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Fractional cosine and sine transforms

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The fractional cosine and sine transforms – closely related to the fractional Fourier transform, which is now actively used in optics and signal processing – are introduced and their main properties and possible applications are discussed.

1. Introduction

The fractional Fourier transform, which is a generalization of the ordinary Fourier transform (FT), was introduced 70 years ago, but only in the last decade has it been actively applied in signal processing, optics and quantum mechanics. The fractional FT gives a more complete representation of the signal in phase space and enlarges the number of applications of the ordinary FT.[1]

In addition to the FT, the cosine and sine transforms (CT, ST), which are based on half-range expansions of a function over cosine and sine basis functions, respectively, are also important tools in signal processing. Despite of some lack of elegance in their properties with respect to the FT, the CT and ST have their own areas of applications. The idea of fractionalization of the CT and ST was proposed in.[2] There the real and imaginary parts of the fractional FT kernel were chosen as the kernels for a fractional CT and a fractional ST, respectively. Nevertheless, the authors note that their fractional transforms are not index additive and, in our point of view, cannot be considered as a fractional version of the CT and ST.

In this paper we introduce fractional cosine and sine transforms that are additive on the index and preserve the similar relationships with the fractional FT as the ordinary CT and ST have with the FT. We derive the main properties of the fractional CT and ST and show, as examples, the fractional CT of some selected signals. Note that, although there are different ways for the fractionalization of cyclic transforms[3] like the FT, the CT, and the ST, in this paper we consider the fractional CT and ST in relation to the fractional FT, which is more useful for signal analysis because the fractional FT corresponds to a rotation of the Wigner distribution and the ambiguity function.

2. Fractional Fourier transform

The fractional FT $F^\alpha(u)$ of a function $f(x)$ is defined as[1]

$$F^\alpha(u) = R_F^\alpha[f(x)](u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k_\alpha(x,u)\, f(x) \exp(-jux/\sin\alpha)dx, \tag{1}$$

where

$$k_\alpha(x,u) = \frac{\exp(j\frac{1}{2}\alpha)}{\sqrt{j\sin\alpha}} \exp[\tfrac{1}{2} j(x^2+u^2)\cot\alpha]. \tag{2}$$

Note that for $\alpha = \frac{1}{2}\pi$, for which $k_{\pi/2}(x,u) = 1$, we have the normal FT, while for $\alpha \to 0$ we have the identity transformation: $F^0(x) = f(x)$. Note, moreover, that $k_{\alpha+\pi}(x,u) = k_\alpha(x,u)$, and hence $F^{\alpha+\pi}(u) = F^\alpha(-u)$, that $k_{-\alpha}(x,u) = k_\alpha^*(x,u)$, that $k_\alpha(x,u) = k_\alpha(u,x)$, and that $k_\alpha(x,u)$ is an even function of both x and u: $k_\alpha(\pm x,u) = k_\alpha(x,u) = k_\alpha(x,\pm u)$.

From the linearity of the fractional FT and the reversion property

$$R_F^\alpha[f(-x)](u) = R_F^\alpha[f(x)](-u) = F^\alpha(-u), \tag{3}$$

we have

$$R_F^\alpha[f(x) \pm f(-x)](u) = F^\alpha(u) \pm F^\alpha(-u), \tag{4}$$

and we conclude that the fractional FT of an even function is even, while the fractional FT of an odd function is odd.

3. Fractional cosine and sine transforms

We now restrict ourselves to a one-sided function $f(x)$, with $f(x) = 0$ for $x < 0$, and define the fractional CT and ST as

$$\begin{aligned} F_c^\alpha(u) &= R_c^\alpha[f(x)](u) = R_F^\alpha[f(x) + f(-x)](u) = F^\alpha(u) + F^\alpha(-u) \\ &= \sqrt{\frac{2}{\pi}} \int_0^\infty k_\alpha(x,u)\, f(x) \cos(ux/\sin\alpha) dx, \qquad (u \geq 0) \end{aligned} \tag{5}$$

$$\begin{aligned} F_s^\alpha(u) &= R_s^\alpha[f(x)](u) = \exp(j\alpha)\; R_F^\alpha[f(x) - f(-x)](u) = \exp(j\alpha)\; [F^\alpha(u) - F^\alpha(-u)] \\ &= -j \exp(j\alpha) \sqrt{\frac{2}{\pi}} \int_0^\infty k_\alpha(x,u)\, f(x) \sin(ux/\sin\alpha) dx, \qquad (u \geq 0) \end{aligned} \tag{6}$$

respectively, which reduce to the normal CT and ST[4] for $\alpha = \frac{1}{2}\pi$. To express the relationship between the fractional FT of a one-sided function to the fractional CT and ST of this function in a different way, we can write

$$2F^\alpha(\pm u) = F_c^\alpha(u) \pm \exp(-j\alpha) F_s^\alpha(u). \tag{7}$$

Considering the kernels of the fractional transforms R^α,

$$\begin{aligned} R_F^\alpha &: (1/\sqrt{2\pi}) k_\alpha(x,u) \exp(-jux/\sin\alpha), \\ R_c^\alpha &: (2/\sqrt{2\pi}) k_\alpha(x,u) \cos(ux/\sin\alpha), \\ j\exp(-j\alpha) R_s^\alpha &: (2/\sqrt{2\pi}) k_\alpha(x,u) \sin(ux/\sin\alpha), \end{aligned}$$

we can say that R_c^α is related to the even part of R_F^α, while R_s^α is related to the odd part of it. In general we conclude that, to determine the fractional CT of a one-sided function $f(x)$, we can as well determine the fractional FT of the evenly extended two-sided function $f(x) + f(-x)$; similarly, to determine the fractional ST of a one-sided function, we can as well determine the fractional FT of the oddly extended two-sided function $\exp(j\alpha)[f(x) - f(-x)]$. And in both cases we restrict ourselves to $u \geq 0$.

4. Some basic properties

From the general observations made in the previous section, we conclude that many properties of the fractional FT immediately translate to the fractional CT and ST. In particular, for all fractional transforms the additivity property for the angle α holds,

$$R^{\alpha_1} R^{\alpha_2}[f(x)](u) = R^{\alpha_1+\alpha_2}[f(x)](u), \tag{8}$$

from which we conclude that the inverse of any fractional transform corresponds to the transform with the negative angle.

With $\tan\beta = \lambda^2 \tan\alpha$ (and with the additional condition $\lambda > 0$ in the case of the fractional CT and ST) and with C defined as

$$C = \sqrt{\frac{\cos\beta}{\cos\alpha}}\, \frac{\exp(j\frac{1}{2}\alpha)}{\exp(j\frac{1}{2}\beta)} \exp\left[j\tfrac{1}{2}u^2 \cot\alpha \left(1 - \frac{\cos^2\beta}{\cos^2\alpha}\right)\right],$$

we have the same scaling property for the fractional CT as we have for the fractional FT,

$$R_c^\alpha[f(\lambda x)](u) = C R_c^\beta[f(x)]\left(\frac{u}{\lambda}\frac{\sin\beta}{\sin\alpha}\right), \tag{9}$$

and a slightly modified one for the fractional ST:

$$\exp(-j\alpha) R_s^\alpha[f(\lambda x)](u) = C \exp(-j\beta) R_s^\beta[f(x)]\left(\frac{u}{\lambda}\frac{\sin\beta}{\sin\alpha}\right). \tag{10}$$

If we shift a one-sided function $f(x)$ away from the origin, $f(x) \to f(x - x_o)$ with $x_o > 0$, we have the same shifting property for the fractional CT and ST (with the additional condition $\cos\alpha \geq 0$ in these cases) as we have for the fractional FT:

$$R^\alpha[f(x - x_o)](u) = \exp[-jx_o \sin\alpha(u - \tfrac{1}{2}x_o\cos\alpha)] R^\alpha[f(x)](u - x_o\cos\alpha). \tag{11}$$

As far as modulation, or shifting in the u-domain, is concerned, we have

$$R_F^\alpha[f(x)](u - u_o\sin\alpha) = \exp[-ju_o\cos\alpha(u - \tfrac{1}{2}u_o\sin\alpha)] R_F^\alpha[f(x)\exp(ju_o x)](u) \tag{12}$$

for the fractional FT, while for the fractional CT and ST we have

$$\begin{aligned} R_{c,s}^\alpha[f(x)](u - u_o\sin\alpha) &= \exp[-ju_o\cos\alpha(u - \tfrac{1}{2}u_o\sin\alpha)] \\ &\times \left\{ R_{c,s}^\alpha[f(x)\cos(u_o x)](u) + j\exp(\mp j\alpha) R_{s,c}^\alpha[f(x)\sin(u_o x)](u)\right\}, \end{aligned} \tag{13}$$

where the first subscript corresponds to the upper ($-$) sign and the second subscript to the lower ($+$) sign.

Parseval's unitarity relation for one-sided functions reads

$$\int_0^\infty f(x) g^*(x) dx = \int_{-\infty}^\infty F^\alpha(u)[G^\alpha(u)]^* du = \int_0^\infty F_{c,s}^\alpha(u)[G_{c,s}^\alpha(u)]^* du. \tag{14}$$

All three fractional transforms satisfy the symmetry relation

$$R^{-\alpha}[f(x)](u) = \left\{R^\alpha[f^*(x)](u)\right\}^*. \tag{15}$$

And while the fractional FT is periodic in α with period 2π and satisfies the half-period relation $R_F^{\alpha+\pi}[f(x)](u) = R_F^\alpha[f(x)](-u)$, the fractional CT and ST are periodic with period π: $R_{c,s}^{\alpha+\pi}[f(x)](u) = R_{c,s}^\alpha[f(x)](u)$.

5. Eigenfunctions and eigenvalues

With $\Psi_n(x)$ the Hermite-Gauss functions,

$$\Psi_n(x) = (\sqrt{\pi}2^n n!)^{-\frac{1}{2}} \exp(-\tfrac{1}{2}x^2) H_n(x), \tag{16}$$

where $H_n(x)$ are the Hermite polynomials, we have

$$
\begin{aligned}
&(1/\sqrt{2\pi})k_\alpha(x,u)\exp(-jux/\sin\alpha) \\
&\qquad = \frac{\exp(j\frac{1}{2}\alpha)}{\sqrt{j\sin\alpha}}\exp[\tfrac{1}{2}j(x^2+u^2)\cot\alpha]\exp(-jux/\sin\alpha) = \sum_{n=0}^{\infty}\Psi_n^*(x)\Psi_n(u)\exp(-jn\alpha).
\end{aligned}
\tag{17}
$$

Thus

$$
\begin{aligned}
&(2/\sqrt{2\pi})k_\alpha(x,u)\cos(ux/\sin\alpha) \\
&\qquad = (1/\sqrt{2\pi})k_\alpha(x,u)[\exp(-jux/\sin\alpha)+\exp(jux/\sin\alpha)] \\
&\qquad = (1/\sqrt{2\pi})\left[k_\alpha(x,u)\exp(-jux/\sin\alpha)+k_\alpha(x,-u)\exp(-j(-u)ux/\sin\alpha)\right] \\
&\qquad = \sum_{n=0}^{\infty}\Psi_n^*(x)\Psi_n(u)\exp(-jn\alpha)+\sum_{n=0}^{\infty}\Psi_n^*(x)\Psi_n(-u)\exp(-jn\alpha) \\
&\qquad = \sum_{n=0}^{\infty}\Psi_n^*(x)[\Psi_n(u)+\Psi_n(-u)]\exp(-jn\alpha) = 2\sum_{n=0}^{\infty}\Psi_{2n}^*(x)\Psi_{2n}(u)\exp(-j2n\alpha)
\end{aligned}
\tag{18}
$$

and, similarly,

$$
-j\exp(j\alpha)(2/\sqrt{2\pi})k_\alpha(x,u)\sin(ux/\sin\alpha) = 2\sum_{n=0}^{\infty}\Psi_{2n+1}^*(x)\Psi_{2n+1}(u)\exp(-j2n\alpha). \tag{19}
$$

We conclude that, while the Hermite-Gauss functions $\Psi_n(x)$ are the eigenfunctions of the fractional FT with eigenvalues $\exp(-jn\alpha)$, the even-order Hermite-Gauss functions $\sqrt{2}\Psi_{2n}(x)$ are the eigenfunctions of the fractional CT and the odd-order Hermite-Gauss functions $\sqrt{2}\Psi_{2n+1}(x)$ the eigenfunctions of the fractional ST, in both cases with eigenvalues $\exp(-j2n\alpha)$. Note that the fractional CT eigenfunctions $\sqrt{2}\Psi_{2n}(x)$ are orthonormal on the half range,

$$
2\int_0^\infty \Psi_{2n}(x)\Psi_{2m}(x)dx = \delta_{n,m},
$$

and that the same holds for the fractional ST eigenfunctions $\sqrt{2}\Psi_{2n+1}(x)$.

6. Possible applications

It is well known that the fractional FT kernel is, except for a constant phase shift, the propagator of the nonstationary Schrödinger equation for the harmonic oscillator. This equation also describes, in the paraxial approximation of the diffraction theory, the beam propagation through a quadratic refractive index medium, an optical fiber or a thin lens, for example. In a similar way, the fractional CT and ST are closely related to the propagator of the nonstationary Schrödinger equation for the half of the quadratic potential

$$
V(x) = \begin{cases} \infty & x<0 \\ \frac{1}{2}kx^2 & x>0 \end{cases} \tag{20}
$$

The fractional CT and ST satisfy the different boundary conditions for $u=0$: $F_s^\alpha(u)=0$ and $\partial F_c^\alpha(u)/\partial u = 0$, respectively.

The fractional CT and ST can be used for separately processing the even and odd parts of two-sided functions. Indeed, with $f_e(x)$ and $f_o(x)$ $(x \geq 0)$ defined as the even and the odd part of the two-sided function $f(x)$, respectively, we can write $f(\pm x) = f_e(x) \pm f_o(x) \quad (x \geq 0)$ and the fractional FT of $f(x)$ can be expressed as

$$R_F^\alpha[f(x)](\pm u) = R_c^\alpha[f_e(x)](u) \pm \exp(-j\alpha) R_s^\alpha[f_o(x)](u) \quad (u \geq 0). \tag{21}$$

In a similar way as for the fractional FT, the fractional Hartley transform of $f(x)$, $R_H^\alpha[f(x)](u)$, can be expressed in terms of the fractional CT and ST,[3]

$$R_H^\alpha[f(x)](\pm u) = R_c^\alpha[f_e(x)](u) \pm j\exp(-j\alpha) R_s^\alpha[f_o(x)](u) \quad (u \geq 0), \tag{22}$$

and its kernel thus reads $(1/\sqrt{2\pi})k_\alpha(x,u)[\cos(ux/\sin\alpha) + \sin(ux/\sin\alpha)]$.

7. Selected fractional cosine and sine transforms

We now consider the fractional CT and ST of some selected functions.

Dirac function

The fractional CT and ST of the Dirac function $f(x) = \delta(x - \xi)$ correspond to the respective kernels:

$$R_c^\alpha[\delta(x-\xi)](u) = \sqrt{2/\pi}\, k_\alpha(\xi,u)\cos(u\xi/\sin\alpha), \tag{23}$$
$$R_s^\alpha[\delta(x-\xi)](u) = -j\exp(j\alpha)\sqrt{2/\pi}\, k_\alpha(\xi,u)\sin(u\xi/\sin\alpha). \tag{24}$$

Cosine and sine functions

Substituting $\alpha = \frac{1}{2}\pi$ into Eqs. (23) and (24) yields

$$\sqrt{2/\pi}\cos(u\xi) = R_c^{\pi/2}[\delta(x-\xi)](u) = R_c^{\pi/2}[\delta(x-u)](\xi),$$
$$-j\exp(j\alpha)\sqrt{2/\pi}\sin(u\xi) = R_s^{\pi/2}[\delta(x-\xi)](u) = R_s^{\pi/2}[\delta(x-u)](\xi).$$

From the additivity property of the fractional transforms we can then write

$$R_c^\alpha[\cos(u_o\xi)](u) = \sqrt{\pi/2}\, R_c^{\alpha+\pi/2}[\delta(x-u_o)](u) = k_{\alpha+\pi/2}(u_o,u)\cos[uu_o/\sin(\alpha+\pi/2)],$$
$$R_s^\alpha[\sin(u_o\xi)](u) = j\exp(-j\alpha)\sqrt{\pi/2}\, R_c^{\alpha+\pi/2}[\delta(x-u_o)](u) = k_{\alpha+\pi/2}(u_o,u)\sin[uu_o/\sin(\alpha+\pi/2)],$$

and the fractional CT and ST of $\cos(u_o x)$ and $\sin(u_o x)$, respectively, take the form

$$R_c^\alpha[\cos(u_o x)](u) = k_{\alpha+\pi/2}(u_o,u)\cos(uu_o/\cos\alpha), \tag{25}$$
$$R_s^\alpha[\sin(u_o x)](u) = k_{\alpha+\pi/2}(u_o,u)\sin(uu_o/\cos\alpha). \tag{26}$$

Constant function

The fractional CT of a constant $f(x) = c$ results in a chirp,

$$R_c^\alpha[c](u) = c\sqrt{\frac{\exp(j\alpha)}{\cos\alpha}}\exp(-\tfrac{1}{2}ju^2\tan\alpha), \tag{27}$$

which reduces to the Dirac function $\sqrt{2\pi}\delta(u)$ for $\alpha \to \frac{1}{2}\pi$. The fractional ST is related to the fractional CT by

$$R_s^\alpha[c](u) = -\exp(j\alpha)\Phi\left(u/\sqrt{j\sin 2\alpha}\right) R_c^\alpha[c](u), \tag{28}$$

where $\Phi(z) = \dfrac{2}{\sqrt{\pi}} \int_o^z \exp(-t^2)dt$ is the probability integral.

The fractional transforms (27) and (28) can be derived from the general relationship[5]

$$\int_0^\infty x^{\mu-1} \exp(-\gamma x - \tfrac{1}{2}\beta x^2)dx = \frac{\Gamma(\mu)}{\beta^{\mu/2}} \exp\left(\frac{\gamma^2}{4\beta}\right) D_{-\mu}\left(\frac{\gamma}{\beta^{1/2}}\right), \tag{29}$$

where [Re $\mu > 0$, Re $\beta > 0$] or [Re $\mu > 0$, Re $\gamma > 0$, Re $\beta = 0$] or [$0 <$ Re $\mu < 2$, Re $\beta =$ Re $\gamma = 0$, Im $\beta \neq 0$], and where $D_p(z)$ is the parabolic cylinder function. Note that for the fractional transform of a constant the parabolic cylinder function $D_p(z)$ for $p = -1$ arises and we have $D_{-1}(z) = \exp(\frac{1}{4}z^2)\sqrt{\pi/2}[1 - \Phi(z/\sqrt{2})]$.

Complex Gaussian function

The fractional CT of the complex Gaussian function $f(x) = \exp(-\frac{1}{2}\beta x^2)$ with Re $\beta \geq 0$ reads

$$R_c^\alpha[\exp(-\tfrac{1}{2}\beta x^2)](u) = \sqrt{\frac{\exp(j\alpha)}{j\eta \sin\alpha}} \exp\left[\tfrac{1}{2}u^2\left(j\cot\alpha - \frac{1}{\eta\sin^2\alpha}\right)\right], \tag{30}$$

where $\eta = \beta - j\cot\alpha$. The fractional ST is related to the fractional CT by

$$R_s^\alpha[\exp(-\tfrac{1}{2}\beta x^2)](u) = -\exp(j\alpha)\Phi\left(ju/\sin\alpha\sqrt{2\eta}\right) R_c^\alpha[\exp(-\tfrac{1}{2}\beta x^2)](u). \tag{31}$$

Note that for $\beta = 1$ the Gaussian function $\exp(-\frac{1}{2}x^2)$ is indeed an eigenfunction of the fractional CT, and that for $\beta = 0$ the fractional transforms (30) and (31) reduce to Eqs. (27) and (28), respectively.

Chirp function

For $\beta = jc$, and hence $\eta = j(c - \cot\alpha)$, the complex Gaussian reduces to a chirp, $f(x) = \exp(-\frac{1}{2}jcx^2)$, for which the fractional CT takes the form of a chirp again:

$$R_c^\alpha[\exp(-\tfrac{1}{2}jcx^2)](u) = \sqrt{\frac{\exp(j\alpha)}{\cos\alpha - c\sin\alpha}} \exp\left[\tfrac{1}{2}ju^2\frac{c+\tan\alpha}{c\tan\alpha - 1}\right]. \tag{32}$$

The ST in this case takes the form

$$R_s^\alpha[\exp(-\tfrac{1}{2}jcx^2)](u) = -\exp(j\alpha)\Phi\left(ju/\sin\alpha\sqrt{2(c-\cot\alpha)}\right) R_c^\alpha[\exp(-\tfrac{1}{2}jcx^2)](u). \tag{33}$$

The function $f(x) = x^{\mu-1}\exp(-\gamma x - \frac{1}{2}\beta x^2)$

Most of the functions considered above belong to the general class of functions of the form $f(x) = x^{\mu-1}\exp(-\gamma x - \frac{1}{2}\beta x^2)$, with the constraints on μ, γ, β as given before: [Re $\mu > 0$, Re $\beta > 0$] or [Re $\mu > 0$, Re $\gamma > 0$, Re $\beta = 0$] or [$0 <$ Re $\mu < 2$, Re $\beta =$ Re $\gamma = 0$, Im $\beta \neq 0$]. From the general relationship (29) we get

$$\begin{aligned} R_c^\alpha[x^{\mu-1}\exp(-\gamma x - \tfrac{1}{2}\beta x^2)](u) &= \sqrt{\frac{\exp(j\alpha)}{2\pi j\eta^\mu \sin\alpha}}\,\Gamma(\mu)\exp\left[\tfrac{1}{2}u^2\left(j\cot\alpha - \frac{1}{2\eta\sin^2\alpha}\right) + \frac{\gamma^2}{4\eta}\right] \\ &\times \left\{\exp\left(-\frac{j\gamma u}{2\eta\sin\alpha}\right) D_{-\mu}\left(\frac{\gamma - ju/\sin\alpha}{\sqrt{\eta}}\right) + \exp\left(\frac{j\gamma u}{2\eta\sin\alpha}\right) D_{-\mu}\left(\frac{\gamma + ju/\sin\alpha}{\sqrt{\eta}}\right)\right\} \end{aligned} \tag{34}$$

and

$$R_s^\alpha[x^{\mu-1}\exp(-\gamma x-\tfrac{1}{2}\beta x^2)](u)=\sqrt{\frac{\exp(j\alpha)}{2\pi j\eta^\mu\sin\alpha}}\,\Gamma(\mu)\exp\left[\tfrac{1}{2}u^2\left(j\cot\alpha-\frac{1}{2\eta\sin^2\alpha}\right)+\frac{\gamma^2}{4\eta}-j\alpha\right]$$
$$\times\left\{\exp\left(-\frac{j\gamma u}{2\eta\sin\alpha}\right)D_{-\mu}\left(\frac{\gamma-ju/\sin\alpha}{\sqrt{\eta}}\right)-\exp\left(\frac{j\gamma u}{2\eta\sin\alpha}\right)D_{-\mu}\left(\frac{\gamma+ju/\sin\alpha}{\sqrt{\eta}}\right)\right\},\quad (35)$$

with $\eta=\beta-j\cot\alpha$.

8. Conclusion

We have introduced the fractional cosine and sine transforms, which are additive with respect the parameter α and which are closely related to the fractional FT. The main properties and possible application of these transforms were discussed. Although we have considered only the one-dimensional case, the generalization to the two- and higher-dimensional cases is straightforward. The fractional CT and ST can be considered as elementary fractional transforms, which are able to treat the even and odd parts of the signal separately, and which, when combined, lead to the fractional Fourier and Hartley transforms.

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