

# Finite mode analysis by means of intensity information in fractional optical systems

Tatiana Alieva and Martin J. Bastiaans

Technische Universiteit Eindhoven, Faculteit Elektrotechniek,  
Postbus 513, 5600 MB Eindhoven, Netherlands

*M.J.Bastiaans@osa.org*

It is shown how a coherent optical signal that contains only a finite number of Hermite-Gauss modes, can be reconstructed from the knowledge of its Radon-Wigner transform – associated with the intensity distribution in a fractional Fourier transform optical system – at only two transversal points. The proposed method can be generalized to any fractional system whose generator transform has a complete orthogonal set of eigenfunctions. © 2001 Optical Society of America

**OCIS codes:** 100.5070, 070.1170, 070.2590, 070.6020, 200.1130

## 1. Introduction

The phase retrieval from intensity distributions is a very old problem, which can be solved in different ways, see for example [1, 2, 3, 4, 5]. In order to create a more optimal algorithm, all possible additional knowledge about an optical signal has to be taken into account. In this paper we propose a new method of phase reconstruction, which can be applied in the case of a coherent field containing a finite number of Hermite-Gauss modes.

It was shown in [6, 7] that the Hermite-Gauss mode content of an optical signal can be determined from the knowledge of the complex field amplitude propagating through a fractional Fourier transform (FT) system, measured at one transversal point. In this paper we show that, under the assumption that the number of modes is finite, the mode content can be determined from the evolution of the *intensity distribution* in the fractional FT system [8, 9] taken at two transversal points. This implies that the phase of a coherent field with  $N + 1$  modes can be reconstructed from measurements of the intensity distributions at  $2(N + 1)$  sensor points taken over two lines parallel to the optical axis of the fractional FT system.

Recently several methods of phase retrieval based on the properties of the fractional FT were proposed: iterative and recursive algorithms for phase recovering from the intensity distributions in two fractional FT domains were discussed in [4] and [5], respectively. In the present paper we consider the possibility of phase retrieval from the evolution of the intensity distribution in the fractional FT system along two lines parallel to the optical axis. Therefore, our method of phase retrieval is based on other initial information than the Gerchberg-Saxton algorithm [4] or the recursive algorithm for phase retrieval in the fractional FT domain [5], which both use two fractional Fourier power spectra for phase reconstruction. Moreover, this approach can be generalized to any fractional transform system [10] whose generator transform has a complete orthogonal set of eigenfunctions (modes). As a useful example of such a transform in optics, we can mention the fractional Hankel transform with Laguerre-Gauss eigenfunctions.

## 2. Intensity evolution under fractional Fourier transformation

Fractional FT optical systems [8, 9] were actively discussed over the last decade as useful tools for optical signal analysis. The evolution of the complex field amplitude in the paraxial approximation of the scalar diffraction theory during propagation through a quadratic refractive index medium, is described by the fractional FT.

The fractional FT of a function  $f(x)$  can be defined as [11]

$$R^\alpha [f(x)](u) = F_\alpha(u) = \int_{-\infty}^{\infty} K(\alpha, x, u) f(x) dx, \quad (1)$$

where the kernel  $K(\alpha, x, u)$  is given by

$$K(\alpha, x, u) = \frac{\exp(i\frac{1}{2}\alpha)}{\sqrt{i2\pi \sin \alpha}} \exp\left(i \frac{(x^2 + u^2) \cos \alpha - 2ux}{2 \sin \alpha}\right). \quad (2)$$

Note that, in particular,  $F_0(u) = f(u)$ , and that  $F_{\pi/2}(u)$  corresponds to the normal FT of  $f(x)$ . The fractional FT is continuous, additive, and periodic with respect to the parameter  $\alpha$ , which can be considered as a rotation angle in the phase plane. Note, moreover, that we use dimensionless variables  $x$  and  $u$ .

The fractional power spectrum, i.e., the squared modulus of the fractional FT  $|F_\alpha(u)|^2$  associated with an intensity distribution, is also periodic in  $\alpha$  and can thus be represented as a Fourier series,

$$|F_\alpha(u)|^2 = \sum_{n=-\infty}^{\infty} Q_n(u) \exp(in\alpha), \quad (3)$$

where

$$Q_n(u) = \frac{1}{2\pi} \int_0^{2\pi} |F_\alpha(u)|^2 \exp(-in\alpha) d\alpha. \quad (4)$$

Since  $|F_\alpha(u)|^2$  is a real function, its FTs with respect to the variables  $u$  or  $\alpha$  are Hermitian; in particular we thus have  $Q_{-n}(u) = Q_n^*(u)$ , where the asterisk  $*$  denotes complex conjugation. Note that the knowledge of the two-dimensional function  $|F_\alpha(u)|^2$  of  $\alpha$  and  $u$ , which is called the Radon-Wigner transform, permits to reconstruct the Wigner distribution of the signal, and then to determine its phase [2] or, in the case of a partially coherent field, its two-point correlation function. In this paper we introduce a method that significantly reduces the number of measurements, under the assumption that the signal is coherent and contains a finite number of modes.

The Hermite-Gauss functions  $\Psi_n(x)$ , defined by

$$\Psi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} \exp(-\frac{1}{2}x^2) H_n(x) \quad (5)$$

with  $H_n(x)$  the Hermite polynomials, form a complete orthonormal set  $\{\Psi_n(x), n = 0, 1, \dots, \infty\}$  on the entire  $x$ -axis. We can thus represent a function  $f(x)$  in the form

$$f(x) = \sum_{n=0}^{\infty} f_n \Psi_n(x). \quad (6)$$

Since the Hermite-Gauss functions are the eigenfunctions for the fractional FT operator with eigenvalue  $\exp(-in\alpha)$ ,

$$K(\alpha, x, u) = \sum_{n=0}^{\infty} \Psi_n^*(x) \Psi_n(u) \exp(-in\alpha), \quad (7)$$

the fractional FT of  $f(x)$  can be expressed as [cf. Eq. (6)]

$$F_\alpha(u) = \sum_{n=0}^{\infty} f_n \exp(-in\alpha) \Psi_n(u). \quad (8)$$

### 3. A finite number of modes

Suppose that an optical signal contains only a finite number of  $N + 1$  successive modes. This implies that there are not more than  $N + 1$  non-zero coefficients  $f_n$  in the mode expansion (6) and that we can write

$$f(x) = \sum_{n=L}^{L+N} f_n \Psi_n(x) \quad (9)$$

with  $L$  ( $L \geq 0$ ) the number of the lowest-order, non-zero mode coefficient  $f_L$  and with  $L + N$  ( $N \geq 0$ ) the number of the highest-order, non-zero mode coefficient  $f_{L+N}$ . The fractional FT now takes the form [cf. Eq. (8)]

$$F_\alpha(u) = \sum_{n=L}^{L+N} f_n \exp(-in\alpha) \Psi_n(u), \quad (10)$$

and when we substitute this expression into Eq. (4), it is easy to see that

$$Q_n(u) = Q_{-n}^*(u) = \sum_{m=L}^{L+N-n} f_m f_{m+n}^* \Psi_m(u) \Psi_{m+n}^*(u) \quad (n = 0, \dots, N), \quad (11)$$

while  $Q_n(u) = 0$  for  $|n| > N$ . Note that in general  $Q_n(u)$  has a complex value, except for  $n = 0$ , for which its value is real.

If the coefficients  $Q_n(u)$  ( $n = 0, \dots, N$ ) are considered for only one value of  $u$ , the system (11) contains  $2N + 1$  real equations for  $2N + 2$  real variables  $\text{Re}f_n$  and  $\text{Im}f_n$  ( $n = L, \dots, L + N$ ). Nevertheless, to solve this system is difficult due to its nonlinearity. If the coefficients  $Q_n(u)$  ( $n = 0, \dots, N$ ) are known at two points  $u_1$  and  $u_2$  such that  $\Psi_n(u_1) \neq 0$  and  $\Psi_n(u_2) \neq 0$  for  $n = L$  and for  $n = L + N$ , the system (11) can be transformed into a recursive set of  $N$  subsystems consisting of two linear equations for two variables, which permits to determine the mode coefficients  $f_{L+n}$  ( $n = 0, \dots, N$ ) up to a constant phase factor.

To show this, we first define the normalized mode coefficients

$$\xi_m = \frac{f_{L+m}}{f_L}, \quad \eta_m = \frac{f_{L+N-m}}{f_{L+N}}, \quad (m = 1, \dots, N) \quad (12)$$

and the normalized constants

$$q_m(u) = \frac{Q_{N-m}(u)}{Q_N(u)}, \quad \psi_m(u) = \frac{\Psi_{L+m}(u)}{\Psi_L(u)}, \quad \phi_m(u) = \frac{\Psi_{L+N-m}^*(u)}{\Psi_{L+N}^*(u)}, \quad (m = 1, \dots, N). \quad (13)$$

Note that the mode coefficients  $f_{L+m} = \xi_m f_L$  ( $m = 1, \dots, N$ ) follow from the normalized mode coefficients  $\xi_m$  ( $m = 1, \dots, N$ ) and the lowest-order, non-zero mode coefficient  $f_L$ .

To determine the normalized mode coefficients  $\xi_m$  and  $\eta_m$  ( $m = 1, \dots, N$ ), we divide all rows (except the  $n = N$  one) of the system (11) by the  $n = N$  row to get

$$\xi_k \psi_k(u) + \eta_k^* \phi_k(u) = q_k(u) - c_k(u) \quad (k = 1, \dots, N) \quad (14)$$

with

$$\begin{cases} c_1(u) &= 0, \\ c_k(u) &= \sum_{m=1}^{k-1} \xi_m \eta_{k-m}^* \psi_m(u) \phi_{k-m}(u) \quad (k = 2, \dots, N). \end{cases} \quad (15)$$

It is easy to see that every next equation of the system (14), let us say the  $k$ -th one, contains only two new mode coefficients  $\xi_k$  and  $\eta_k$  compared to the  $k - 1$  previous equations, and the system can thus be

solved recursively. Note that from their definitions (12) it follows that  $\xi_N = \xi_m/\eta_{N-m} = 1/\eta_N$ ; the highest-order normalized mode coefficient  $\xi_N = 1/\eta_N$  thus follows directly from any two  $\xi_m$  and  $\eta_{N-m}$  ( $m = 1, \dots, N-1$ ), and the last equation of the system (14), the one for  $k = N$ , could in fact be omitted.

If we consider the system (14) for two points  $u_1$  and  $u_2$ , all the normalized mode coefficients  $\xi_k$  ( $k = 1, \dots, N$ ) can be consecutively recovered from the system of linear equations

$$\begin{bmatrix} \psi_k(u_1) & \phi_k(u_1) \\ \psi_k(u_2) & \phi_k(u_2) \end{bmatrix} \begin{bmatrix} \xi_k \\ \eta_k^* \end{bmatrix} = \begin{bmatrix} q_k(u_1) - c_k(u_1) \\ q_k(u_2) - c_k(u_2) \end{bmatrix} \quad (k = 1, \dots, N), \quad (16)$$

provided that  $u_1$  and  $u_2$  are chosen such that each  $2 \times 2$ -matrix constructed from  $\psi_k(u_1)$ ,  $\phi_k(u_1)$ ,  $\psi_k(u_2)$ , and  $\phi_k(u_2)$  ( $k = 1, \dots, N$ ) is non-singular.

Furthermore, from the  $n = N$  row of the system (11) and substituting from the definitions (12), we have

$$\frac{Q_N(u)}{\Psi_L(u)\Psi_{L+N}^*(u)} = |f_L|^2 \xi_N^*, \quad (17)$$

and we can determine the lowest-order, non-zero mode coefficient  $f_L$  up to a constant phase factor. We can thus conclude that if the intensity distributions  $|F_\alpha(u_1)|^2$  and  $|F_\alpha(u_2)|^2$  along two lines parallel to the optical axis are measured, all  $N+1$  complex mode coefficients  $f_n$  ( $n = L, \dots, L+N$ ) and, hence, the signal itself, can be determined up to this common constant phase factor.

Note that the constants  $q_k(u)$ ,  $\psi_k(u)$ , and  $\phi_k(u)$  depend on the number  $L$  of the lowest-order, non-zero mode. If this number is known in advance, the procedure described above can be applied without any problem. In the case that this number is not known in advance, the procedure can be applied with the arbitrary choice  $L = L_0$  first. If the reconstructed intensity distribution does not fit to the measured intensity distribution  $|f(x)|^2$  for  $L = L_0$ , we have to repeat the procedure for other values of  $L$  until a proper fit will occur.

In summary, the proposed algorithm can be divided into the following steps:

1. Calculate  $Q_n(u_1)$  and  $Q_n(u_2)$  ( $n = 0, \dots, N$ ) as the Fourier transforms with respect to  $\alpha$  of the intensity measurements  $|F_\alpha(u_1)|^2$  and  $|F_\alpha(u_2)|^2$ , see Eq. (4). We suppose that  $|F_\alpha(u)|^2$  is a bandlimited signal and that its Fourier spectrum does not contain more than  $2N+1$  nonzero elements:  $Q_n(u) = 0$  for  $|n| > N$ . Note that  $N$  can always be found if the number of sensor points is sufficiently high; then the cutoff frequency of the Fourier transform of  $|F_\alpha(u)|^2$  forms an indication of  $N$ .
2. With  $L$  known – or arbitrarily set to its lowest possible value  $L = L_0$  – form the normalized constants  $q_m(u_i)$ ,  $\psi_m(u_i)$  and  $\phi_m(u_i)$  ( $i = 1, 2; m = 1, \dots, N$ ) according to Eq. (13).
3. Determine the normalized coefficients  $\xi_k$  and  $\eta_k^*$  ( $k = 1, \dots, N$ ) by recursively solving Eq. (16), starting from  $k = 1$  and  $c_1(u_1) = c_1(u_2) = 0$ .
4. Find the mode coefficient  $f_L$  up to a constant phase factor using Eq. (17).
5. Determine the mode coefficients  $f_{L+m}$  using Eq. (12) for  $m = 1, \dots, N$ .

Note that for the implementation of the algorithm the range of the modes (with lower bound  $L$  and upper bound  $L+N$ ) has to be known *a priori*. In the case that only the number of modes  $N$  is known, but not the lower bound  $L$ , an initial value  $L = L_0$  should be chosen in step 2 to start the algorithm, and the algorithm has to be completed by some additional steps and some additional measurements of the intensity distributions, for example  $|f(x)|^2$ . The additional calculations are then the following:

6. Calculate  $|f(x_i)|^2$  for several points  $x_i$  and compare them with the measured values. In the case they do not fit, replace  $L$  by  $L+1$  and repeat the steps 2–6 until a proper fitting occurs.

In order to calculate the coefficients, we have to choose, in an optimal way, the positions of the sensors along the two lines parallel to the optical axis, and the values  $u_1$  and  $u_2$  transversal to the optical axis. To perform the simplest algorithm for the calculation of the FT (step 1), the sensor positions  $\alpha_k$  ( $k = 0, \dots, M - 1$ ) along the two lines parallel to the optical axis have to be equally spaced over  $\alpha$ , the angle parameter of the fractional FT. This implies that  $\alpha_k = 2\pi k/M$ , where  $M$  is the number of sensors. Although, theoretically, the algorithm could be implemented for  $M = N + 1$  sensor points over each line, the occurrence of noise requires a larger number  $M$ ; usually  $M$  has to be more than  $2(N + 1)$ . The choice of the transversal points  $u_1$  and  $u_2$  is based on the following considerations. First, they have to be close to the optical axis, where the paraxial approximation of the diffraction theory applies. Moreover, in order to solve the system (16), the Hermite-Gauss functions  $\Psi_n(u_i)$  ( $i = 1, 2; n = L, \dots, L + N$ ) should never lead to a singular  $2 \times 2$  matrix. Finally, using the normalized constants (13) requires that  $\Psi_n(u_1)$  and  $\Psi_n(u_2)$  for  $n = L$  and  $n = L + N$  differ from zero.

The computational complexity of the main algorithm (steps 1–5) is composed of (i) two times a calculation of the FT (corresponding to  $2M \log_2 M$  multiplication operations if the Fast FT is used, where  $M$  is the number of sensor points along the lines parallel to the optical axis), (ii) the calculation of  $N + 1$  Hermite-Gauss functions at two points (leading to  $N^2 + 5N$  multiplication operations), and (iii) the normalization procedure, the solution of the system of linear equations, and the determination of the mode coefficients (corresponding to  $12N$  multiplication operations).

As an example let us consider the reconstruction of the optical field described by

$$f(x) = 0.54 x^4 \exp(-x^2/2)(1 + ix^2\sqrt{3}), \quad (18)$$

for which the lowest-order, non-zero Hermite-Gauss mode is  $\Psi_0(x)$  and the highest-order is  $\Psi_6(x)$ ; hence  $L = 0$  and  $N = 6$ . The evolution of the intensity distribution taken in the transversal points  $u_1 = -1$  and  $u_2 = 0$  was chosen for further processing; the number of sensor points along the lines parallel to the optical axis was taken equal to 16. The intensity distribution was corrupted by additive white Gaussian noise with an intensity of 5% of its current value. The amplitude and the phase of the original and reconstructed signals are represented in Figures 1 and 2, respectively. The solid line in these figures corresponds to the original function, while the signs  $*$  and  $\circ$  correspond to the function reconstructed from one particular realization and from the average of 30 realizations, respectively. Note that the phase might not be reconstructed accurately, if the amplitude is low.

#### 4. Conclusions

Although this paper was restricted to the reconstruction of a signal with a finite number of Hermite-Gauss functions, the same method can be used in the more general case that a signal can be decomposed into a finite set of orthogonal functions that are the eigenfunctions of a cyclic transform, i.e., a transform that produces the identity transform after  $M$  consecutive operations. It was shown in [10] that one of the possibilities to define a fractional transform that is associated with a given cyclic generator transform, has as its kernel

$$K(\alpha, x, u) = \sum_{n=0}^{\infty} \Phi_n^*(x) \Phi_n(u) \exp(in\alpha l), \quad (19)$$

where  $\{\Phi_n(x)\}$  is the orthonormal set of the eigenfunctions of the generator transform  $K(2\pi/M, x, u)$ , and  $l$  is an integer. For the fractional FT defined by Eq. (2),  $l = -1$  and  $\Phi_n(x)$  are the Hermite-Gauss functions; for the fractional Hankel transform,  $l = +1$  and  $\Phi_n(x)$  are the Laguerre-Gauss functions [12]. If a signal now contains only a finite number of the modes  $\Phi_n(x)$ , it can be reconstructed from the evolution of the intensity distributions in the corresponding fractional system taken at two transversal points. A useful example of the fractional optical system is the fractional Hankel transform system: it permits to reconstruct the Laguerre-Gauss mode content of rotationally symmetric optical beams.

## References

1. M. R. Teague, "Deterministic phase retrieval: a Green's function solution," *J. Opt. Soc. Am.* **73**, 1434-1441 (1983).
2. M. G. Raymer, M. Beck, and D. F. McAlister "Complex wave-field reconstruction using phase-space tomography," *Phys. Rev. Lett.* **72**, 1137-1140 (1994).
3. J. Tu, S. Tamura, "Analytic relation for recovering the mutual intensity by means of intensity information," *J. Opt. Soc. Am. A* **15**, 202-206 (1998).
4. Z. Zalevsky and R. G. Dorsch, "Gerchberg-Saxton algorithm applied in the fractional Fourier or the Fresnel domain," *Opt. Lett.* **21**, 842-844 (1996).
5. W. X. Cong, N. X. Chen, and B. Y. Gu, "Recursive algorithms for phase retrieval in the fractional Fourier transform domain," *Appl. Opt.* **37**, 6906-6910 (1998).
6. T. Alieva and M. J. Bastiaans, "Mode analysis through the fractional transforms in optics," *Opt. Lett.* **24**, 1206-1208 (1999).
7. T. Alieva and K. B. Wolf, "Finite mode analysis through harmonic waveguides," *J. Opt. Soc. Am. A* **17**, 1482-1484 (2000).
8. A. W. Lohmann, D. Mendlovic, Z. Zalevsky, "Fractional transformation in optics," in *Progress in Optics*, vol. **XXXVIII**, ed. E. Wolf, Elsevier, Amsterdam, pp. 263-342 (1998).
9. H. M. Ozaktas, M. A. Kutay, and D. Mendlovic, "Introduction to the fractional Fourier transform and its applications," in *Advances in Imaging and Electron Physics*, vol. **106**, ed. P. W. Hawkes, Academic Press, San Diego, CA, pp. 239-291 (1999).
10. T. Alieva and M. L. Calvo, "Fractionalization of the cyclic transforms," *J. Opt. Soc. Am. A* **17**, 2330-2338 (2000).
11. L. B. Almeida, "The fractional Fourier transform and time-frequency representations," *IEEE Trans. Signal Process.* **42**, 3084-3091 (1994).
12. L. Yu, W. D. Huang, M. C. Huang, Z. Z. Zhu, X. M. Zeng, W. Ji, "The Laguerre-Gaussian series representation of 2-dimensional fractional Fourier transform," *J. Phys. A: Math. and Gen.* **31**, 9353-9357 (1998).

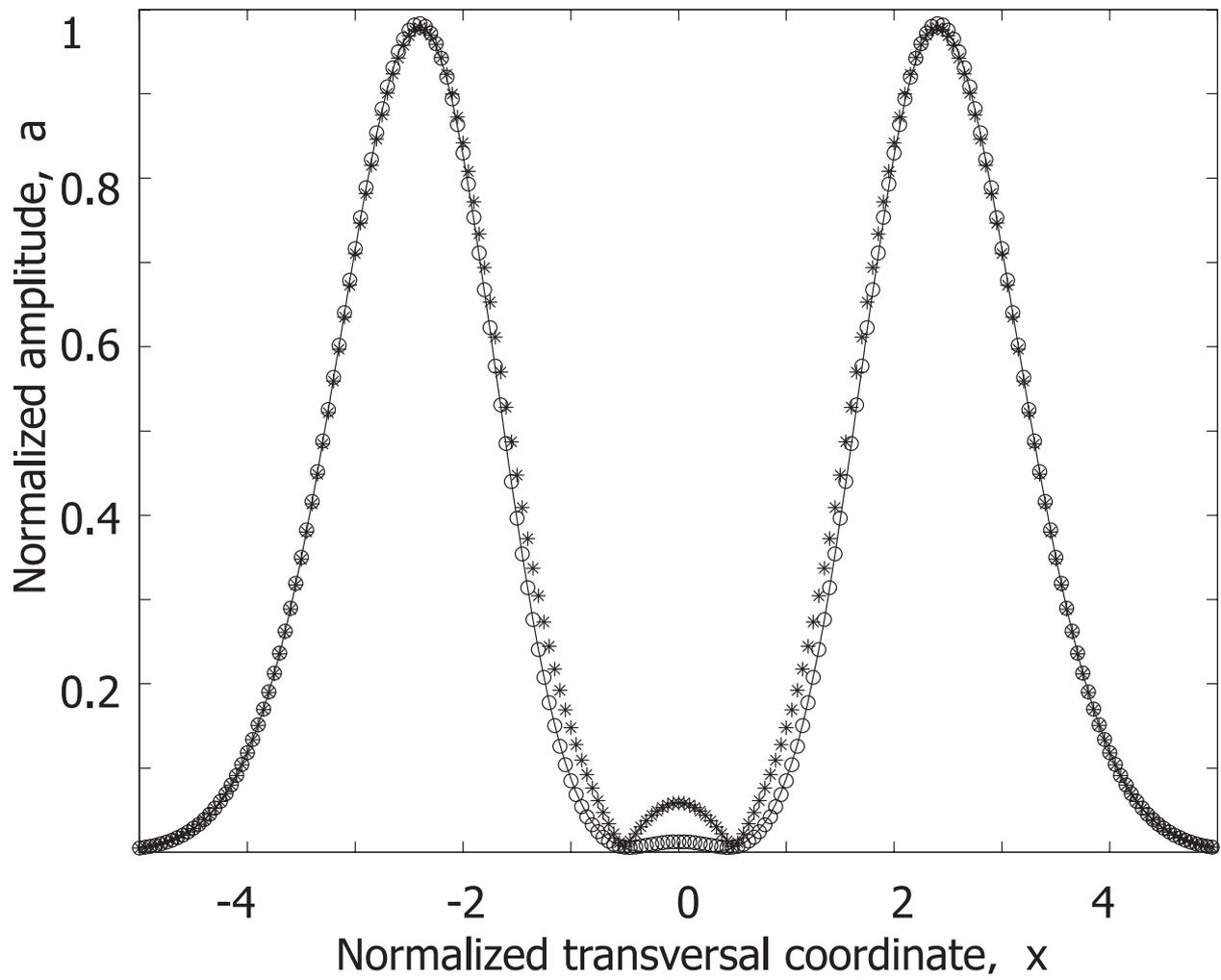


Fig. 1. The amplitude of  $f(x)$ .

Figure 1

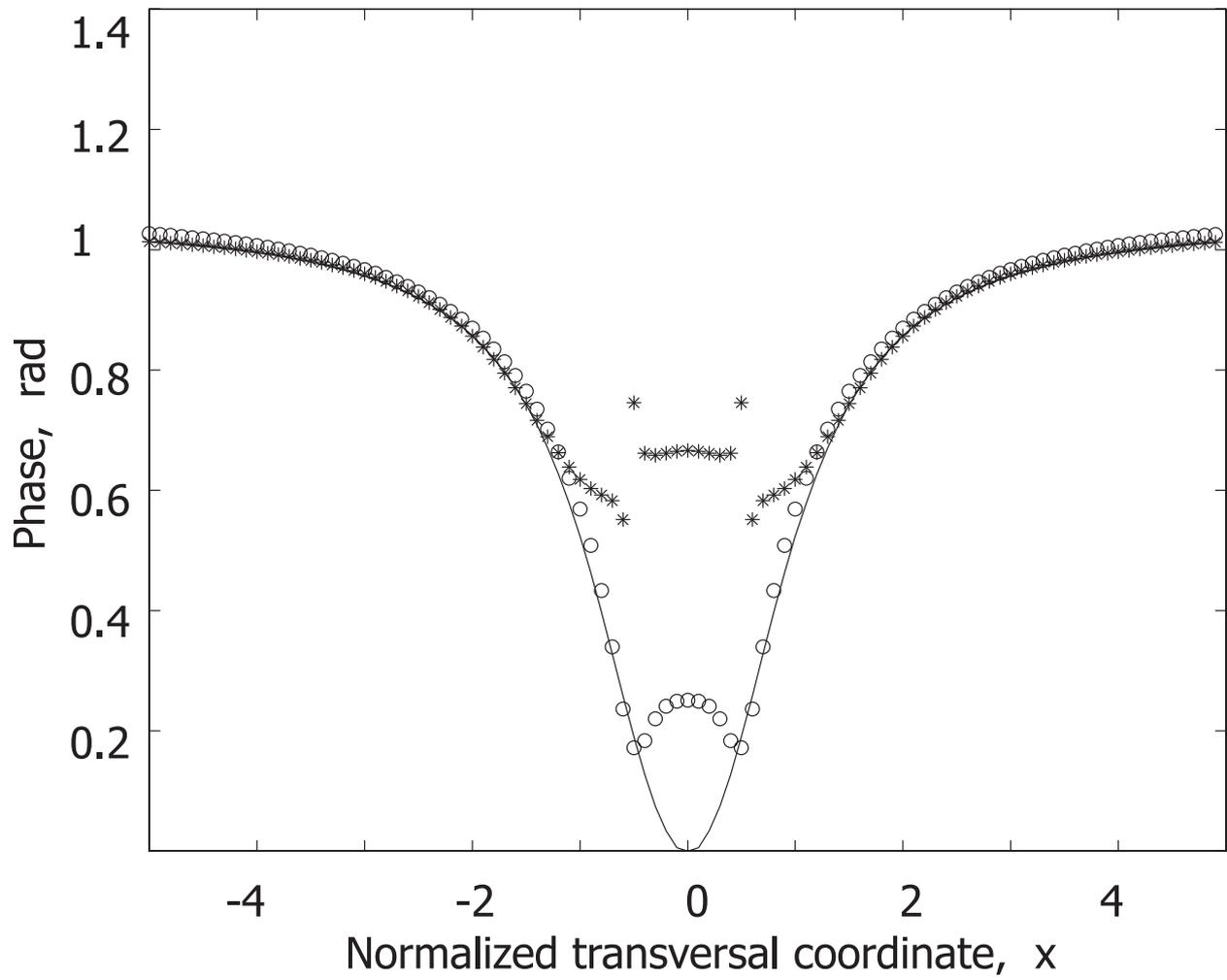


Fig. 2. The phase of  $f(x)$ .

Figure 2