^{\frac{1}{4}}e^{-\pi([n+\frac{1}{2}]/N)^2}\theta_3(\pi\zeta^*;\,e^{-\pi}) \quad (\text{with } \zeta = \theta/\Theta + j[n+\tfrac{1}{2}]/N) \tag{5.16}$$

for odd and even N, respectively. It is important to note that in the discrete-time case the zeros of the theta function do not occur on a raster point! Hence, the Zak transform $\tilde{g}(n,\theta;\,N)$ of a Gaussian elementary signal has no zeros.

Let us, for the sake of simplicity, take N odd, which implies that there is a sampling point at $t = 0$. The corresponding window function $w[n]$ then reads [cf. Eq. (3.15)]

$$Nw[n] = 2^{-\frac{1}{4}}e^{\pi(n/N)^2}\left(\frac{K_o}{\pi}\right)^{-3/2}\sum_{p+\frac{1}{2}\geq|n/N|}^{\infty}(-1)^p e^{-\pi(p+\frac{1}{2})^2}, \tag{5.17}$$

or, in a different form [cf. Eq. (3.16)],

$$Nw[n] = 2^{-\frac{1}{4}}\left(\frac{K_o}{\pi}\right)^{-3/2}(-1)^m e^{\pi[(n/N)^2-(m+\frac{1}{2})^2]} \times \sum_{p=m}^{\infty}(-1)^{p-m}e^{-\pi[(p+\frac{1}{2})^2-(m+\frac{1}{2})^2]}, \tag{5.18}$$

which form can be approximated by [cf. Eq. (3.17)]

$$Nw[n] \simeq 2^{-\frac{1}{4}}\left(\frac{K_o}{\pi}\right)^{-3/2}(-1)^m e^{\pi[(n/N)^2-(m+\frac{1}{2})^2]}, \tag{5.19}$$

with m defined by $(m-\frac{1}{2})N < |n| \leq (m+\frac{1}{2})N$. We remark that for large positive r, the extrema of the window function read

$$Nw[\pm([r+\tfrac{1}{2}]N-\tfrac{1}{2})] \simeq 2^{-\frac{1}{4}}\left(\frac{K_o}{\pi}\right)^{-3/2}(-1)^r e^{\pi[(r+\frac{1}{2}-1/2N)^2-(r+\frac{1}{2})^2]} =$$

$$= 2^{-\frac{1}{4}}\left(\frac{K_o}{\pi}\right)^{-3/2}(-1)^r e^{\pi[(1/2N)^2-(r+\frac{1}{2})/N]} \simeq$$

$$\simeq 2^{-\frac{1}{4}}\left(\frac{K_o}{\pi}\right)^{-3/2}(-1)^r e^{-\pi(r/N)}. \tag{5.20}$$

Unlike in the continuous-time case, where $|w(t)|$ did not decrease with increasing value of $|t|$ [see Eq. (3.18)], the function $|w[n]|$ does eventually decrease exponentially with increasing value of $|n|$. A similar result holds when N is chosen even.

6 Discrete Gabor transform and discrete Zak transform

In this section we will introduce *discrete versions* of the Gabor and the Zak transform [2, 3, 22, 23, 25, 26, 27, 30] by *sampling* the *continuous* variable θ that arises both in the Fourier transform (5.6) and in the Zak transform (5.8) of discrete-time signals. We start with the Fourier transform (5.6)

$$\bar{a}\left(\frac{n}{N}, \frac{\theta}{\Theta}\right) = \sum_{m} \sum_{k=<N>} a_{mk} e^{-j(m\theta N - k\Theta n)},$$

which is periodic in the *discrete* time index n with period N and periodic in the *continuous* frequency variable θ with period Θ. We define the array $\bar{a}[n, l; N, M]$ as *samples* of this Fourier transform $\bar{a}(n/N, \theta/\Theta)$

$$\bar{a}[n, l; N, M] = \bar{a}\left(\frac{n}{N}, \frac{l}{M}\right) = \sum_{m} \sum_{k=<N>} a_{mk} e^{-j[m(2\pi/M)l - k(2\pi/N)n]}; \tag{6.1}$$

thus we have sampled the frequency axis such that in each period of length Θ there appear M equally-spaced sampling points a distance Θ/M apart. We then define the array A_{mk} as a kind of *inverse* Fourier transform [cf. Eq. (5.7)] of the array $\bar{a}[n, l; N, M]$ by

$$A_{mk} = \frac{1}{MN} \sum_{n=<N>} \sum_{l=<M>} \bar{a}[n, l; N, M] e^{j[m(2\pi/M)l - k(2\pi/N)n]}. \tag{6.2}$$

We remark that, whereas the array a_{mk} is periodic in the frequency variable k with period N but does not have a periodicity with respect to the time index m, the array A_{mk} is periodic in both the frequency variable k *and* the time index m with periods N and M, respectively. It can easily be shown that the relation between A_{mk} and a_{mk} reads

$$A_{mk} = \sum_{r} a_{m+rM,k}, \tag{6.3}$$

from which relation we conclude that A_{mk} is a *summation of the array a_{mk} and all its replicas shifted along the m-direction over distances rM* ($r = -\infty, \ldots, +\infty$). Of course, the array $\bar{a}[n, l; N, M]$ can directly be expressed in the array A_{mk} through the relationship [cf. Eq. (6.1)]

$$\bar{a}[n, l; N, M] = \sum_{m=<M>} \sum_{k=<N>} A_{mk} e^{-j[m(2\pi/M)l - k(2\pi/N)n]}. \tag{6.4}$$

The latter relationship (6.4) between the arrays $\bar{a}[n, l; N, M]$ and A_{mk} is known as the *discrete Fourier transform*, whereas the relation (6.2) is consequently called the *inverse* discrete Fourier transform. It is important to note that if $a_{mk} \neq 0$ in an m-interval of length M and vanishes outside that interval, the array a_{mk} is just one period of the periodic array A_{mk} and can thus be reconstructed from A_{mk} and, hence, from $\bar{a}[n, l; N, M]$.

We will now perform an analogous procedure for the Zak transform (5.8)

$$\tilde{\varphi}(n, \theta; N) = \sum_m \varphi[n + mN] e^{-jm\theta N},$$

which, like the Fourier transform (5.6), is periodic in the *continuous* frequency variable θ with period Θ, as well. We define the array $\tilde{\varphi}[n, l; N, M]$ as *samples* of this Zak transform $\tilde{\varphi}(n, \theta; N)$

$$\tilde{\varphi}[n, l; N, M] = \tilde{\varphi}\left(n, \frac{\Theta}{M} l; N\right) = \sum_m \varphi[n + mN] e^{-jm(2\pi/M)l}; \qquad (6.5)$$

thus we have again sampled the frequency axis such that in each period of length Θ there appear M equally-spaced sampling points a distance Θ/M apart. We then define the function $\Phi[n + mN]$ as a kind of *inverse* Zak transform [cf. Eq. (5.10)] of the array $\tilde{\varphi}[n, l; N, M]$ by

$$\Phi[n + mN] = \frac{1}{M} \sum_{l=<M>} \tilde{\varphi}[n, l; N, M] e^{jm(2\pi/M)l}. \qquad (6.6)$$

It can easily be shown that the relationship

$$\Phi[n] = \sum_r \varphi[n + rMN] \qquad (6.7)$$

holds; thus, $\Phi[n]$ is a *summation of the signal* $\varphi[n]$ *and all its replicas shifted over distances* rMN ($r = -\infty, \ldots, +\infty$). The sequence $\Phi[n]$ is thus periodic in the time index n with period MN. Of course, the array $\tilde{\varphi}[n, l; N, M]$ can directly be expressed in the sequence $\Phi[n]$ through the relationship [cf. Eq. (6.5)]

$$\tilde{\varphi}[n, l; N, M] = \sum_{m=<M>} \Phi[n + mN] e^{-jm(2\pi/M)l}. \qquad (6.8)$$

On the analogy of the discrete Fourier transform (6.4), we shall call the latter relationship (6.8) between the array $\tilde{\varphi}[n, l; N, M]$ and the sequence $\Phi[n]$ the *discrete Zak transform*, whereas the relation (6.6) will consequently be called the *inverse* discrete Zak transform. It is important again to note that if $\varphi[n] \neq 0$ in an interval of length MN and vanishes outside that interval, the signal $\varphi[n]$ is just one period of the periodic sequence $\Phi[n]$ and can thus be reconstructed from $\Phi[n]$ and, hence, from $\tilde{\varphi}[n, l; N, M]$.

We are now prepared to sample the product form (5.11) of the Gabor transform for discrete-time signals, leading to

$$\bar{a}[n, l; N, M] = N\tilde{\varphi}[n, l; N, M]\tilde{w}^*[n, l; N, M]. \qquad (6.9)$$

Upon substituting the latter expression into the inverse discrete Fourier transform (6.2), we get

$$A_{mk} = \frac{1}{MN} \sum_{n=<N>} \sum_{l=<M>} N\tilde{\varphi}[n, l; N, M]\tilde{w}^*[n, l; N, M]$$
$$\times e^{j[m(2\pi/M)l - k(2\pi/N)n]},$$

in which relation we substitute from the discrete Zak transform (6.8) for $\Phi[n]$

$$A_{mk} = \frac{1}{M} \sum_{n=<N>} \sum_{l=<M>} \left[\sum_{r=<M>} \Phi[n + rN] e^{-jr(2\pi/M)l} \right]$$
$$\times \tilde{w}^*[n, l; N, M] e^{j[m(2\pi/M)l - k(2\pi/N)n]}.$$

We rearrange factors

$$A_{mk} = \sum_{n=<N>} \sum_{r=<M>} \Phi[n + rN] \left[\frac{1}{M} \sum_{l=<M>} \tilde{w}[n, l; N, M] e^{j(r-m)(2\pi/M)l} \right]^*$$
$$\times e^{-jk(2\pi/N)n}$$

and recognize [cf. Eq. (6.6)] the inverse discrete Zak transform of $W[n]$

$$A_{mk} = \sum_{n=<N>} \sum_{r=<M>} \Phi[n + rN] W^*[n + rN - mN] e^{-jk(2\pi/N)n}.$$

If we introduce, on the analogy of Eq. (5.4), the short-hand notation

$$W_{mk}[n] = W[n - mN] e^{jk\Theta n} = W[n - mN] e^{jk(2\pi/N)n}, \tag{6.10}$$

the coefficients A_{mk} take the form

$$A_{mk} = \sum_{n=<N>} \sum_{r=<M>} \Phi[n + rN] W^*_{mk}[n + rN],$$

which finally leads to

$$A_{mk} = \sum_{n=<MN>} \Phi[n] W^*_{mk}[n]. \tag{6.11}$$

The latter relationship will be called the *discrete Gabor transform.* As we remarked before, the discrete Gabor transform A_{mk} shows – apart from the usual periodicity with period N in the k-direction, due to the discrete nature of the signal – a *periodicity with period M in the m-direction.*

The *inverse* of the discrete Gabor transform results from sampling the product form (5.12) of the inverse of the Gabor transform for discrete-time signals, leading to

$$\tilde{\varphi}[n, l; N, M] = \bar{a}[n, l; N, M]\tilde{g}[n, l; N, M]. \tag{6.12}$$

Upon substituting the latter expression into the inverse discrete Zak transform (6.6), we get

$$\Phi[n] = \frac{1}{M} \sum_{l=<M>} \bar{a}[n, l; N, M]\tilde{g}[n, l; N, M],$$

in which relation we substitute from the discrete Fourier transformation (6.4)

$$\Phi[n] = \frac{1}{M} \sum_{l=<M>} \left[\sum_{m=<M>} \sum_{k=<N>} A_{mk} e^{-j[m(2\pi/M)l - k(2\pi/N)n]} \right] \times \tilde{g}[n, l; N, M].$$

We rearrange factors

$$\Phi[n] = \sum_{m=<M>} \sum_{k=<N>} A_{mk} \left[\frac{1}{M} \sum_{l=<M>} \tilde{g}[n, l; N, M] e^{-jm(2\pi/M)l} \right] e^{jk(2\pi/N)n}$$

and recognize [cf. Eq. (6.6)] the inverse discrete Zak transform of $G[n]$

$$\Phi[n] = \sum_{m=<M>} \sum_{k=<N>} A_{mk} G[n - mN] e^{jk(2\pi/N)n}.$$

With the short-hand notation [cf. Eq. (6.10)]

$$G_{mk}[n] = G[n - mN] e^{jk(2\pi/N)n},$$

the inverse discrete Gabor transform thus reads

$$\Phi[n] = \sum_{m=<M>} \sum_{k=<N>} A_{mk} G_{mk}[n]. \tag{6.13}$$

What is the importance of having a *discrete* Gabor transform (6.11), whereas we are in fact only interested in the coefficients of Gabor's signal expansion and thus in the *normal* Gabor transform (5.3)? Assume that the signal $\varphi[n] \neq 0$ in an interval of length N_φ and vanishes outside that interval, and that the window function $w[n] \neq 0$ in an interval of length N_w and vanishes outside that interval; then the coefficients of the (normal) Gabor expansion (5.3)

$$a_{mk} = \sum_n \varphi[n] w^*_{mk}[n]$$

can only be $\neq 0$ in an m-interval of length M, say, where M is the smallest integer for which the relation $MN \geq N_\varphi + N_w - 1$ holds. Now take M such that $MN \geq N_\varphi + N_w - 1$ and construct the periodic signal sequence $\Phi[n] = \sum_r \varphi[n + rMN]$ and the periodic window sequence $W[n] = \sum_r w[n + rMN]$ according to Eq. (6.7). In that case the array a_{mk} of the (normal) Gabor transform (5.3) can be identified with one period of the array A_{mk} of the discrete Gabor transform (6.11)

$$A_{mk} = \sum_{n=<MN>} \Phi[n] W^*_{mk}[n].$$

The array A_{mk} (and thus a_{mk}) can be computed via the discrete Zak transform and the inverse discrete Fourier transform, and in computing these transforms we can use a fast computer algorithm known as the *fast Fourier transform* (FFT). Calculating the discrete Gabor transform A_{mk} in such a way could be called the *fast discrete Gabor transform* and is equivalent to the *fast convolution* well known in digital signal processing. In detail we proceed as follows:

- from the signal $\varphi[n]$ and the window function $w[n]$ we determine – with the use of a fast computer algorithm – their discrete Zak transforms $\tilde{\varphi}[n, l;\, N,\, M]$ and $\tilde{w}[n, l;\, N,\, M]$, respectively, via Eq. (6.5);

- the discrete Fourier transform $\bar{a}[n, l;\, N,\, M]$ follows from the product form (6.9) of the discrete Gabor transform;

- from the array $\bar{a}[n, l;\, N,\, M]$ we determine – with the use of a fast computer algorithm again – the inverse discrete Fourier transform A_{mk}, according to Eq. (6.2);

- the array of Gabor expansion coefficients a_{mk} then follows as one period of the periodic array A_{mk}.

In general the signal $\varphi[n]$ does not vanish outside a certain interval or, if it does, the interval can be too large. In that case we can apply *overlap-add* techniques by splitting the signal $\varphi[n]$ in parts and treating all parts separately. In detail we proceed as follows. We represent the signal $\varphi[n]$ as a sequence of partial signals $\varphi^{(r)}[n]$, where each partial signal vanishes outside an interval K; hence,

$$\varphi[n] = \sum_r \varphi^{(r)}[n] \tag{6.14}$$

with

$$\varphi^{(r)}[n] = \begin{cases} \varphi[n] & \text{for } r\,N_\varphi \le n \le (r+1)\,N_\varphi - 1 \\ 0 & \text{elsewhere.} \end{cases} \tag{6.15}$$

Upon substituting the expansion (6.14) into the Gabor transform (5.3) we get

$$a_{mk} = \sum_n \left[\sum_r \varphi^{(r)}[n] \right] w^*_{mk}[n] = \sum_r \left[\sum_n \varphi^{(r)}[n] w^*_{mk}[n] \right] = \sum_r a^{(r)}_{mk}, \tag{6.16}$$

where each partial Gabor transform

$$a^{(r)}_{mk} = \sum_n \varphi^{(r)}[n] w^*_{mk}[n]$$

can be evaluated along the lines described in the previous paragraph. The last summation in Eq. (6.16) must take into account, of course, the overlap between the partial Gabor transforms.

7 Oversampling by a rational factor

In his original paper, Gabor restricted himself to a *critical* sampling of the time-frequency domain, where the expansion coefficients can be interpreted as independent data, i.e., degrees of freedom of a signal. It is the aim of this section to extend Gabor's concepts to the case of *oversampling*, in which case the expansion coefficients are no longer independent.

Let us consider an elementary signal $g(t)$ again, but let us now construct a discrete set of shifted and modulated versions defined as [cf. Eq. (2.2)]

$$g(t - m\alpha T)e^{jk\beta\Omega t}, \tag{7.1}$$

where the time shift αT and the frequency shift $\beta\Omega$ satisfy the relationships $\Omega T = 2\pi$ and $\alpha\beta \leq 1$, and where m and k may take all integer values. Gabor's signal expansion would then read [cf. Eq. (2.3)]

$$\varphi(t) = \sum_m \sum_k a_{mk} g(t - m\alpha T)e^{jk\beta\Omega t}. \tag{7.2}$$

Gabor's original signal expansion was restricted to the special case $\alpha\beta = 1$, and more particular $\alpha = \beta = 1$, in which case the expansion coefficients a_{mk} can be identified as degrees of freedom of the signal. For $\alpha\beta > 1$, the set of shifted and modulated versions of the elementary signal is not complete and thus cannot represent any arbitrary signal, while for $\alpha\beta < 1$, the set is overcomplete which implies that Gabor's expansion coefficients become dependent and can no longer be identified as degrees of freedom. In the special case $\alpha\beta = 1$, it has been shown in Section 3 how a window function $w(t)$ can be found such that the expansion coefficients can be determined via the so-called Gabor transform, which now takes the form [cf. Eq. (2.6)]

$$a_{mk} = \int \varphi(t) w^*(t - m\alpha T)e^{-jk\beta\Omega t} dt. \tag{7.3}$$

It is the aim of this section to show how a window function can be found when the parameters α and β satisfy the relation $\alpha\beta = q/p \leq 1$, where p and q are positive integers, $p \geq q \geq 1$.

In the case of *oversampling* (i.e., $\alpha\beta < 1$), the relationship between the window function $w(t)$ and the elementary signal $g(t)$ follows from substituting from the Gabor transform (7.3) into Gabor's signal expansion (7.2). After some manipulation (see Appendix B) we get the condition

$$\frac{T}{\beta} \sum_m w^* \left(t + k\frac{T}{\beta} - m\alpha T \right) g(t - m\alpha T) = \delta_k. \tag{7.4}$$

In principle, a window function can be derived from a given elementary signal with the help of Eq. (7.4); the way in which we should proceed, however, is not clear.

That a window function can be found in the case of *oversampling* is not surprising. To see this let us consider the *continuous* analogues of Gabor's signal

expansion and the Gabor transform. The Gabor transform (7.3) can be considered as a sampled version of the sliding-window spectrum $s(t, \omega)$ [7, 10] of the signal $\varphi(t)$, defined as

$$s(t, \omega) = \int \varphi(\tau) w^*(\tau - t) e^{-j\omega\tau} d\tau, \tag{7.5}$$

where the sampling appears on the time-frequency lattice ($t = m\alpha T, \omega = k\beta\Omega$). Gabor's expansion coefficients follow from the sliding-window spectrum through the relation $a_{mk} = s(m\alpha T, k\beta\Omega)$. It is well known that the signal $\varphi(t)$ can be reconstructed from its sliding-window spectrum $s(t, \omega)$ in many different ways, one of them reading

$$\varphi(\tau) \int |w(t)|^2 dt = \frac{1}{2\pi} \iint s(t, \omega) w(\tau - t) e^{j\omega\tau} dt d\omega. \tag{7.6}$$

It is not difficult to see that the latter signal representation is a continuous analogue of Gabor's signal expansion (7.2), and that it can be derived from this expansion by letting the time step αT and the frequency step $\beta\Omega$ tend to zero. In fact, the signal representation (7.6) is identical to Gabor's signal expansion (7.2) with an *infinitely dense* sampling lattice. We conclude that in that limiting case, the window function function $w(t)$ may be chosen proportional to the elementary signal $g(t)$. Later on in this section we will derive a detailed transition from Gabor's signal expansion to its continuous analogue, by letting $\alpha\beta \downarrow 0$.

Using the Fourier transform [cf. Eq. (2.11)] and the Zak transform [cf. Eq. (3.1)], it can be shown (see Appendix C) that the Gabor transform (7.3) can be transformed into the *sum-of-products form*

$$\bar{a}\left(x, y + \frac{r}{p}\right) = \frac{\alpha p T}{q}$$

$$\times \sum_{s=<q>} \tilde{\varphi}\left((x+s)\frac{\alpha p T}{q}, y\frac{\Omega}{\alpha}; \alpha p T\right) \tilde{w}^*\left((x+s)\frac{\alpha p T}{q}, \left[y + \frac{r}{p}\right]\frac{\Omega}{\alpha}; \alpha T\right), \tag{7.7}$$

where the expression $s = < q >$ is used throughout as a short-hand notation for an interval of q successive integers ($s = 0, 1, 2, ..., q-1$, for instance); due to the periodicity of the Fourier transform and the Zak transform, any sequence of q successive integers can be chosen. Note that in Gabor's original case of critical sampling ($p = q = 1$), Eq. (7.7) takes the simple *product form* [cf. Eq. (4.1)]

$$\bar{a}(x, y) = \alpha T \tilde{\varphi}\left(x\alpha T, y\frac{\Omega}{\alpha}; \alpha T\right) \tilde{w}^*\left(x\alpha T, y\frac{\Omega}{\alpha}; \alpha T\right).$$

By using the Fourier transform and the Zak transform, it can also be shown (see Appendix D) that Gabor's signal expansion (7.2) can be transformed into the *sum-of-products form*

$$\tilde{\varphi}\left((x+s)\frac{\alpha p T}{q}, y\frac{\Omega}{\alpha}; \alpha p T\right) =$$

$$= \frac{1}{p} \sum_{r=<p>} \bar{a}\left(x, y + \frac{r}{p}\right) \tilde{g}\left((x+s)\frac{\alpha p T}{q}, \left[y + \frac{r}{p}\right]\frac{\Omega}{\alpha}; \alpha T\right), \tag{7.8}$$

where the expression $r = < p >$ is used throughout as a short-hand notation for an interval of p successive integers ($r = 0, 1, 2, ..., p-1$, for instance); due to the periodicity of the Fourier transform and the Zak transform, any sequence of p successive integers can be chosen. Note that in Gabor's original case of critical sampling ($p = q = 1$), Eq. (7.8) takes the simple *product form* [cf. Eq. (4.2)]

$$\tilde{\varphi}\left(x\alpha T, y\frac{\Omega}{\alpha}; \alpha T\right) = \bar{a}(x, y)\tilde{g}\left(x\alpha T, y\frac{\Omega}{\alpha}; \alpha T\right).$$

We remark that, with $r = < p >$, the Fourier transform $\bar{a}(x, y)$ is completely determined by the p functions

$$a_r(x, y) = \bar{a}\left(x, y + \frac{r}{p}\right), \tag{7.9}$$

where the variable y extends over an interval of length $1/p$. Likewise, with $s = < q >$, the Zak transform $\tilde{\varphi}(x\alpha p T/q, y\Omega/\alpha; \alpha p T)$ is completely determined by the q functions

$$\varphi_s(x, y) = \tilde{\varphi}\left((x+s)\frac{\alpha p T}{q}, y\frac{\Omega}{\alpha}; \alpha p T\right), \tag{7.10}$$

where the variable x extends over an interval of length 1. Moreover, with $r = < p >$ and $s = < q >$, the Zak transforms $\tilde{g}(x\alpha p T/q, y\Omega/\alpha; \alpha T)$ and $\tilde{w}(x\alpha p T/q, y\Omega/\alpha; \alpha T)$ are completely determined by the $q \times p$ functions

$$g_{sr}(x, y) = \tilde{g}\left((x+s)\frac{\alpha p T}{q}, \left[y + \frac{r}{p}\right]\frac{\Omega}{\alpha}; \alpha T\right) \tag{7.11}$$

and

$$w_{sr}(x, y) = \tilde{w}\left((x+s)\frac{\alpha p T}{q}, \left[y + \frac{r}{p}\right]\frac{\Omega}{\alpha}; \alpha T\right), \tag{7.12}$$

respectively, where x extends over an interval of length 1 and y over an interval of length $1/p$.

Let us now, for convenience, choose $r = 0, 1, ..., p-1$; the p functions $a_r(x, y)$ can then be combined into a p-dimensional column vector of functions

$$\mathbf{a} = [a_0(x, y)\ a_1(x, y)\ ...\ a_{p-1}(x, y)]^T. \tag{7.13}$$

Likewise, with $s = 0, 1, ..., q-1$, the q functions $\varphi_s(x, y)$ can be combined into a q-dimensional column vector of functions

$$\phi = [\varphi_0(x, y)\ \varphi_1(x, y)\ ...\ \varphi_{q-1}(x, y)]^T. \tag{7.14}$$

Moreover, the $q \times p$ functions $g_{sr}(x, y)$ and $w_{sr}(x, y)$ can be combined into the $(q \times p)$-dimensional matrices of functions

$$\mathbf{G} = \begin{bmatrix} g_{00}(x,y) & g_{01}(x,y) & \dots & g_{0,p-1}(x,y) \\ g_{10}(x,y) & g_{11}(x,y) & \dots & g_{1,p-1}(x,y) \\ . & . & \dots & . \\ . & . & \dots & . \\ g_{q-1,0}(x,y) & g_{q-1,1}(x,y) & \dots & g_{q-1,p-1}(x,y) \end{bmatrix} \tag{7.15}$$

and

$$\mathbf{W} = \begin{bmatrix} w_{00}(x,y) & w_{01}(x,y) & \dots & w_{0,p-1}(x,y) \\ w_{10}(x,y) & w_{11}(x,y) & \dots & w_{1,p-1}(x,y) \\ . & . & \dots & . \\ . & . & \dots & . \\ w_{q-1,0}(x,y) & w_{q-1,1}(x,y) & \dots & w_{q-1,p-1}(x,y) \end{bmatrix}, \tag{7.16}$$

respectively. With the help of these vectors and matrices, Eqs. (7.7) and (7.8) can now be expressed in the elegant matrix-vector products

$$\mathbf{a} = \frac{\alpha p T}{q} \mathbf{W}^* \boldsymbol{\phi} \tag{7.17}$$

and

$$\boldsymbol{\phi} = \frac{1}{p} \mathbf{G} \mathbf{a}, \tag{7.18}$$

where, as usual, the asterisk in connection with vectors and matrices denotes complex conjugation *and* transposition. Note that Eq. (7.17) represents p equations in q unknowns, whereas Eq. (7.18) represents q equations in p unknowns. In the case of oversampling ($p > q \geq 1$) the latter set of equations is thus *underdetermined*.

We will now prove that Gabor's signal expansion (7.2) and the Gabor transform (7.3) form a transform pair, by showing that for any elementary signal $g(t)$ a window function $w(t)$ can be constructed. Instead of doing this by combining Gabor's signal expansion and the Gabor transform, which lead to Eq. (7.4), we will use the results (7.17) and (7.18) derived above.

If we substitute from Eq. (7.17) into Eq. (7.18) we get

$$\boldsymbol{\phi} = \frac{\alpha T}{q} \mathbf{G} \mathbf{W}^* \boldsymbol{\phi},$$

which relation should hold for any arbitrary vector $\boldsymbol{\phi}$ [i.e., for any arbitrary signal $\varphi(t)$]. This condition immediately leads to the relationship

$$\frac{\alpha T}{q} \mathbf{G} \mathbf{W}^* = \mathbf{I}_q, \tag{7.19}$$

where $\mathbf{I}_q$ is the $(q \times q)$-dimensional identity matrix.

We remark that in Gabor's original case of critical sampling ($p = q = 1$), the matrices $\mathbf{G}$ and $\mathbf{W}$ reduce to scalars g_{00} and w_{00}, respectively, and that Eq. (7.19) takes the simple product form [cf. Eq. (3.8)]

$$\alpha T \; g_{00} w_{00}^{*} = \alpha T \tilde{g}\left(x\alpha T, y\frac{\Omega}{\alpha}; \alpha T\right) \tilde{w}^{*}\left(x\alpha T, y\frac{\Omega}{\alpha}; \alpha T\right) = 1.$$

In this case of critical sampling, the Zak transform of the window function could thus be easily found, in principle, as the inverse of the Zak transform of the elementary signal, and the resulting window function would be *unique*. The occurrence of zeros in the Zak transform of the elementary signal, however, prohibits such an easy procedure. The problems caused by these zeros can be overcome by oversampling.

In the case of oversampling, the window function that corresponds to a given elementary signal is *not unique*. This is in accordance with the fact that in the case of oversampling the set of shifted and modulated versions of the elementary signal is overcomplete, and that Gabor's expansion coefficients are dependent and can no longer be considered as degrees of freedom, as we have mentioned before. In the case of oversampling, the general condition (7.19) enables us to construct a window function $w(t)$ for a given elementary signal $g(t)$, by solving a set of $q \times q$ equations in $q \times p$ unknowns. Since $q < p$, this set of equations is again underdetermined.

Let us now consider Eqs. (7.18) and (7.19) in the general case of oversampling. In that case we have $q < p$, which implies that $\mathbf{G}$ is not a square matrix and does not have a normal inverse $\mathbf{G}^{-1}$, and that Eqs. (7.18) and (7.19) do not have unique solutions. It is well known that, under the condition that rank($\mathbf{G}$) = q, the *optimum solutions* in the sense of the *minimum L_2 norm* can now be found with the help of the so-called *generalized (Moore-Penrose) inverse* [12] $\mathbf{G}^{\dagger}$, defined by

$$\mathbf{G}^{\dagger} = \mathbf{G}^{*}(\mathbf{G}\mathbf{G}^{*})^{-1}; \tag{7.20}$$

note that $\mathbf{G}\mathbf{G}^{\dagger} = \mathbf{I}_q$ and that $\mathbf{G}^{\dagger}\mathbf{G}\mathbf{G}^{*} = \mathbf{G}^{*}$. The optimum solution $\mathbf{W}_{opt}$ then reads

$$\mathbf{W}_{opt} = \frac{q}{\alpha T}(\mathbf{G}^{\dagger})^{*} \tag{7.21}$$

and the optimum solution $\mathbf{a}_{opt}$ reads

$$\mathbf{a}_{opt} = p \; \mathbf{G}^{\dagger}\phi = \frac{\alpha p T}{q}\mathbf{W}_{opt}^{*}\phi. \tag{7.22}$$

Of course, if we proceed in this way, we will find, for any x and y, the minimum L_2 norm solutions for the matrix $\mathbf{W}$ and the vector $\mathbf{a}$. It is not difficult to show, however, that the minimum L_2 norms of $\mathbf{W}$ and $\mathbf{a}$ correspond to the minimum L_2 norms of the Zak transform $\tilde{w}(x\alpha pT/q, y\Omega/\alpha; \alpha T)$ and the Fourier transform $\bar{a}(x, y)$, respectively, and thus, with the help of Parseval's energy theorem, to the minimum L_2 norms of the window function $w(t)$ and the array of Gabor coefficients a_{mk}, respectively.

Instead of looking for the optimum solution $\mathbf{W}_{opt}$ in the sense of the minimum L_2 norm of $\mathbf{W}$, we could as well look for the optimum solution $\mathbf{W}_F$ in the sense

of the minimum L_2 norm of the difference $\mathbf{W} - \mathbf{F}$; in this way we would find the matrix $\mathbf{W}$ that resembles best the matrix $\mathbf{F}$. To find $\mathbf{W}_F$ we proceed as follows

$$\frac{\alpha T}{q}\mathbf{G}\mathbf{W}^* = \mathbf{I}_q,$$

$$\frac{\alpha T}{q}\mathbf{G}(\mathbf{W} - \mathbf{F})^* = \mathbf{I}_q - \frac{\alpha T}{q}\mathbf{G}\mathbf{F}^*,$$

$$\frac{\alpha T}{q}(\mathbf{W}_F - \mathbf{F})^* = \mathbf{G}^\dagger\left(\mathbf{I}_q - \frac{\alpha T}{q}\mathbf{G}\mathbf{F}^*\right),$$

$$\frac{\alpha T}{q}\mathbf{W}_F^* = \mathbf{G}^\dagger + \frac{\alpha T}{q}(\mathbf{I}_p - \mathbf{G}^\dagger\mathbf{G})\mathbf{F}^*$$

and hence

$$\mathbf{W}_F^* = \mathbf{W}_{opt}^* + (\mathbf{I}_p - \mathbf{G}^\dagger\mathbf{G})\mathbf{F}^*, \tag{7.23}$$

where $\mathbf{I}_p$ is the $(p \times p)$-dimensional identity matrix.

An obvious choice for the matrix $\mathbf{F}$ would be a matrix that is proportional to the matrix $\mathbf{G}$. From Eq. (7.23) we then have

$$\mathbf{W}_G^* = \mathbf{W}_{opt}^* + (\mathbf{I}_p - \mathbf{G}^\dagger\mathbf{G})\mathbf{G}^*,$$

but the second term in the right-hand side of this relationship vanishes, due to the fact that

$$(\mathbf{I}_p - \mathbf{G}^\dagger\mathbf{G})\mathbf{G}^* = \mathbf{G}^* - \mathbf{G}^\dagger\mathbf{G}\mathbf{G}^* = \mathbf{G}^* - \mathbf{G}^* = \mathbf{0}.$$

We thus reach the important conclusion that $\mathbf{W}_G = \mathbf{W}_{opt}$; hence, the window function $w_{opt}(t)$ that has the minimum L_2 norm is the same as the window function $w_g(t)$ whose difference from the elementary signal $g(t)$ has the minimum L_2 norm, and resembles best this elementary signal.

In Fig. 4 we have depicted the Zak transforms of some window functions that correspond to the Gaussian elementary signal (2.1) for different values of α and β, resulting in different values of oversampling p/q, while in Fig. 5 we have depicted these window functions themselves. We remark that the resemblance between the window function and the elementary signal increases with decreasing $\alpha\beta$.

Let us finally consider the special case of integer oversampling ($p > q = 1$). In that case the matrices $\mathbf{G}$ and $\mathbf{W}$ reduce to row vectors $\mathbf{g}^T$ and $\mathbf{w}^T$, respectively, and the L_2 norm of $\mathbf{G}$ equals $\mathbf{g}^*\mathbf{g}$. Moreover, $\mathbf{G}^\dagger = (\mathbf{g}^*\mathbf{g})^{-1}\mathbf{g}^* = \alpha T\mathbf{w}_{opt}^*$. Let us consider the L_2 norm $\mathbf{g}^*\mathbf{g}$ in the limiting case that $\alpha T \downarrow 0$ and $p \to \infty$:

$$\mathbf{g}^*\mathbf{g} = \sum_{r=0}^{p-1} |g_{0r}(x, y)|^2 = \sum_{r=0}^{p-1} \left|\tilde{g}\left(x\alpha pT, \left[y + \frac{r}{p}\right]\frac{\Omega}{\alpha}; \alpha T\right)\right|^2.$$

Since $\tilde{g}(t, \omega; \alpha T)$ is almost independent of t for small values of αT, we might as well write

$$\mathbf{g}^*\mathbf{g} \simeq \sum_{r=0}^{p-1} \left|\tilde{g}\left(t, \left[y + \frac{r}{p}\right]\frac{\Omega}{\alpha}; \alpha T\right)\right|^2,$$

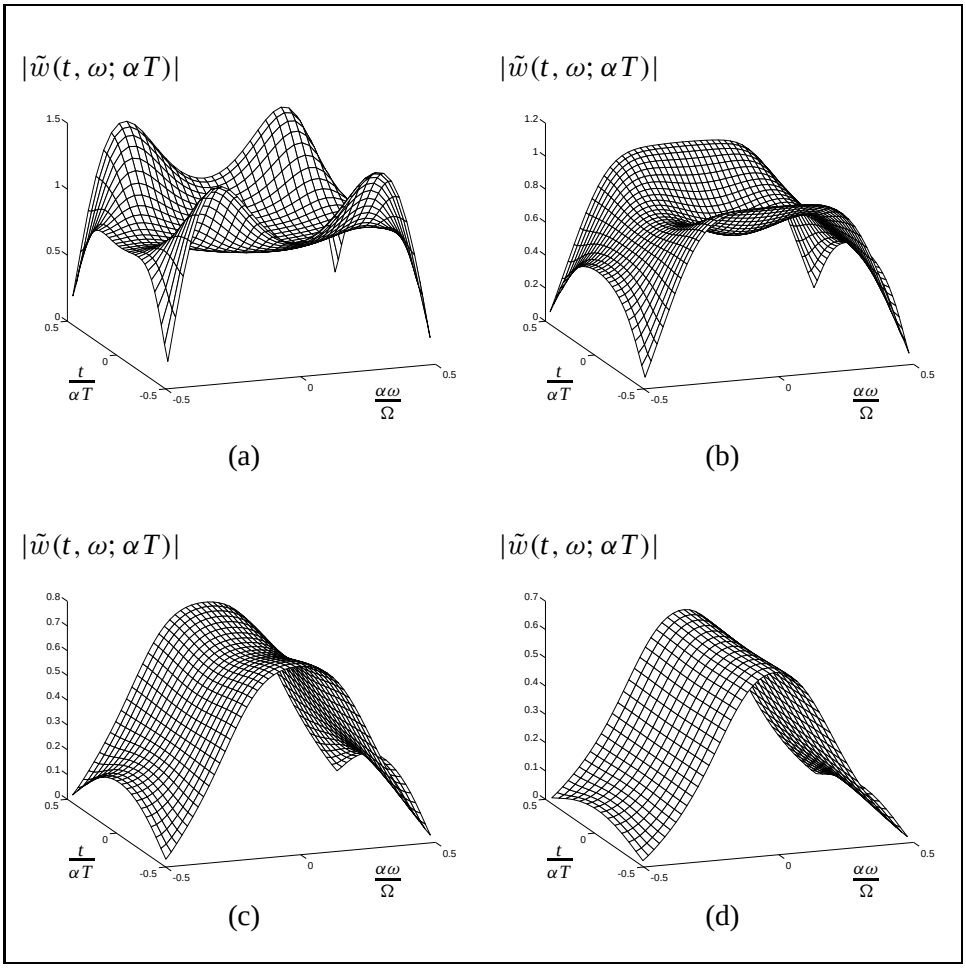

Figure 4: The Zak transform $\tilde{w}(t,\omega;\alpha T)$ in the case of a Gaussian elementary signal $g(t) = 2^{\frac{1}{4}} e^{-\pi(t/T)^2}$ for different values of oversampling:
(a) $\alpha = \beta = \sqrt{6/7}$, $p/q = 7/6$,
(b) $\alpha = \beta = \sqrt{2/3}$, $p/q = 3/2$,
(c) $\alpha = \beta = \sqrt{2/5}$, $p/q = 5/2$,
(d) $\alpha = \beta = \sqrt{2/7}$, $p/q = 7/2$.

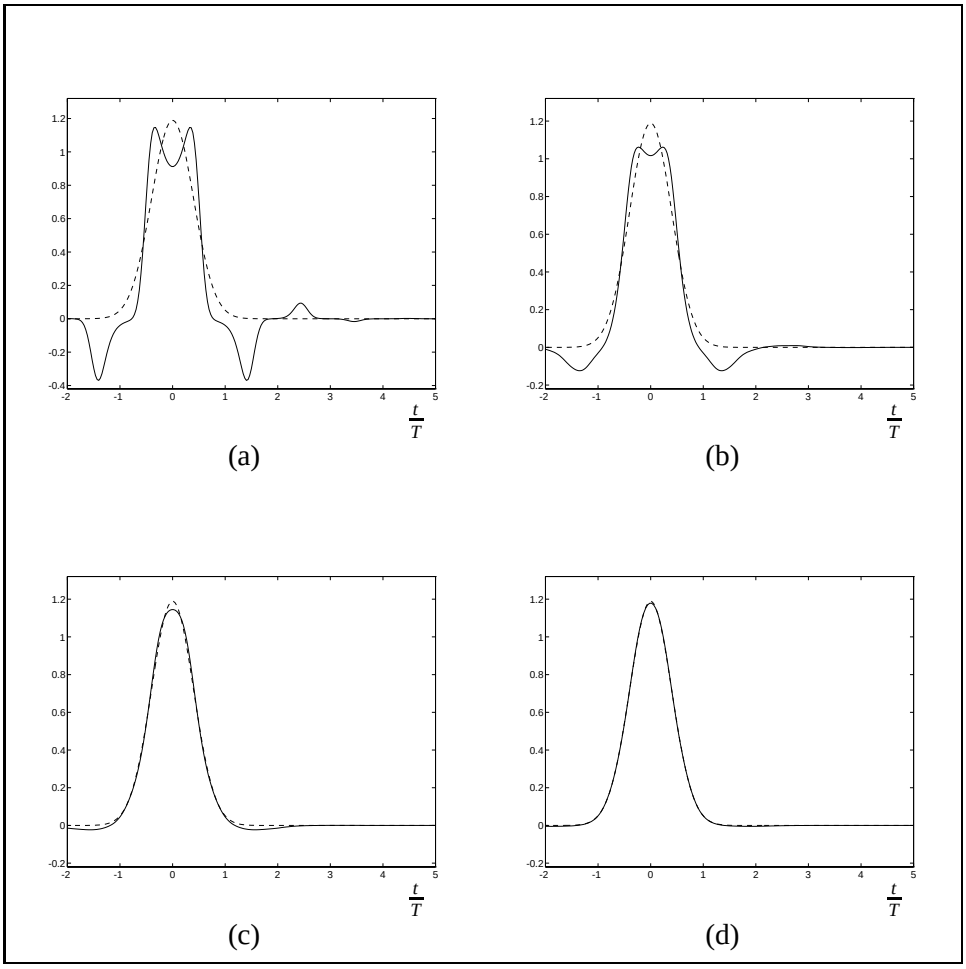

Figure 5: A Gaussian elementary signal $g(t) = 2^{\frac{1}{4}} e^{-\pi (t/T)^2}$ (dashed line) and its corresponding window function $(\alpha T/q) w(t)$ (solid line) for different values of oversampling:
(a) $\alpha = \beta = \sqrt{6/7}$, $p/q = 7/6$,
(b) $\alpha = \beta = \sqrt{2/3}$, $p/q = 3/2$,
(c) $\alpha = \beta = \sqrt{2/5}$, $p/q = 5/2$,
(d) $\alpha = \beta = \sqrt{2/7}$, $p/q = 7/2$.

and for large p this might as well be written as

$$\mathbf{g}^*\mathbf{g} \simeq \sum_{r=0}^{p-1} \left| \tilde{g}\left(t, \frac{r}{p}\frac{\Omega}{\alpha}; \alpha T\right)\right|^2 .$$

In the limit of $\alpha T \downarrow 0$ and $p \to \infty$, we might as well write the latter expression in the form

$$\mathbf{g}^*\mathbf{g} \simeq \frac{p}{2\pi}\int_{\alpha T}\int_{\Omega/\alpha} |\tilde{g}(t,\omega;\alpha T)|^2 dt d\omega,$$

which, with the help of Parseval's energy theorem (3.4), can be expressed in the form

$$\mathbf{g}^*\mathbf{g} \simeq \frac{p}{\alpha T}\int |g(t)|^2 dt.$$

Since

$$\mathbf{g} = \alpha T(\mathbf{g}^*\mathbf{g})\mathbf{w}_{opt},$$

we finally conclude that

$$g(t) \simeq \left[p\int |g(t)|^2 dt\right] w_{opt}(t),$$

or

$$g(t) \simeq \frac{w_{opt}(t)}{p\int |w_{opt}(t)|^2 dt}. \tag{7.24}$$

We can link these results to the continuous analogue (7.6) of Gabor's signal expansion. Approximating the double integration in this continuous analogue by a double summation we get

$$\varphi(\tau) \simeq \frac{\alpha\beta}{\int |w(t)|^2 dt}\sum_m\sum_k s(m\alpha T, k\beta\Omega) w(\tau - m\alpha T) e^{jk\beta\Omega\tau},$$

which expression has indeed the form of a Gabor expansion [cf. Eq. (7.2)] with expansion coefficients $a_{mk} = s(m\alpha T, k\beta\Omega)$ [cf. Eq. (7.5)] and with an elementary signal $g(t)$ that is proportional to the window function [cf. Eq. (7.24)].

8 Discrete Gabor and Zak transforms with oversampling

In this section we apply the ideas of oversampling to the discrete Gabor expansion and the discrete Gabor transform. In the general case of oversampling, the (discrete) Gabor expansion for a discrete-time signal $\varphi[n]$ takes the form

$$\varphi[n] = \sum_m \sum_{k=<K>} a_{mk} g[n - mN] e^{j2\pi kn/K} \tag{8.1}$$

and the (discrete) Gabor transform reads

$$a_{mk} = \sum_n \varphi[n] w^*[n - mN] e^{-j2\pi kn/K}. \tag{8.2}$$

These formulas are similar to previous expressions for the Gabor expansion [cf. Eqs. (5.5) and (6.13)] and the Gabor transform [cf. Eqs. (5.3) and (6.11)], except that we have now chosen $\Theta = 2\pi/K$ instead of $\Theta = 2\pi/N$, with $K \geq N$. The case $K = N$ corresponds to critical sampling, while for $K > N$ we are dealing with oversampling. For convenience, we introduce two integers p and q ($p \geq q \geq 1$) that do not have common factors and for which the relationship $pN = qK$ holds; note that $K/N = p/q \geq 1$ represents the degree of oversampling.

It is the aim of this section to derive a relationship between the window function $w[n]$ and the elementary signal $g[n]$ and to show how the array of Gabor coefficients a_{mk} can be determined in the case of oversampling where, moreover, the window function $w[n]$ has a finite support N_w. For convenience, we consider signals $\varphi[n]$ that have a finite support N_φ, too; in the case that we are dealing with signals of longer (or even infinite) support, we can always split the signal in parts that do have a finite support N_φ and treat all these parts separately, as we have mentioned already in Section 6. Under these conditions of finite support, the array a_{mk}, which is periodic in the k-variable, has a finite support M in the m-variable, where the support M satisfies the condition

$$MN \geq N_\varphi + N_w - 1. \tag{8.3}$$

We introduce, as before, the periodized version A_{mk} of the array a_{mk} according to [see Eq. (6.3)]

$$A_{mk} = \sum_r a_{m+rM,k}; \tag{8.4}$$

note again that the periodized array A_{mk} is periodic in m (and k) with period M (and K), and that we can identify a_{mk} (which does not show a periodicity with respect to m) as one period of A_{mk}. Furthermore we introduce, as before, the periodized version $W[n]$ of the (finite support) window function $w[n]$ according to [cf. Eq. (6.7)]

$$W[n] = \sum_r w[n + rMN]; \tag{8.5}$$

note that the periodized window function $W[n]$ is periodic with period MN and that we can identify $w[n]$ as one period of $W[n]$. It is easy to see that the periodized

array A_{mk} can be expressed in terms of the periodized window function $W[n]$ via a kind of Gabor transform [cf. Eq. (8.2)]:

$$A_{mk} = \sum_n \varphi[n] W^*[n - mN] e^{-j2\pi kn/K}. \tag{8.6}$$

We could as well periodize the (finite support) signal $\varphi[n]$ according to [see Eq. (6.7)]

$$\Phi[n] = \sum_r \varphi[n + rMN]; \tag{8.7}$$

note that the periodized signal $\Phi[n]$ is periodic with period MN and that we can identify $\varphi[n]$ as one period of $\Phi[n]$. If we substitute from Gabor's signal expansion (8.1) into Eq. (8.7) we get

$$\Phi[n] = \sum_r \left[\sum_m \sum_{k=<K>} a_{mk} g[n + rMN - mN] e^{j2\pi k(n + rMN)/K} \right].$$

After arranging factors and requiring that K is a divisor of MN (and hence $e^{j2\pi kr(MN/K)} = 1$), we can write

$$\Phi[n] = \sum_m \sum_{k=<K>} a_{mk} \left[\sum_r g[n + rMN - mN] \right] e^{j2\pi kn/K}.$$

After introducing the periodized version $G[n]$ of the elementary signal $g[n]$ [cf. Eq. (8.5)], we can write

$$\Phi[n] = \sum_m \sum_{k=<K>} a_{mk} G[n - mN] e^{j2\pi kn/K}, \tag{8.8}$$

which relationship has the form of a Gabor expansion [cf. Eq. (8.1)].

It is not difficult to show that we also have the relationships [cf. Eq. (6.13)]

$$\Phi[n] = \sum_{m=<M>} \sum_{k=<K>} A_{mk} G[n - mN] e^{j2\pi kn/K} \tag{8.9}$$

and [cf. Eq. (6.11)]

$$A_{mk} = \sum_{n=<MN>} \Phi[n] W^*[n - mN] e^{-j2\pi kn/K}, \tag{8.10}$$

which are fully periodized versions of Gabor's signal expansion (8.1) and the Gabor transform (8.2), respectively. Equation (8.9) is known as the discrete Gabor expansion [just like Eq. (6.13)], while Eq. (8.10) is known as the discrete Gabor transform [just like Eq. (6.11)]; the difference with the previous definitions is only caused by the oversampling $K \geq N$.

In the general case of oversampling, the discrete Fourier transform $\bar{a}[n, l; K, M]$ of the periodic array A_{mk} is defined according to [cf. Eqs. (6.1) and (6.4)]

$$\bar{a}[n, l; K, M] = \bar{a}\left(\frac{n}{K}, \frac{l}{M}\right) = \sum_{m=<M>} \sum_{k=<K>} A_{mk} e^{-j2\pi(ml/M - kn/K)}; \tag{8.11}$$

note that the discrete Fourier transform $\bar{a}[n, l; K, M]$ is periodic in the variables n and l with periods K and M, respectively. The inverse discrete Fourier transform then reads [cf. Eq. (6.2)]

$$A_{mk} = \frac{1}{MK} \sum_{n=<K>} \sum_{l=<M>} \bar{a}[n, l; K, M] e^{j2\pi(ml/M - kn/K)}. \tag{8.12}$$

As before, the discrete Zak transform $\tilde{w}[n, l; N, M]$ of the periodized window function $W[n]$ is defined as a one-dimensional discrete Fourier transform of the sequence $W[n + mN]$ (with $m = < M >$ and n being a mere parameter), hence [cf. Eq. (6.8)]

$$\tilde{w}[n, l; N, M] = \sum_{m=<M>} W[n + mN] e^{-j2\pi ml/M}; \tag{8.13}$$

after substituting from Eq. (8.5), it can easily be shown that the discrete Zak transform can also be represented in the form [cf. Eq. (6.5)]

$$\tilde{w}[n, l; N, M] = \sum_{m} w[n + mN] e^{-j2\pi ml/M}. \tag{8.14}$$

We remark again that the discrete Zak transform $\tilde{w}[n, l; N, M]$ is periodic in the frequency variable l with period M and quasi-periodic in the time variable n with quasi-period N:

$$\tilde{w}[n + mN, l + kM; N, M] = \tilde{w}[n, l; N, M] e^{j2\pi ml/M}. \tag{8.15}$$

The inverse relationship of the discrete Zak transform takes the form [cf. Eq. (6.6)]

$$W[n + mN] = \frac{1}{M} \sum_{l=<M>} \tilde{w}[n, l; N, M] e^{j2\pi ml/M}. \tag{8.16}$$

We also introduce the discrete Zak transform of $\Phi[n]$

$$\tilde{\varphi}[n, l; pN, M/p] = \sum_{m=<M/p>} \Phi[n + mpN] e^{-j2\pi mpl/M}, \tag{8.17}$$

where the condition that p is a divisor of M should hold. If we substitute from Eq. (8.7), it is not difficult to see that we also have the relationship

$$\tilde{\varphi}[n, l; pN, M/p] = \sum_{m} \varphi[n + mpN] e^{-j2\pi mpl/M}. \tag{8.18}$$

From the condition that p is a divisor of M, we can write $M = pL$, where L is an integer. From $pN = qK$ and $M = pL$ we conclude that $MN = qKL$, which implies that K is a divisor of MN. The latter condition is exactly the condition that should hold to be able to derive Eq. (8.8). Moreover, from $pN = qK$ and knowing that p and q do not have common factors, we also conclude that p is a divisor of K; hence we can write $K = pJ$, where J is an integer. We thus conclude that K,

M, and N can be expressed in terms of the integers p, q (with $p \geq q \geq 1$, and p and q not having common factors), J and L: $K = pJ$, $M = pL$, and $N = qJ$.

Using the discrete Fourier transform and the discrete Zak transform, it can be shown (see Appendix E) that the discrete Gabor transform can be transformed into the sum-of-products form

$$\bar{a}[n, l + rM/p;\, K, M] =$$

$$= K \sum_{s=<q>} \tilde{\varphi}[n + sK, l;\, pN, M/p]\tilde{w}^*[n + sK, l + rM/p;\, N, M], \qquad (8.19)$$

which is the discrete counterpart of Eq. (7.7). Note that in Gabor's original case of critical sampling ($p = q = 1$, $K = N$), Eq. (8.19) takes the simple product form [see Eq. (6.9)]

$$\bar{a}[n, l;\, N, M] = N\tilde{\varphi}[n, l;\, N, M]\tilde{w}^*[n, l;\, N, M].$$

We remark that Eq. (8.19) enables us to calculate Gabor's expansion coefficients a_{mk} in a way that is completely different from the original definition (8.2).

- We first determine the discrete Zak transform $\tilde{\varphi}[n + sK, l;\, pN, M/p]$ of the signal $\varphi[n]$ by means of its definition (8.18). Since the discrete Zak transform is essentially a discrete Fourier transform, this can be done with a fast algorithm, especially if we choose $M/p = L$ equal to a power of 2.

- We then multiply the discrete Zak transform $\tilde{\varphi}[n + sK, l;\, pN, M/p]$ by the discrete Zak transform $\tilde{w}^*[n + sK, l + rM/p;\, N, M]$ of the window function $w[n]$. Note that the determination of the discrete Zak transform of the window function needs to be done only once, and that it can also be done with a fast algorithm.

- We sum over $s = < q >$. Note that this summation is not necessary if we choose $q = 1$, i.e., in the case of *integer* oversampling $K/N = p$.

- We then determine the periodized array A_{mk} by means of the inverse discrete Fourier transformation (8.12). Note that this can be done with a fast algorithm again, especially if we choose $K = pJ$ and $M = pL$ equal to powers of 2.

- We finally recognize the array of Gabor coefficients a_{mk} as one period of the periodic array A_{mk}.

By using the discrete Fourier transform and the discrete Zak transform, it can also be shown (see Appendix F) that the discrete Gabor expansion can be transformed into the sum-of-products form

$$\tilde{\varphi}[n + sK, l;\, pN, M/p] =$$

$$= \frac{1}{p} \sum_{r=<p>} \bar{a}[n, l + rM/p;\, K, M]\tilde{g}[n + sK, l + rM/p;\, N, M], \qquad (8.20)$$

which is the discrete counterpart of Eq. (7.8). Note that in Gabor's original case of critical sampling ($p = q = 1$, $K = N$), Eq. (8.20) takes the simple product form [see Eq. (6.12)]

$$\tilde{\varphi}[n, l; N, M] = \bar{a}[n, l; N, M]\tilde{g}[n, l; N, M].$$

We remark that, with $r =< p >$, the discrete Fourier transform $\bar{a}[n, l; K, M]$ is completely determined by the p functions

$$a_r[n, l] = \bar{a}[n, l + rM/p; K, M], \tag{8.21}$$

where the variable l extends over an interval of length M/p. Likewise, with $s =< q >$, the discrete Zak transform $\tilde{\varphi}[n, l; pN, M/p]$ is completely determined by the q functions

$$\varphi_s[n, l] = \tilde{\varphi}[n + sK, l; pN, M/p], \tag{8.22}$$

where the variable n extends over an interval of length K. Moreover, with $r =< p >$ and $s =< q >$, the discrete Zak transforms $\tilde{g}[n, l; N, M]$ and $\tilde{w}[n, l; N, M]$ are completely determined by the $q \times p$ functions

$$g_{sr} = \tilde{g}[n + sK, l + rM/p; N, M] \tag{8.23}$$

and

$$w_{sr} = \tilde{w}[n + sK, l + rM/p; N, M], \tag{8.24}$$

respectively.

Let us now, for convenience, choose $r = 0, 1, ..., p-1$; the p functions $a_r[n, l]$ can then be combined into a p-dimensional column vector of functions

$$\mathbf{a} = (a_0[n, l]\ a_1[n, l]\ ...\ a_{p-1}[n, l])^T. \tag{8.25}$$

Likewise, with $s = 0, 1, ..., q-1$, the q functions $\varphi_s[n, l]$ can be combined into a q-dimensional column vector of functions

$$\phi = (\varphi_0[n, l]\ \varphi_1[n, l]\ ...\ \varphi_{q-1}[n, l])^T. \tag{8.26}$$

Moreover, the $q \times p$ functions $g_{sr}[n, l]$ and $w_{sr}[n, l]$ can be combined into the $(q \times p)$-dimensional matrices of functions

$$\mathbf{G} = \begin{bmatrix} g_{00}[n, l] & g_{01}[n, l] & \dots & g_{0,p-1}[n, l] \\ g_{10}[n, l] & g_{11}[n, l] & \dots & g_{1,p-1}[n, l] \\ . & . & \dots & . \\ . & . & \dots & . \\ g_{q-1,0}[n, l] & g_{q-1,1}[n, l] & \dots & g_{q-1,p-1}[n, l] \end{bmatrix} \tag{8.27}$$

and

$$\mathbf{W} = \begin{bmatrix} w_{00}[n, l] & w_{01}[n, l] & \dots & w_{0,p-1}[n, l] \\ w_{10}[n, l] & w_{11}[n, l] & \dots & w_{1,p-1}[n, l] \\ . & . & \dots & . \\ . & . & \dots & . \\ w_{q-1,0}[n, l] & w_{q-1,1}[n, l] & \dots & w_{q-1,p-1}[n, l] \end{bmatrix}, \tag{8.28}$$

respectively. With the help of these vectors and matrices, Eqs. (8.19) and (8.20) can be expressed in the elegant matrix-vector products [cf. Eqs. (7.17) and (7.18)]

$$\mathbf{a} = K\ \mathbf{W}^*\boldsymbol{\phi} \tag{8.29}$$

and

$$\boldsymbol{\phi} = \frac{1}{p}\mathbf{G}\mathbf{a}, \tag{8.30}$$

respectively. Note that Eq. (8.29) represents p equations in q unknowns, whereas Eq. (8.30) represents q equations in p unknowns. In the case of oversampling ($p > q \geq 1$) the latter set of equations is thus underdetermined.

We can now proceed along the same lines as the ones that we followed in Section 7; we only have to replace the factor $\alpha pT/q$ by K [compare Eqs. (7.17) and (8.29)], which is the same as replacing the (continuous-time) shifting distance αT by the (discrete-time) shifting distance N. We thus find, for example, that the optimum elementary signal $g_{opt}[n]$ that corresponds to a given (finite-support) window function $w[n]$ can be found with the help of the relation $K\mathbf{G}_{opt} = p(\mathbf{W}^\dagger)^*$, and that this optimum elementary signal $g_{opt}[n]$ resembles best the window function $w[n]$.

Let us, as an example, choose a Gaussian window function $w[n]$ that is symmetrical around the point $\frac{1}{2}(N-1)$:

$$w[n] = e^{-(\pi/pN^2)(n-\frac{1}{2}[N-1])^2}. \tag{8.31}$$

The discrete Zak transform $\tilde{w}[n, l;\, N, M]$ of this function reads

$$\tilde{w}[n, l;\, N, M] = e^{-(\pi/pN^2)(n-\frac{1}{2}[N-1])^2}\theta_3\left(z; e^{-\pi/p}\right), \tag{8.32}$$

where we have set

$$z = \pi\frac{l}{M} - j\pi\frac{1}{p}\left(\frac{n-\frac{1}{2}[N-1]}{N}\right).$$

We remark that the theta function $\theta_3(z; e^{-\pi/p})$ has zeros for $z = \pi(k+\frac{1}{2}) - j\pi(m+\frac{1}{2})/p$; hence, although a zero will not be reached for integer values of n, the value of $\tilde{w}[n, l;\, N, M]$ will be very small for l in the neighbourhood of $(k+\frac{1}{2})M$ and n in the neighbourhood of $mN - \frac{1}{2}$. In Fig. 6 we have depicted the discrete Zak transform $\tilde{w}[n, l;\, N, M]$ for several values of the parameter p, with $N = M = 24$.

The width of the Gaussian window function (8.31) is, roughly, $N\sqrt{p}$. It will be clear that when we truncate the window function to an interval of length N_w where N_w is much larger than $N\sqrt{p}$, the discrete Zak transform of this truncated window function will almost be equal to the one of the untruncated window function.

We now apply the techniques outlined in this section to determine the elementary signal $g_{opt}[n]$ that corresponds to a truncated Gaussian window function $w[n]$ [cf. Eq. (8.31)]. For convenience, we restrict ourselves to integer oversampling:

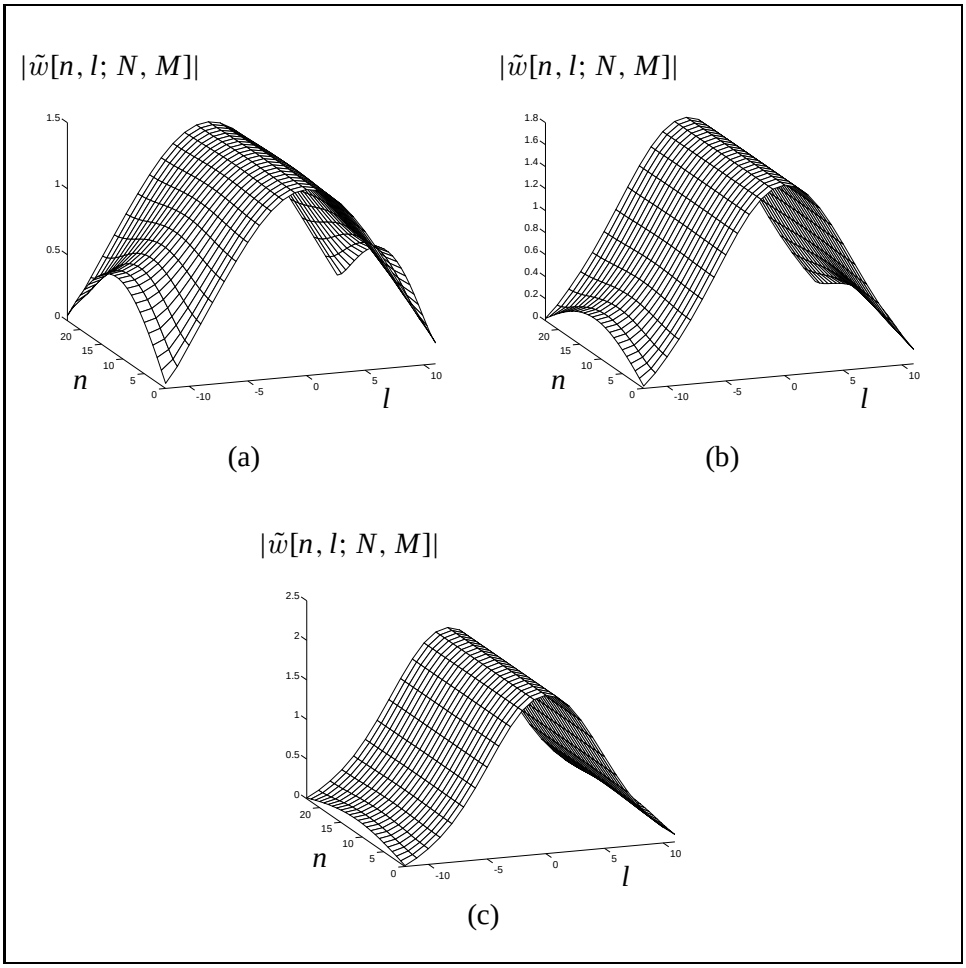

Figure 6: The discrete Zak transform $\tilde{w}[n, l; N, M]$ in the case of a Gaussian window function $w[n] = e^{-(\pi/pN)^2(n-\frac{1}{2}[N-1])^2}$ (with $N = M = 24$) for different values of p: (a) $p = 2$, (b) $p = 3$, (c) $p = 4$.

$K/N = p \geq q = 1$. Moreover, we choose $N = M = 24$. In Fig. 7 we have depicted the discrete Zak transforms $\tilde{g}_{opt}[n, l; N, M]$ of the optimum elementary signals $g_{opt}[n]$ for different values of the oversampling parameter p, while in Fig. 8 we have depicted these elementary signals themselves. We remark that the resemblance between the elementary signal and the window function increases with increasing value of the oversampling parameter p.

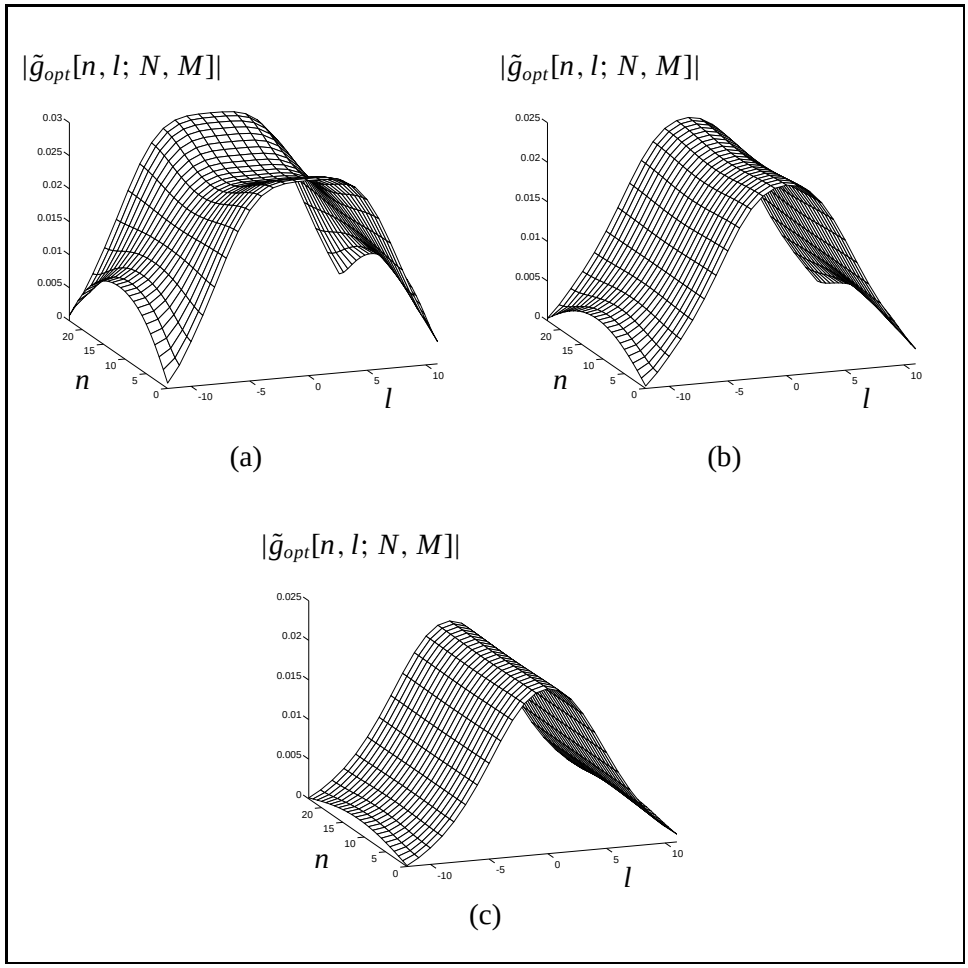

Figure 7: The discrete Zak transform $\tilde{g}_{opt}[n, l; N, M]$ in the case of a Gaussian window function $w[n] = e^{-(\pi/pN)^2(n - \frac{1}{2}[N-1])^2}$ (with $N = M = 24$) for different values of oversampling $p = K/N$: (a) $p = 2$, (b) $p = 3$, (c) $p = 4$.

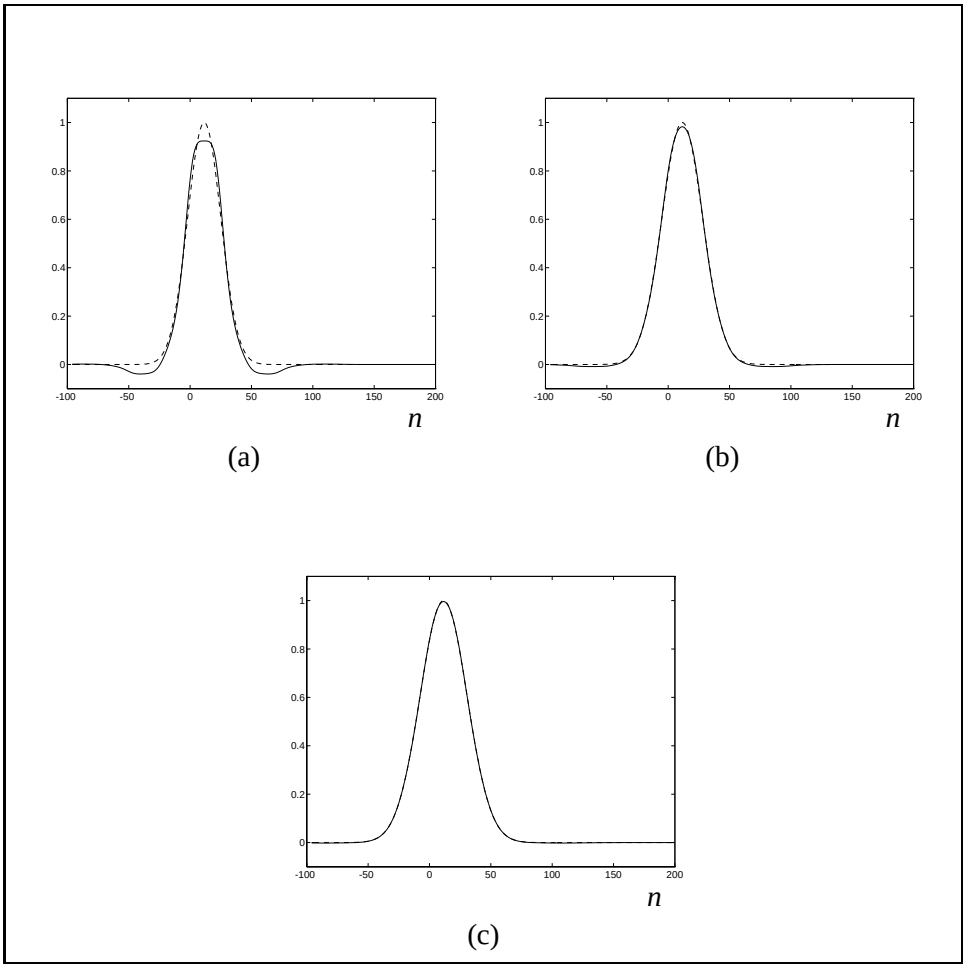

Figure 8: A Gaussian window function $w[n] = e^{-(\pi/pN)^2(n-\frac{1}{2}[N-1])^2}$ (with $N = 24$, dashed line) and its corresponding optimum elementary signal $g_{opt}[n]$ (solid line) for different values of oversampling $p = K/N$:
(a) $p = 2$, (b) $p = 3$, (c) $p = 4$.

9 Coherent-optical set-up

We finally describe a coherent-optical set-up with which Gabor's expansion coefficients of a rastered, one-dimensional signal can be generated.

We first note that Eq. (7.7) allows an easy determination of the array of Gabor coefficients a_{mk} via the Zak transform, especially in the case of *integer* oversampling (i.e. $q = 1$). Since we are in fact only dealing with Fourier transformations, Eq. (7.7) enables a coherent-optical implementation. Let us therefore consider the optical arrangement depicted in Fig. 6, and let us identify the two variables t and ω in the Zak transforms and the Fourier transform, as the x and y coordinates. Moreover, let us, for the sake of convenience, take $\alpha = 1/p$ and $\beta = q = 1$, with which Eq. (7.7) reduces to

$$\bar{a}\left(\frac{t}{T}, \frac{\omega}{\Omega}\right) = T\tilde{\varphi}(t, p\omega; T)\tilde{w}^*\left(t, p\omega; \frac{T}{p}\right). \tag{9.1}$$

Figure 9: Coherent-optical arrangement to generate Gabor's expansion coefficients of a rastered, one-dimensional signal.

A plane wave of monochromatic laser light is normally incident upon a transparency situated in the input plane. The transparency contains the signal $\varphi(x)$ in a rastered format. With X being the width of this raster and $p\mu X$ (with $\mu > 0$)

being the spacing between the raster lines, the light amplitude $\varphi_i(x_i, y_i)$ just behind the transparency reads

$$\varphi_i(x_i, y_i) = \text{rect}\left(\frac{x_i}{X}\right) \sum_n \varphi(x_i + nX)\delta(y_i - np\mu X). \tag{9.2}$$

An anamorphic optical system between the input plane and the middle plane performs a Fourier transformation in the y-direction and an ideal imaging (with inversion) in the x-direction. Such an anamorphic system can be realized as shown, for instance, using a combination of a spherical and a cylindrical lens. The anamorphic operation results in the light amplitude

$$\varphi_1(x, y) = \iint \varphi_i(x_i, y_i) e^{-j\gamma_i y y_i} \delta(x - x_i) dx_i dy_i =$$

$$= \text{rect}\left(\frac{x}{X}\right) \tilde{\varphi}(x, p\mu\gamma_i y; X) \tag{9.3}$$

just in front of the middle plane; the parameter γ_i contains the effect of the wave length λ of the laser light and the focal length f_i of the spherical lens: $\gamma_i = 2\pi/\lambda f_i$. Note that in Eq. (9.3) we have introduced the Zak transform of $\varphi(x)$, defined by Eq. (3.1) with t replaced by x, ω replaced by $p\mu\gamma_i y$, and τ replaced by X.

A transparency with amplitude transmittance

$$m(x, y) = \text{rect}\left(\frac{x}{X}\right) \text{rect}\left(\frac{y}{Y}\right) X\tilde{w}^*\left(x, p\mu\gamma_i y; \frac{X}{p}\right), \tag{9.4}$$

where $\mu\gamma_i Y = 2\pi/X$, is situated in the middle plane. Just behind this transparency, the light amplitude takes the form

$$\varphi_2(x, y) = m(x, y)\varphi_1(x, y) = \text{rect}\left(\frac{x}{X}\right) \text{rect}\left(\frac{y}{Y}\right) \bar{a}\left(\frac{x}{X}, \frac{y}{Y}\right), \tag{9.5}$$

where use has been made of Eq. (9.1), with t replaced by x, ω replaced by $\mu\gamma_i y$, and T replaced by X. Note that the aperture $\text{rect}(x/X)\text{rect}(y/Y)$ contains *one* period of the periodic Fourier transform $\bar{a}(x/X, y/Y)$, p periods of the (periodic) Zak transform $\tilde{\varphi}(x, p\mu\gamma_i y; X)$, and p quasi-periods of the (quasi-periodic) Zak transform $\tilde{w}^*(x, p\mu\gamma_i y; X/p)$.

One of the reasons for oversampling is the additional freedom in choosing the window function $w(t)$. In particular, a window function can be chosen such that it is mathematically well-behaved; this is usually not the case for the original Gabor expansion with critical sampling. Indeed, since the Zak transform $\tilde{g}$ of a continuous, square integrable elementary signal $g(t)$ has zeros [18], the Zak transform $\tilde{w} = 1/T\tilde{g}^*$ [cf. Eq. (3.8)] of the corresponding window function $w(t)$ has poles, which reflects the bad behaviour of the window function. We remark that in the case of oversampling ($p > 1$) the Zak transform $\tilde{w}(x, p\mu\gamma_i y; X/p)$ can be constructed such that it has no poles, and, hence, that a practical transparency can indeed be fabricated!

Finally, a two-dimensional Fourier transformation is performed between the middle plane and the output plane. Such a Fourier transformation can be realized

as shown, for instance, using a spherical lens. The light amplitude in the output plane then takes the form

$$\varphi_o(x_o, y_o) = \frac{1}{XY} \iint \varphi_2(x, y) e^{-j\gamma_o(x_o x - y_o y)} dx dy =$$

$$= \sum_m \sum_k a_{mk} \operatorname{sinc}\left(\frac{\gamma_o}{\gamma_i}\frac{x_o}{\mu Y} - k\right) \operatorname{sinc}\left(\frac{\gamma_o}{\gamma_i}\frac{y_o}{\mu X} - m\right), \tag{9.6}$$

where the sinc-function $\operatorname{sinc}(z) = \sin(\pi z)/(\pi z)$ has been introduced; the parameter γ_o, again, contains the effects of the wave length λ of the laser light and the focal length f_o of the spherical lens: $\gamma_o = 2\pi/\lambda f_o$. We conclude that Gabor's expansion coefficients appear on a rectangular lattice of points

$$a_{mk} = \varphi_o\left(k\frac{\gamma_i}{\gamma_o}\mu Y, m\frac{\gamma_i}{\gamma_o}\mu X\right) \tag{9.7}$$

in the output plane.

We remark that it is not an essential requirement that the input transparency consists of Dirac functions. When we replace the practically unrealizable Dirac functions $\delta(y - np\mu X)$ by realizable functions $d(y - np\mu X)$, say, then Eq. (9.2) reads

$$\varphi_i(x_i, y_i) = \operatorname{rect}\left(\frac{x_i}{X}\right) \sum_n \varphi(x_i + nX) d(y_i - np\mu X) \tag{9.8}$$

and the light amplitude $\varphi_1(x, y)$ just in front of the middle plane takes the form [cf. Eq. (9.3)]

$$\varphi_1(x, y) = \operatorname{rect}\left(\frac{x}{X}\right) \tilde{\varphi}(x, p\mu\gamma_i y; X) \bar{d}(\gamma_i y). \tag{9.9}$$

The additional factor $\bar{d}(\gamma_i y)$ – the Fourier transform of $d(y)$ – can easily be compensated for by means of a transparency in the middle plane.

The technique described in this section to generate Gabor's expansion coefficients, fully utilizes the two-dimensional nature of the optical system, its parallel processing features, and the large space-bandwidth product possible in optical processing. The technique exhibits a resemblance to *folded spectrum* techniques, where space-bandwidth products in the order of 300 000 are reported [13]. In the case of speech processing, where speech recognition and speaker identification are important problems, such a space-bandwidth product would allow us to process speech fragments of about 1 minute.

10 Conclusion

In this paper we have presented Gabor's expansion of a signal into a discrete set of properly shifted and modulated versions of an elementary signal. We have also described the inverse operation – the Gabor transform – with which Gabor's expansion coefficients can be determined, and we have shown how the expansion coefficients can be determined, even in the case that the set of elementary signals is not orthonormal. The key solution was the construction of a window function such that the discrete set of shifted and modulated versions of the window function is bi-orthonormal to the corresponding set of elementary signals. Thus, we have shown a strong relationship between Gabor's expansion of a signal on the one hand and sampling of the sliding-window spectrum of the signal on the other.

The Gabor lattice played a key role in the first part of this paper. It is the regular lattice ($t = mT, \omega = k\Omega$) with $\Omega T = 2\pi$ in the time-frequency domain, in which each cell occupies an area of 2π. The density of the Gabor lattice is thus equal to the *Nyquist density* $1/2\pi$, which, as is well known in information theory, is the minimum time-frequency density needed for full transmission of information. Gabor's expansion coefficients can then be interpreted as degrees of freedom of the signal.

We have introduced the Zak transform and we have shown its intimate relation to the Gabor transform. Not only did we consider the Gabor transform and the Zak transform in the continuous-time case, but we have also considered the discrete-time case. Furthermore, we have introduced the *discrete* Gabor transform and the *discrete* Zak transform, by sampling the continuous frequency variable that still occurred in the discrete-time case. The discrete transforms enable us to determine Gabor's expansion coefficients via a fast computer algorithm, analogous to the fast Fourier transform algorithm well known in digital signal processing.

Using the Zak transform, we have seen that – at least for critically sampling the time-frequency domain on the lattice $(mT, k\Omega)$ – the Gabor transform, as well as Gabor's signal expansion itself, can be transformed into a product form. Determination of the expansion coefficients via the product forms may be difficult, however, because of the occurrence of zeros in the Zak transform. One way of avoiding the problems that arise from these zeros is to sample the time-frequency domain on a denser lattice $(m\alpha T, k\beta\Omega)$, with the product $\alpha\beta$ smaller than 1. In this paper we have considered the special case $\alpha\beta = q/p$, with $p \geq q \geq 1$. We have shown that in this case of oversampling by a rational factor, the Gabor transform and Gabor's signal expansion can be transformed into sum-of-products forms. Using these sum-of-product forms, it was possible to show that the Gabor transform and Gabor's signal expansion form a transform pair. The properties of denser lattices have already been studied in many other papers [2, 3, 14, 22, 23, 25, 26, 27, 30, 35].

The process of oversampling introduces dependence between the Gabor coefficients; whereas these coefficients can be considered as degrees of freedom in the case of critical sampling, they can no longer be given such an interpretation in the case of oversampling. In controlling the dependence between the Gabor coefficients, we were able to avoid the problems that arise from the occurrence

of zeros in the Zak transform. In particular it was shown how the window function that appears in the Gabor transform, can be constructed from the elementary signal that is used in Gabor's signal expansion. The additional freedom caused by oversampling, allowed us to construct the window function in such a way that it is mathematically well-behaved. Moreover, it was shown that for a very large oversampling the window function can become proportional to the elementary signal; this result is in accordance with the continuous analogue of Gabor's signal expansion.

Finally, a coherent-optical arrangement was described which is able to generate Gabor's expansion coefficients of a rastered, one-dimensional signal via the Zak transform. The technique described there – which resembles folded spectrum techniques – fully utilizes the two-dimensional nature of the optical system, its parallel processing features, and the large space-bandwidth product possible in optical processing. Due to the possibility of avoiding the problems that arise from the occurrence of zeros in the Zak transform, the required optical transparency can indeed be fabricated.

We conclude this paper by drawing attention to some related topics: the *wavelet transform* of a signal and the way of representing a signal as a discrete set of *wavelets*. There is some resemblance between these topics and the ones that were presented in this paper. But, whereas the Gabor transform and the sliding-window spectrum lead to a *time-frequency* representation of the signal, the wavelet transform leads to a *time-scale* representation. And whereas the Gabor lattice is linear both in the time and the frequency variable, the lattice that is used in the wavelet representation is nonlinear. An excellent review of the wavelet transform can be found in [14].

Appendix A. The second bi-orthonormality condition (2.8)

We will show that the first bi-orthonormality condition (2.7), which in product form reads [see Eq. (3.8)]

$$T\tilde{w}^*(t,\omega;T)\tilde{g}(t,\omega;T)=1,$$

implies the second bi-orthonormality condition (2.8)

$$\sum_m\sum_k w^*_{mk}(t_1)g_{mk}(t_2)=\delta(t_2-t_1).$$

We start with the left-hand side of the second bi-orthonormality condition (2.8)

$$\sum_m\sum_k w^*_{mk}(t_1)g_{mk}(t_2)=\sum_m\sum_k w^*(t_1-mT)e^{-jk\Omega t_1}g(t_2-mT)e^{jk\Omega t_2}.$$

After expressing the window function $w(t_1-mT)$ by means of its Zak transform $\tilde{w}(t_1,\omega;T)$ [cf. Eq. (3.3)], the latter expression takes the form

$$\sum_m\sum_k\left[\frac{1}{\Omega}\int_\Omega \tilde{w}(t_1,\omega;T)e^{-jm\omega T}d\omega\right]^* e^{-jk\Omega t_1}g(t_2-mT)e^{jk\Omega t_2}.$$

We rearrange factors

$$\sum_k e^{jk\Omega(t_2-t_1)}\frac{1}{\Omega}\int_\Omega \tilde{w}^*(t_1,\omega;T)\left[\sum_m g(t_2-mT)e^{jm\omega T}\right]d\omega$$

and recognize [cf. Eq. (3.1)] the Zak transform $\tilde{g}(t_2,\omega;T)$ of the function $g(t_2)$

$$\sum_k e^{jk\Omega(t_2-t_1)}\frac{1}{\Omega}\int_\Omega \tilde{w}^*(t_1,\omega;T)\tilde{g}(t_2,\omega;T)d\omega.$$

We replace the sum of exponentials by a sum of Dirac functions

$$T\sum_n \delta(t_2-t_1-nT)\frac{1}{\Omega}\int_\Omega \tilde{w}^*(t_1,\omega;T)\tilde{g}(t_2,\omega;T)d\omega$$

and replace the variable t_2 in the integral by t_1+nT

$$T\sum_n \delta(t_2-t_1-nT)\frac{1}{\Omega}\int_\Omega \tilde{w}^*(t_1,\omega;T)\tilde{g}(t_1+nT,\omega;T)d\omega.$$

We use the quasi-periodicity property [cf. Eq. (3.2)] of the Zak transform $\tilde{g}(t_1,\omega;T)$

$$\sum_n \delta(t_2-t_1-nT)\frac{1}{\Omega}\int_\Omega T\tilde{w}^*(t_1,\omega;T)\tilde{g}(t_1,\omega;T)e^{jn\omega T}d\omega$$

and substitute from the product form (3.8) of the first bi-orthonormality condition

$$\sum_n \delta(t_2-t_1-nT)\frac{1}{\Omega}\int_\Omega e^{jn\omega T}d\omega.$$

We evaluate the integral

$$\sum_n \delta(t_2 - t_1 - nT)\delta_n$$

and conclude that the expression reduces to the required Dirac function

$$\delta(t_2 - t_1).$$

Appendix B. Derivation of condition (7.4)

In Gabor's signal expansion (7.2) we substitute from the Gabor transform (7.3)

$$\varphi(t) = \sum_m \sum_k \left[\int \varphi(t') w^*(t' - m\alpha T) e^{-jk\beta\Omega t'} dt' \right] g(t - m\alpha T) e^{jk\beta\Omega t}$$

and require that this relationship should hold for any arbitrary signal $\varphi(t)$. We rearrange factors

$$\varphi(t) = \int \varphi(t') \left[\sum_m w^*(t' - m\alpha T) g(t - m\alpha T) \right] \left[\sum_k e^{-jk\beta\Omega(t' - t)} \right] dt'$$

and replace the sum of exponentials by a sum of Dirac functions

$$\varphi(t) = \int \varphi(t') \left[\sum_m w^*(t' - m\alpha T) g(t - m\alpha T) \right] \left[\frac{T}{\beta} \sum_k \delta\left(t' - t - k\frac{T}{\beta} \right) \right] dt'.$$

We rearrange factors again

$$\varphi(t) = \frac{T}{\beta} \sum_k \int \varphi(t') \left[\sum_m w^*(t' - m\alpha T) g(t - m\alpha T) \right] \delta\left(t' - t - k\frac{T}{\beta} \right) dt'$$

and evaluate the integral

$$\varphi(t) = \frac{T}{\beta} \sum_k \varphi\left(t + k\frac{T}{\beta} \right) \left[\sum_m w^*\left(t + k\frac{T}{\beta} - m\alpha T \right) g(t - m\alpha T) \right].$$

The latter relationship holds for any signal $\varphi(t)$ if and only if the $k = 0$ term in the summation over k is the only nonvanishing term, which immediately leads to the condition

$$\frac{T}{\beta} \sum_m w^*\left(t + k\frac{T}{\beta} - m\alpha T \right) g(t - m\alpha T) = \delta_k,$$

where δ_k is a Kronecker delta, with $\delta_0 = 1$ and $\delta_k = 0$ for $k \neq 0$.

Appendix C. The sum-of-products form (7.7)

In the Fourier transform (2.11) of the array a_{mk}, we substitute from the Gabor transform (7.3)

$$\bar{a}(x,y) = \sum_m \sum_k \left[\int \varphi(t) w^*(t - m\alpha T) e^{-jk\beta\Omega t} dt \right] e^{-j2\pi(my - kx)}$$

and rearrange factors

$$\bar{a}(x,y) = \sum_m \left[\int \varphi(t) w^*(t - m\alpha T) \left\{ \sum_k e^{-jk(\beta\Omega t - 2\pi x)} \right\} dt \right] e^{-j2\pi my}.$$

We replace the sum of exponentials by a sum of Dirac functions

$$\bar{a}(x,y) = \sum_m \left[\int \varphi(t) w^*(t - m\alpha T) \left\{ \frac{T}{\beta} \sum_k \delta\left(t - (x+k)\frac{T}{\beta} \right) \right\} dt \right] e^{-j2\pi my}$$

and rearrange factors again

$$\bar{a}(x,y) = \frac{T}{\beta} \sum_m \left[\sum_k \int \varphi(t) w^*(t - m\alpha T) \delta\left(t - (x+k)\frac{T}{\beta} \right) dt \right] e^{-j2\pi my}.$$

We evaluate the integral

$$\bar{a}(x,y) = \frac{T}{\beta} \sum_m \left[\sum_k \varphi\left((x+k)\frac{T}{\beta} \right) w^*\left((x+k)\frac{T}{\beta} - m\alpha T \right) \right] e^{-j2\pi my}$$

and rearrange factors again

$$\bar{a}(x,y) = \frac{T}{\beta} \sum_k \varphi\left(x\frac{T}{\beta} + k\frac{T}{\beta} \right) e^{-jk(T/\beta)y(\Omega/\alpha)}$$

$$\times \left[\sum_m w^*\left(x\frac{T}{\beta} - \left[m - \frac{k}{\alpha\beta} \right] \alpha T \right) e^{-j[m - k/\alpha\beta]\alpha T y(\Omega/\alpha)} \right].$$

We replace $1/\alpha\beta$ by p/q

$$\bar{a}(x,y) = \frac{T}{\beta} \sum_k \varphi\left(x\frac{T}{\beta} + k\frac{T}{\beta} \right) e^{-jk(T/\beta)y(\Omega/\alpha)}$$

$$\times \left[\sum_m w\left(x\frac{T}{\beta} - \left[m - k\frac{p}{q} \right] \alpha T \right) e^{j[m - k(p/q)]\alpha T y(\Omega/\alpha)} \right]^*$$

and replace the summation over k by a double summation over s and n through the substitution $k = nq + s$, where s extends over an interval of length q

$$\bar{a}(x,y) = \frac{T}{\beta} \sum_n \sum_{s=<q>} \varphi\left(x\frac{T}{\beta} + (nq+s)\frac{T}{\beta}\right) e^{-j(nq+s)(T/\beta)y(\Omega/\alpha)}$$

$$\times \left[\sum_m w\left(x\frac{T}{\beta} - \left[m-(nq+s)\frac{p}{q}\right]\alpha T\right) e^{j[m-(nq+s)(p/q)]\alpha T y(\Omega/\alpha)}\right]^*.$$

We substitute $m - np$ by $-k$

$$\bar{a}(x,y) = \frac{T}{\beta} \sum_n \sum_{s=<q>} \varphi\left(x\frac{T}{\beta} + \left[n+\frac{s}{q}\right] q\frac{T}{\beta}\right) e^{-j[n+(s/q)](qT/\beta)y(\Omega/\alpha)}$$

$$\times \left[\sum_k w\left(x\frac{T}{\beta} + \left[k+s\frac{p}{q}\right]\alpha T\right) e^{-j[k+s(p/q)]\alpha T y(\Omega/\alpha)}\right]^*$$

and rearrange factors, while using the relation $\alpha p = q/\beta$

$$\bar{a}(x,y) = \frac{T}{\beta} \sum_{s=<q>} \sum_n \varphi\left((x+s)\frac{T}{\beta} + n\alpha pT\right) e^{-jn\alpha pTy(\Omega/\alpha)}$$

$$\times \left[\sum_k w\left((x+s)\frac{T}{\beta} + k\alpha T\right) e^{-jk\alpha Ty(\Omega/\alpha)}\right]^*.$$

In the last expression we recognize the definitions [Eq. (3.1)] for the Zak transforms $\tilde{\varphi}(xT/\beta, y\Omega/\alpha; \alpha pT)$ and $\tilde{w}(xT/\beta, y\Omega/\alpha; \alpha T)$ of the signal $\varphi(t)$ and the window function $w(t)$, respectively, and can write

$$\bar{a}(x,y) = \frac{T}{\beta} \sum_{s=<q>} \tilde{\varphi}\left((x+s)\frac{T}{\beta}, y\frac{\Omega}{\alpha}; \alpha pT\right) \tilde{w}^*\left((x+s)\frac{T}{\beta}, y\frac{\Omega}{\alpha}; \alpha T\right)$$

or, with $1/\beta = \alpha p/q$,

$$\bar{a}(x,y) = \frac{\alpha pT}{q} \sum_{s=<q>} \tilde{\varphi}\left((x+s)\frac{\alpha pT}{q}, y\frac{\Omega}{\alpha}; \alpha pT\right) \tilde{w}^*\left((x+s)\frac{\alpha pT}{q}, y\frac{\Omega}{\alpha}; \alpha T\right).$$

We finally replace y by $y + r/p$ and use the periodicity of the Zak transform $\tilde{\varphi}(x\alpha pT/q, y\Omega/\alpha; \alpha pT)$, which leads to the result

$$\bar{a}\left(x, y+\frac{r}{p}\right) = \frac{\alpha pT}{q}$$

$$\times \sum_{s=<q>} \tilde{\varphi}\left((x+s)\frac{\alpha pT}{q}, y\frac{\Omega}{\alpha}; \alpha pT\right) \tilde{w}^*\left((x+s)\frac{\alpha pT}{q}, \left[y+\frac{r}{p}\right]\frac{\Omega}{\alpha}; \alpha T\right).$$

Appendix D. The sum-of-products form (7.8)

In Gabor's signal expansion (7.2) we substitute from the inverse Fourier transform (2.13)

$$\varphi(t) = \sum_m \sum_k \left[\int_1 \int_1 \bar{a}(x, y) e^{j2\pi(my - kx)} dx dy \right] g(t - m\alpha T) e^{jk\beta\Omega t}$$

and rearrange factors

$$\varphi(t) =$$

$$= \int_1 \int_1 \bar{a}(x, y) \left[\sum_m g(t - m\alpha T) e^{j2\pi my} \right] \left[\sum_k e^{-jk(2\pi x - \beta\Omega t)} \right] dx dy.$$

We replace the sum of exponentials by a sum of Dirac functions and recognize the Zak transform of the elementary signal $g(t)$ [cf. Eq. (3.1)]:

$$\varphi(t) = \int_1 \int_1 \bar{a}(x, y) \tilde{g}\left(t, y\frac{\Omega}{\alpha}; \alpha T\right) \left[\sum_k \delta\left(x - \frac{\beta}{T}t + k\right) \right] dx dy.$$

We rearrange factors again and substitute from the periodicity property (2.12) of $\bar{a}(x, y)$

$$\varphi(t) = \int_1 \left[\sum_k \int_1 \bar{a}(x + k, y) \delta\left(x + k - \frac{\beta}{T}t\right) dx \right] \tilde{g}\left(t, y\frac{\Omega}{\alpha}; \alpha T\right) dy$$

and we replace the summation over k together with the integral over the finite x-interval by an integral over the entire x-axis

$$\varphi(t) = \int_1 \left[\int \bar{a}(x, y) \delta\left(x - \frac{\beta}{T}t\right) dx \right] \tilde{g}\left(t, y\frac{\Omega}{\alpha}; \alpha T\right) dy.$$

Evaluation of the resulting integral over x yields the intermediate result

$$\varphi(t) = \int_1 \bar{a}\left(\frac{\beta}{T}t, y\right) \tilde{g}\left(t, y\frac{\Omega}{\alpha}; \alpha T\right) dy.$$

We now write down the definition of the Zak transform [cf. Eq. (3.1)]

$$\tilde{\varphi}\left(x\frac{T}{\beta}, y_o\frac{\Omega}{\alpha}; \alpha p T\right) =$$

$$= \tilde{\varphi}\left(x\frac{T}{\beta}, y_o\frac{\Omega}{\alpha}; q\frac{T}{\beta}\right) = \sum_n \varphi\left((x + nq)\frac{T}{\beta}\right) e^{-jnq(T/\beta)y_o(\Omega/\alpha)}$$

and substitute from the intermediate result above

$$\tilde{\varphi}\left(x\frac{T}{\beta}, y_o\frac{\Omega}{\alpha}; \alpha p T\right) =$$

$$= \sum_n \left[\int_1 \bar{a}\left(\frac{\beta}{T}(x+nq)\frac{T}{\beta}, y\right) \tilde{g}\left((x+nq)\frac{T}{\beta}, y\frac{\Omega}{\alpha}; \alpha T\right) dy \right]$$
$$\times e^{-jnq(T/\beta)y_o(\Omega/\alpha)}.$$

We rearrange things, using the relation $1/\beta = (p/q)\alpha$,

$$\tilde{\varphi}\left(x\frac{T}{\beta}, y_o\frac{\Omega}{\alpha}; \alpha pT\right) =$$

$$= \sum_n \left[\int_1 \bar{a}(x+nq, y)\tilde{g}\left(x\frac{T}{\beta} + pn\alpha T, y\frac{\Omega}{\alpha}; \alpha T\right) dy \right] e^{-j2\pi pny_o}$$

and use the periodicity property (2.12) of $\bar{a}(x, y)$ and the quasi-periodicity property of $\tilde{g}(xT/\beta, y\Omega/\alpha; \alpha T)$ [cf. Eq. (3.2)]

$$\tilde{\varphi}\left(x\frac{T}{\beta}, y_o\frac{\Omega}{\alpha}; \alpha pT\right) =$$

$$= \sum_n \left[\int_1 \bar{a}(x, y)\tilde{g}\left(x\frac{T}{\beta}, y\frac{\Omega}{\alpha}; \alpha T\right) e^{jpny(\Omega/\alpha)\alpha T} dy \right] e^{-j2\pi pny_o}.$$

We rearrange factors

$$\tilde{\varphi}\left(x\frac{T}{\beta}, y_o\frac{\Omega}{\alpha}; \alpha pT\right) = \int_1 \bar{a}(x, y)\tilde{g}\left(x\frac{T}{\beta}, y\frac{\Omega}{\alpha}; \alpha T\right) \left[\sum_n e^{j2\pi np(y-y_o)} \right] dy$$

and replace the sum of exponentials by a sum of Dirac functions

$$\tilde{\varphi}\left(x\frac{T}{\beta}, y_o\frac{\Omega}{\alpha}; \alpha pT\right) =$$

$$= \int_1 \bar{a}(x, y)\tilde{g}\left(x\frac{T}{\beta}, y\frac{\Omega}{\alpha}; \alpha T\right) \left[\frac{1}{p} \sum_n \delta\left(y - y_o - \frac{n}{p}\right) \right] dy.$$

We replace the summation over n by a double summation over r and k through the substitution $n = kp + r$, where r extends over an interval of length p

$$\tilde{\varphi}\left(x\frac{T}{\beta}, y_o\frac{\Omega}{\alpha}; \alpha pT\right) =$$

$$= \int_1 \bar{a}(x, y)\tilde{g}\left(x\frac{T}{\beta}, y\frac{\Omega}{\alpha}; \alpha T\right) \left[\frac{1}{p} \sum_k \sum_{r=<p>} \delta\left(y - y_o - k - \frac{r}{p}\right) \right] dy$$

and rearrange factors

$$\tilde{\varphi}\left(x\frac{T}{\beta}, y_o\frac{\Omega}{\alpha}; \alpha pT\right) =$$

$$= \frac{1}{p} \sum_{r=<p>} \left[\sum_k \int_1 \bar{a}(x, y)\tilde{g}\left(x\frac{T}{\beta}, y\frac{\Omega}{\alpha}; \alpha T\right) \delta\left(y - k - y_o - \frac{r}{p}\right) dy \right].$$

We use the periodicity of $\bar{a}(x, y)$ and $\tilde{g}(xT/\beta, y\Omega/\alpha; \alpha T)$

$$\tilde{\varphi}\left(x\frac{T}{\beta}, y_o\frac{\Omega}{\alpha}; \alpha pT\right) =$$

$$= \frac{1}{p}\sum_{r=<p>}\left[\sum_k \int_1 \bar{a}(x, y-k)\tilde{g}\left(x\frac{T}{\beta}, (y-k)\frac{\Omega}{\alpha}; \alpha T\right)\delta\left(y-k-y_o-\frac{r}{p}\right)dy\right]$$

and replace the summation over k together with the integral over the finite y-interval by an integral over the entire y-axis

$$\tilde{\varphi}\left(x\frac{T}{\beta}, y_o\frac{\Omega}{\alpha}; \alpha pT\right) =$$

$$= \frac{1}{p}\sum_{r=<p>}\int \bar{a}(x, y)\tilde{g}\left(x\frac{T}{\beta}, y\frac{\Omega}{\alpha}; \alpha T\right)\delta\left(y-y_o-\frac{r}{p}\right)dy.$$

Evaluation of the integral and replacing y_o by y results in

$$\tilde{\varphi}\left(x\frac{T}{\beta}, y\frac{\Omega}{\alpha}; \alpha pT\right) = \frac{1}{p}\sum_{r=<p>}\bar{a}\left(x, y+\frac{r}{p}\right)\tilde{g}\left(x\frac{T}{\beta}, \left[y+\frac{r}{p}\right]\frac{\Omega}{\alpha}; \alpha T\right)$$

or, with $1/\beta = \alpha p/q$,

$$\tilde{\varphi}\left(x\frac{\alpha pT}{q}, y\frac{\Omega}{\alpha}; \alpha pT\right) = \frac{1}{p}\sum_{r=<p>}\bar{a}\left(x, y+\frac{r}{p}\right)\tilde{g}\left(x\frac{\alpha pT}{q}, \left[y+\frac{r}{p}\right]\frac{\Omega}{\alpha}; \alpha T\right).$$

We finally replace x by $x+s$ and use the periodicity of the Fourier transform $\bar{a}(x, y)$, which leads to the result

$$\tilde{\varphi}\left((x+s)\frac{\alpha pT}{q}, y\frac{\Omega}{\alpha}; \alpha pT\right) =$$

$$= \frac{1}{p}\sum_{r=<p>}\bar{a}\left(x, y+\frac{r}{p}\right)\tilde{g}\left((x+s)\frac{\alpha pT}{q}, \left[y+\frac{r}{p}\right]\frac{\Omega}{\alpha}; \alpha T\right).$$

Appendix E. The sum-of-products form (8.19)

In the discrete Fourier transform (8.11) of the array A_{mk}, we substitute from the Gabor transform (8.6)

$$\bar{a}[n, l; K, M] =$$

$$= \sum_{m=<M>} \sum_{k=<K>} \left[\sum_{n'} \varphi[n'] W^*[n' - mN] e^{-j2\pi kn'/K} \right] e^{-j2\pi(ml/M - kn/K)}$$

and rearrange factors

$$\bar{a}[n, l; K, M] =$$

$$= \sum_{m=<M>} \left[\sum_{n'} \varphi[n'] W^*[n' - mN] \left\{ \sum_{k=<K>} e^{-j2\pi k(n'-n)/K} \right\} \right] e^{-j2\pi ml/M}.$$

We replace the sum of exponentials by a sum of Kronecker deltas

$$\bar{a}[n, l; K, M] =$$

$$= \sum_{m=<M>} \left[\sum_{n'} \varphi[n'] W^*[n' - mN] \left\{ K \sum_{k} \delta[n' - n - kK] \right\} \right] e^{-j2\pi ml/M}$$

and rearrange factors again

$$\bar{a}[n, l; K, M] =$$

$$= K \sum_{m=<M>} \left[\sum_{k} \sum_{n'} \varphi[n'] W^*[n' - mN] \delta[n' - n - kK] \right] e^{-j2\pi ml/M}.$$

We evaluate the summation over n'

$$\bar{a}[n, l; K, M] = K \sum_{m=<M>} \left[\sum_{k} \varphi[n + kK] W^*[n + kK - mN] \right] e^{-j2\pi ml/M}$$

and rearrange factors again

$$\bar{a}[n, l; K, M] = K \sum_{k} \varphi[n + kK] e^{-j2\pi kK(l/MN)}$$

$$\times \left[\sum_{m=<M>} W^*[n - (m - kK/N)N] e^{-j2\pi(m - kK/N)N(l/MN)} \right].$$

We replace K/N by p/q

$$\bar{a}[n, l; K, M] = K \sum_{k} \varphi[n + kK] e^{-j2\pi kK(l/MN)}$$

$$\times \left[\sum_{m=<M>} W[n - (m - kp/q)N] e^{j2\pi(m - kp/q)N(l/MN)} \right]^*$$

and replace the summation over k by a double summation over s and n' through the substitution $k = n'q + s$, where s extends over an interval of length q

$$\bar{a}[n, l; K, M] = K \sum_{s=<q>} \sum_{n'} \varphi[n + sK + n'qK] e^{-j2\pi(n'q+s)K(l/MN)}$$

$$\times \left[\sum_{m=<M>} W[n - (m - n'p - sp/q)N] e^{j2\pi(m - n'p - sp/q)N(l/MN)} \right]^* .$$

We substitute $m - n'p$ by $-k$

$$\bar{a}[n, l; K, M] = K \sum_{s=<q>} \sum_{n'} \varphi[n + sK + n'qK] e^{-j2\pi n'qK(l/MN)}$$

$$\times \left[\sum_{k=<M>} W[n + sK + kN] e^{-j2\pi kN(l/MN)} \right]^*$$

and recognize the definitions (8.13) and (8.18) for the discrete Zak transforms $\tilde{\varphi}[n+sK, l;\ pN,\ M/p]$ and $\tilde{w}[n+sK, l;\ N,\ M]$ of the signal $\varphi[n]$ and the window function $w[n]$, respectively, leading to

$$\bar{a}[n, l; K, M] = K \sum_{s=<q>} \tilde{\varphi}[n + sK, l; pN, M/p] \tilde{w}^*[n + sK, l; N, M].$$

We finally replace l by $l + rM/p$ and use the periodicity property of the discrete Zak transform $\tilde{\varphi}[n + sK, l;\ pN,\ M/p]$, which leads to the result

$$\bar{a}[n, l + rM/p; K, M] =$$

$$= K \sum_{s=<q>} \tilde{\varphi}[n + sK, l; pN, M/p] \tilde{w}^*[n + sK, l + rM/p; N, M].$$

Appendix F. The sum-of-products form (8.20)

In the discrete Gabor expansion (8.9) we substitute from the inverse discrete Fourier transform (8.12)

$$\Phi[n] = \sum_{m=<M>} \sum_{k=<K>} \left[\frac{1}{MK} \sum_{n'=<K>} \sum_{l=<M>} \bar{a}[n', l;\, K, M] e^{j2\pi(ml/M - kn'/K)} \right] \times G[n - mN] e^{j2\pi kn/K}$$

and rearrange factors

$$\Phi[n] = \frac{1}{MK} \sum_{n'=<K>} \sum_{l=<M>} \bar{a}[n', l;\, K, M] \left[\sum_{m=<M>} G[n - mN] e^{j2\pi mN(l/MN)} \right] \times \left[\sum_{k=<K>} e^{-j2\pi k(n'-n)/K)} \right].$$

We replace the sum of exponentials by a sum of Kronecker deltas and recognize the discrete Zak transform of the elementary signal $g[n]$ [cf. Eq. (8.13)]:

$$\Phi[n] = \frac{1}{MK} \sum_{n'=<K>} \sum_{l=<M>} \bar{a}[n', l;\, K, M] \tilde{g}[n, l;\, N, M] \left[K \sum_{k} \delta[n' - n - kK] \right].$$

We rearrange factors again and substitute from the periodicity property of the discrete Fourier transform $\bar{a}[n', l;\, K, M]$

$$\Phi[n] = \frac{1}{M} \sum_{l=<M>} \left[\sum_{k} \sum_{n'=<K>} \bar{a}[n' - kK, l;\, K, M] \delta[n' - kK - n] \right] \tilde{g}[n, l;\, N, M]$$

and we replace the summation over k together with the summation over the finite n'-interval by a summation over all n'

$$\Phi[n] = \frac{1}{M} \sum_{l=<M>} \left[\sum_{n'} \bar{a}[n', l;\, K, M] \delta[n' - n] \right] \tilde{g}[n, l;\, N, M].$$

Evaluation of the resulting summation over n' yields the intermediate result

$$\Phi[n] = \frac{1}{M} \sum_{l=<M>} \bar{a}[n, l;\, K, M] \tilde{g}[n, l;\, N, M].$$

We now write down the definition of the discrete Zak transform (8.17)

$$\tilde{\varphi}[n, l;\, pN, M/p] = \sum_{m=<M/p>} \Phi[n + mpN] e^{-j2\pi mpN(l/MN)}$$

and substitute from the intermediate result above

$$\tilde{\varphi}[n, l;\, pN, M/p] =$$

$$= \sum_{m=<M/p>} \left[\frac{1}{M} \sum_{l'=<M>} \bar{a}[n+mpN, l'; K, M] \tilde{g}[n+mpN, l'; N, M] \right]$$
$$\times e^{-j2\pi mpN(l/MN)}.$$

We rearrange things, using the relation $pN = qK$,

$$\tilde{\varphi}[n, l; pN, M/p] =$$
$$= \sum_{m=<M/p>} \left[\frac{1}{M} \sum_{l'=<M>} \bar{a}[n+mqK, l'; K, M] \tilde{g}[n+mpN, l'; N, M] \right]$$
$$\times e^{-j2\pi mpN(l/MN)}$$

and use the periodicity property of the discrete Fourier transform $\bar{a}[n, l'; K, M]$ and the quasi-periodicity property of the discrete Zak transform $\tilde{g}[n, l'; N, M]$)

$$\tilde{\varphi}[n, l; pN, M/p] =$$
$$= \sum_{m=<M/p>} \left[\frac{1}{M} \sum_{l'=<M>} \bar{a}[n, l'; K, M] \tilde{g}[n, l'; N, M] e^{j2\pi mpN(l'/MN)} \right]$$
$$\times e^{-j2\pi mpN(l/MN)}.$$

We rearrange factors

$$\tilde{\varphi}[n, l; pN, M/p] = \frac{1}{M} \sum_{l'=<M>} \bar{a}[n, l'; K, M] \tilde{g}[n, l'; N, M]$$
$$\times \left[\sum_{m=<M/p>} e^{j2\pi m(l'-l)(p/M)} \right]$$

and replace the sum of exponentials by a sum of Kronecker deltas

$$\tilde{\varphi}[n, l; pN, M/p] = \frac{1}{M} \sum_{l'=<M>} \bar{a}[n, l'; K, M] \tilde{g}[n, l'; N, M]$$
$$\times \left[\frac{M}{p} \sum_{m} \delta[l'-l-mM/p] \right].$$

We replace the summation over m by a double summation over r and k through the substitution $m = kp + r$, where r extends over an interval of length p

$$\tilde{\varphi}[n, l; pN, M/p] = \sum_{l'=<M>} \bar{a}[n, l'; K, M] \tilde{g}[n, l'; N.M]$$
$$\times \left[\frac{1}{p} \sum_{k} \sum_{r=<p>} \delta[l'-l-kM-rM/p] \right]$$

and rearrange factors

$$\tilde{\varphi}[n, l;\, pN, M/p] = \frac{1}{p} \sum_{r=<p>}$$

$$\left[\sum_{k} \sum_{l'=<M>} \bar{a}[n, l';\, K, M] \tilde{g}[n, l';\, N, M] \delta[l' - kM - l - rM/p] \right].$$

We use the periodicity of the discrete Fourier transform $\bar{a}[n, l';\, K, M]$ and the discrete Zak transform $\tilde{g}[n, l';\, N, M]$

$$\tilde{\varphi}[n, l;\, pN, M/p] = \frac{1}{p} \sum_{r=<p>}$$

$$\left[\sum_{k} \sum_{l'=<M>} \bar{a}[n, l' - kM;\, K, M] \tilde{g}[n, l' - kM;\, N, M] \delta[l' - kM - l - rM/p] \right]$$

and replace the summation over k together with the summation over the finite l'-interval by a summation over all l'

$$\tilde{\varphi}[n, l;\, pN, M/p] = \frac{1}{p} \sum_{r=<p>}$$

$$\left[\sum_{l'} \bar{a}[n, l';\, K, M] \tilde{g}[n, l';\, N, M] \delta[l' - l - rM/p] \right].$$

Evaluation of the summation over l' results in

$$\tilde{\varphi}[n, l;\, pN, M/p] = \frac{1}{p} \sum_{r=<p>} \bar{a}[n, l + rM/p;\, K, M] \tilde{g}[n, l + rM/p;\, N.M]$$

We finally replace n by $n + sK$ and use the periodicity property of the discrete Fourier transform $\bar{a}[n, l;\, K, M]$, which yields the result

$$\tilde{\varphi}[n + sK, l;\, pN, M/p] =$$

$$= \frac{1}{p} \sum_{r=<p>} \bar{a}[n, l + rM/p;\, K, M] \tilde{g}[n + sK, l + rM/p;\, N, M].$$

References

[1] Abramowitz, M. and I.A. Stegun, eds.
Handbook of mathematical functions.
New York: Dover, 1970.

[2] Auslander, L. and I.C. Gertner, R. Tolimieri
The discrete Zak transform application to time-frequency analysis and synthesis of nonstationary signals.
IEEE Trans. Signal Process., Vol. **39** (1991), p. 825-835.

[3] Balart, R. and R.S. Orr
Computational accuracy and stability issues for the finite, discrete Gabor transform.
In: *Proc. IEEE-SP International symposium on time-frequency and time-scale analysis*, Victoria, BC, Canada, 4-6 October 1992.
Piscataway, NJ, USA: IEEE, 1992. P. 403-406.

[4] Bargmann, V. and P. Butera, L. Girardello, J.R. Klauder
On the completeness of the coherent states.
Rep. Math. Phys., Vol. **2** (1971), p. 221-228.

[5] Bastiaans, M.J.
Gabor's expansion of a signal into Gaussian elementary signals.
Proc. IEEE, Vol. **68** (1980), p. 538-539.

[6] Bastiaans, M.J.
The expansion of an optical signal into a discrete set of Gaussian beams.
Optik, Vol. **57** (1980), p. 95-102.

[7] Bastiaans, M.J.
Sampling theorem for the complex spectrogram, and Gabor's expansion of a signal in Gaussian elementary signals.
Opt. Eng., Vol. **20** (1981), p. 594-598.

[8] Bastiaans, M.J.
Gabor's signal expansion and degrees of freedom of a signal.
Opt. Acta, Vol. **29** (1982), p. 1223-1229.

[9] Bastiaans, M.J.
On the sliding-window representation in digital signal processing.
IEEE Trans. Acoust., Speech Signal Processing, Vol. **ASSP-33** (1985), p. 868-873.

[10] Bastiaans, M.J.
Gabor's signal expansion and its relation to sampling of the sliding-window spectrum.
In: *Advanced topics in Shannon sampling and interpolation theory*. Ed. by R.J. Marks II.
New York: Springer, 1993. P. 1-37.

[11] Bastiaans, M.J.
Gabor's signal expansion and the Zak transform.
Appl. Opt., Vol. **33** (1981), p. 5241-5255.

[12] Campbell, S.L. and C.D. Meyer, Jr.
Generalized inverses of linear transformations.
London: Pitman, 1979.

[13] Casasent, D.
Optical signal processing.
In: *Optical data processing; Applications*. Ed. by D. Casasent.
Berlin: Springer, 1978. Chap. 8.
Topics in applied physics, Vol. **23**.

[14] Daubechies, I.
The wavelet transform, time-frequency localization and signal analysis.
IEEE Trans. Inform. Theory, Vol. **36** (1990), p. 961-1005.

[15] Gabor, D.
Theory of communication.
J. Inst. Elec. Eng., Vol. **93 (III)** (1946), p. 429-457.

[16] Helstrom, C.W.
An expansion of a signal in Gaussian elementary signals.
IEEE Trans. Inform. Theory, Vol. **IT-12** (1966), p. 81-82.

[17] Janssen, A.J.E.M.
Gabor representation of generalized functions.
J. Math. Anal. Appl., Vol. **83** (1981), p. 377-396.

[18] Janssen, A.J.E.M.
Bargmann transform, Zak transform and coherent states.
J. Math. Phys., Vol. **23** (1982), p. 720-731.

[19] Janssen, A.J.E.M.
The Zak transform: A signal transform for sampled time-continuous signals.
Philips J. Res. Vol. **43** (1988), p. 23-69.

[20] Kritikos, H.N. and P.T. Farnum
Approximate evaluation of Gabor expansions.
IEEE Trans. Syst., Man Cybern., Vol. **SMC-17** (1987), p. 978-981.

[21] Neumann, J. von
Mathematical foundations of quantum mechanics.
Princeton: Princeton University Press, 1955. Chap. 5, Sect. 4.

[22] Orr, R.S.
Finite discrete Gabor transform relations under periodization and sampling.
In: *Proc. IEEE-SP International symposium on time-frequency and time-scale analysis*, Victoria, BC, Canada, 4-6 October 1992.
Piscataway, NJ, USA: IEEE, 1992. P. 395-398.

[23] Orr, R.S.
The order of computation for finite discrete Gabor transforms.
IEEE Trans. Signal Process., Vol. **41** (1993), p. 122-130.

[24] Papoulis, A.
Signal analysis.
New York: McGraw-Hill, 1977.

[25] Qian, S. and K. Chen, S. Li
Optimal biorthogonal sequence for finite discrete-time Gabor expansion.
Signal Process. Vol. **27** (1992), p. 177-185.

[26] Qian, S. and D. Chen
A general solution of biorthogonal analysis window functions for orthogonal-like discrete Gabor transform.
In: *Proc. IEEE-SP International symposium on time-frequency and time-scale analysis*, Victoria, BC, Canada, 4-6 October 1992.
Piscataway, NJ, USA: IEEE, 1992. P. 387-389.

[27] Qian, S. and D. Chen
Discrete Gabor transform.
IEEE Trans. Signal Process., Vol. **41** (1993), p. 2429-2438.

[28] Schempp, W.
Radar ambiguity functions, the Heisenberg group, and holomorphic theta series.
Proc. Am. Math. Soc. Vol. **92** (1984), p. 103-110.

[29] Reed, M. and B. Simon
Analysis of operators.
London: Academic Press, 1978.
Methods of modern mathematical physics, Vol. **4**

[30] Wexler, J. and S. Raz
Discrete Gabor expansions.
Signal Process., Vol. **21** (1990), p. 207-221.

[31] Whittaker, E.T. and G.N. Watson
A course of modern analysis.
London: Cambridge University Press, 1927. Chaps. 20 and 21.

[32] Zak, J.
Finite translations in solid-state physics.
Phys. Rev. Lett., Vol. **19** (1967), p. 1385-1387.

[33] Zak, J.
Dynamics of electrons in solids in external fields.
Phys. Rev., Vol. **168** (1968), p. 686-695.

[34] Zak, J.
The kq-representation in the dynamics of electrons in solids.
Solid State Physics, Vol. **27** (1972), p. 1-62.

[35] Zibulski, M. and Y.Y. Zeevi
Oversampling in the Gabor scheme.
IEEE Trans. Signal Process., Vol. **41** (1993), p. 2679-2687.