Gabor's signal expansion in optics

Martin J. Bastiaans

1 Introduction

In his original paper, Gabor suggested the representation of a time signal in a combined time-frequency domain. Actually he proposed to represent the signal as a superposition of shifted and modulated versions of a so-called elementary signal. Moreover, as an elementary signal he chose a Gaussian signal, because such a signal has a good localization, both in the time domain and in the frequency domain.

In this chapter we will consider some applications of Gabor's ideas in the field of optics. In optics, signals not only depend on the time variable t, but also on the space vector $\boldsymbol{r} = (x, y, z)$. In fact, the space dependence is often much more important than the time dependence. To see how Gabor's ideas can be translated to the spatial domain, we shall confine ourselves in this chapter to strictly time-harmonic optical signals with temporal frequency ω. Such an optical signal can be described, for instance, by $\Re\{\varphi(x, y, z) \exp(-j\omega t)\}$, where $\Re$ denotes the real part and where the complex amplitude $\varphi(x, y, z)$ contains the relevant spatial information of the signal.

Very often the optical signal propagates through some optical medium from a certain input plane $z = z_i$, say, to an output plane $z = z_o$. In that case it suffices to know – in an arbitrary plane z = constant – the complex amplitude $\varphi(x, y)$ that depends on the transverse coordinates x and y only; we will see in Section 2 that the z-dependence of the complete signal $\varphi(x, y, z)$ follows from the properties of the medium in which the optical signal is propagating. In Section 2 we will also show how we can interpret the two-dimensional Fourier transform $\hat{\varphi}(u, v)$ of a function $\varphi(x, y)$ in physical terms. Throughout this entire chapter we will denote the Fourier transform of a function by the same symbol as the function itself, but marked by a hat on top of the symbol.

In Section 3 we will consider Gabor's signal expansion and its inverse – the Gabor transform, with the help of which Gabor's expansion coefficients can be determined – for optical signals. For convenience we will consider optical signals that depend on one transverse coordinate x only, and that do not depend on y. In that case we can restrict ourselves to the one-dimensional function $\varphi(x)$ and its Fourier transform $\hat{\varphi}(u)$. The extension to the more general, two-dimensional case is rather straightforward. We will pay special attention to the case of a Gaussian elementary signal, which is intimately related to the optically important Gaussian beam. Moreover, we will use Section 3 to do some preparatory, theoretical work on critical sampling, on integer oversampling and the product form of the Gabor transform in terms of the Zak transform, and on the windowed Fourier transform, expressed as an interpolation of the Gabor transform; the results of this work will then be used in Sections 4 and 5.

The propagation of an optical signal, described in terms of its Gabor coefficients, will be treated in Section 4. In this section we will restrict ourselves to the case of critical sampling of the space-frequency domain, and we will study in more detail the concept of degrees of freedom of a signal.

Finally, in Section 5 a coherent-optical setup will be considered, with which Gabor's expansion coefficients can be generated.

2 Some optics fundamentals

In this section we will show how the exponential $\exp[j(k_x x + k_y y)]$ that plays a central role in the Fourier transform and in Gabor's expansion of an optical signal, can be given a physical interpretation. We will derive such an interpretation considering only one of the simplest systems in which an optical signal can propagate, viz. vacuum.

Since an optical signal is in essence an electromagnetic phenomenon, described by an electric field vector $\boldsymbol{E}$ and a magnetic field vector $\boldsymbol{H}$, its propagation is governed by *Maxwell's equations*. In vacuum – and in our special case of a harmonic time dependence – these equations take the form (see, for instance, [Goo96])

$$\begin{array}{rcl} \nabla \times \boldsymbol{E} &=& j\omega\mu\boldsymbol{H} \\ \nabla \times \boldsymbol{H} &=& -j\omega\epsilon\boldsymbol{E} \\ \nabla \cdot \epsilon\boldsymbol{E} &=& 0 \\ \nabla \cdot \mu\boldsymbol{H} &=& 0. \end{array} \tag{1.1}$$

Here μ and ϵ are the permeability and permittivity, respectively, of the medium in which the optical signal is propagating, $\times$ and $\cdot$ represent a vector cross product and a vector dot product, respectively, while ∇ represents the gradient operator, which in Cartesian coordinates takes the form $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$.

From Maxwell's equations (1.1) and using the vector identity

$$\nabla \times (\nabla \times \boldsymbol{E}) = \nabla(\nabla\cdot \boldsymbol{E}) - (\nabla\cdot\nabla)\boldsymbol{E},$$

we can easily derive the equation

$$(\nabla^2 + k^2)\boldsymbol{E} = 0 \tag{1.2}$$

for the electric field vector $\boldsymbol{E}$, and a similar one for the magnetic field vector $\boldsymbol{H}$; in Eq. (1.2) we have introduced the Laplacian operator $\nabla^2 = \nabla\cdot\nabla$, and the wave number $k = \omega/c$, with $c = 1/\sqrt{\mu\epsilon}$ the velocity of propagation in vacuum. An equation of the form (1.2) is known as a *Helmholtz equation* (see, for instance, [Goo96]).

We note that an equation of the form (1.2) is obeyed by both $\boldsymbol{E}$ and $\boldsymbol{H}$; hence an identical scalar equation is obeyed by all components of those vectors. Therefore it is possible to summarize the behaviour of all components of $\boldsymbol{E}$ and $\boldsymbol{H}$ through a single *scalar* Helmholtz equation

$$(\nabla^2 + k^2)\varphi(x, y, z) = 0, \tag{1.3}$$

where $\varphi(x, y, z)$ represents any of the components of the electric or the magnetic field vector.

A basic solution of the Helmholtz equation (1.3) is formed by the signal

$$\varphi(x, y, z) = A\, e^{j\boldsymbol{k}\cdot\boldsymbol{r}} = A\, e^{j(k_x x + k_y y + k_z z)}, \tag{1.4}$$

where a *wave vector* $\boldsymbol{k} = (k_x, k_y, k_z)$ has been introduced, which should satisfy the condition

$$\boldsymbol{k}\cdot\boldsymbol{k} = k_x^2 + k_y^2 + k_z^2 = k^2. \tag{1.5}$$

In the important case that the wave vector $\boldsymbol{k}$ has real components, the signal (1.4) represents a (uniform) *plane wave*: on a plane $\boldsymbol{k}\cdot\boldsymbol{r}$ = constant – which plane is perpendicular to the wave vector $\boldsymbol{k}$ – the signal has a constant phase. And, of course, *the direction of the plane wave is determined by the wave vector*. It would be outside the scope of this chapter to consider non-uniform plane waves, for which the wave vector is complex.

We remark that it is sufficient to specify a plane wave in a plane $z = 0$, say, in which case the optical signal (1.4) reduces to

$$\varphi(x, y, 0) = A\, e^{j(k_x x + k_y y)}. \tag{1.6}$$

The complete signal (1.4) follows from including again the additional term $k_z z$ in the exponential, while knowing that k_z is related to k_x and k_y by the relation $k_z^2 = k^2 - (k_x^2 + k_y^2)$, see Eq. (1.5). Without going into more detail, we note that such a behaviour is not restricted to plane waves and not restricted to propagation in vacuum, but holds for any optical signal: if the signal $\varphi(x, y, 0)$ is specified in a plane $z = 0$, the complete signal $\varphi(x, y, z)$ follows from the propagation properties of the medium. It is crucial, however, that we are allowed to represent the optical signal by means of a scalar function instead of a vectorial one. And although this might not always be the case, such a scalar treatment is appropriate in a large number of cases (see, for instance, [Goo96]).

The exponential $\exp[j(k_x x + k_y y)]$ that arises in Eq. (1.6) resembles the exponential $\exp[j2\pi f t]$ that is normally used in the case of time signals. Analogously to the way in which a time signal can be represented in terms of its frequency spectrum via an inverse Fourier transformation, a (two-dimensional) space signal can be represented by an integral of the form

$$\varphi(x, y) = \iint \hat{\varphi}\left(\frac{k_x}{2\pi}, \frac{k_y}{2\pi}\right) e^{j(k_x x + k_y y)} \frac{dk_x}{2\pi} \frac{dk_y}{2\pi}, \tag{1.7}$$

where $\hat{\varphi}(k_x/2\pi, k_y/2\pi)$ is the (two-dimensional) spatial Fourier transform of the space signal $\varphi(x, y)$. Unless otherwise stated, all integrations and summations in this chapter extend from $-\infty$ to $+\infty$; moreover, we will throughout assume that the operands are such that these integrations and summations exist.

The exponential that arises in Eqs. (1.6) and (1.7) represents in fact a (cross-section of a) plane wave [see Eq. (1.4)] with wave vector $\boldsymbol{k} = (k_x, k_y, [k^2 - k_x^2 - k_y^2]^{1/2})$ [see Eq. (1.5)], and the inverse Fourier transformation (1.7) thus represents the optical signal as *a superposition of plane waves*.

3 Gabor's signal expansion in optics

The previous section has lead us to some similarities between time signals, which formed the subject in Gabor's original paper, and space signals, which form the relevant functions in optics. It is now clear that the analogy of *time* is *space*, and the analogy of *temporal frequency* is *spatial frequency*. And where temporal frequency corresponds to the *pitch of a tone*, spatial frequency corresponds to the *direction of a plane wave*.

In the remainder of this chapter we will, for convenience, confine ourselves to space signals that – while propagating in the z-direction – depend on the transverse coordinate x only, and do not depend on y, implying that the y-component k_y of the wave vector $\boldsymbol{k}$ is identically zero. In that case we can restrict ourselves to the one-dimensional function $\varphi(x)$ and its spatial Fourier transform

$$\hat{\varphi}(u) = (F\,\varphi)(u) = \int \varphi(x) e^{-j2\pi ux}\, dx. \tag{1.8}$$

The extension to the more general, two-dimensional case is rather straightforward.

3.1 Gabor's signal expansion and the Gabor transform

Instead of describing a signal in a space domain [by means of $\varphi(x)$] or in a spatial-frequency domain [by means of $\hat{\varphi}(u)$], Gabor's signal expansion represents the signal in a combined space-frequency domain. With $g(x)$ the *elementary signal* or *synthesis window*, and with a space shift X and a spatial-frequency shift U that satisfy the conditions $UX \leq 1$, the expansion reads

$$\varphi(x) = \sum_m \sum_k a_{mk}\, g(x - mX) e^{j2\pi kUx}. \tag{1.9}$$

Throughout this chapter we will consistently use the variable m (and later on M and n) in connection with a space shift (mX, for instance), and the variable k (and later on K and l) in connection with a spatial-frequency shift (kU, for instance).

Suppose that the elementary signal $g(x)$ is a function that is concentrated in the space domain around the position $x = 0$ and that its Fourier transform $\hat{g}(u)$ is concentrated in the spatial-frequency domain around the spatial frequency, or direction, $u = 0$. If we let this function propagate in vacuum, it will behave more or less like an *optical ray*, remaining concentrated around the z-axis. Likewise, the shifted and modulated versions $g(x - mX)\exp(j2\pi kUx)$ of the elementary signal behave like optical rays passing through the positions $x = mX$ and having directions $u = kU$. Gabor's signal expansion can thus be considered as representing an optical signal as a *superposition of rays*; moreover, the Gabor coefficient a_{mk} represents the complex amplitude of the ray passing through the position mX with direction kU.

With $\gamma(x)$ an *analysis window* that corresponds to the synthesis window $g(x)$, the minimal norm Gabor coefficients a_{mk} can be determined by means of the *Gabor transform*

$$a_{mk} = \int \varphi(x)\gamma^*(x - mX)e^{-j2\pi kUx}\,dx. \tag{1.10}$$

The Gabor transform can, of course, be considered as an inner product of the signal $\varphi(x)$ and a shifted and modulated version of the analysis window. Note, however, that the Gabor transform can also be considered as a sampled version of the *windowed Fourier transform* $W_\varphi(x,u)$ of the signal $\varphi(x)$,

$$W_\varphi(x,u) = \int \varphi(x')\gamma^*(x' - x)e^{-j2\pi ux'}\,dx', \tag{1.11}$$

on the rectangular lattice $(x = mX, u = kU)$: $a_{mk} = W_\varphi(mX, kU)$.

3.2 Gaussian elementary signal and Gaussian beams

Gabor's original choice of the elementary signal was a Gaussian

$$g(x) = 2^{\frac{1}{4}}e^{-\pi(x/X)^2}. \tag{1.12}$$

The big advantage of a Gaussian elementary signal is that it has a good localization, both in the space and in the spatial-frequency domain. In optics, a Gaussian choice is very attractive, because the interpretation of such a Gaussian elementary signal as an optical ray is very appealing. And, indeed, much knowledge has been gained in optics (see, for instance, [Sie86]) about the propagation of so-called *Gaussian beams*, i.e., signals of the form

$$\varphi(x,z) = A\sqrt{q(z)}e^{j\pi q(z)x^2}, \tag{1.13}$$

where the imaginary part of the *beam parameter* $q(z)$ is related to the width of the beam in the transversal x-direction and the real part of $q(z)$ is related to the convergence or divergence of the beam in the longitudinal z-direction. For a large class of optical systems, a Gaussian beam retains its Gaussian character (1.13), and the propagation is completely determined by the z-dependent beam parameter $q(z)$.

It is obvious that a Gaussian beam reduces – in the plane $z = 0$, say – to the form of the Gaussian elementary signal (1.12), if the beam parameter $q(0)$ is strictly imaginary with an imaginary part that is positive. Hence, Gabor's signal expansion can thus be considered as representing an optical signal as a superposition of (cross-sections of) Gaussian beams. Since for a large class of optical systems it is not too difficult to determine the propagation of Gaussian beams, it is now straightforward to determine the propagation of an arbitrary optical signal: we let all the Gaussian beams that build up the optical signal in the input plane, propagate through the optical medium, and superpose the beams in the output plane with their proper Gabor coefficients. And although this superposition in the output plane is no longer in the form of a Gabor expansion, it remains possible to determine the optical signal in the output plane.

With these ideas in mind, several authors have expressed the optical signal inside an aperture in terms of a Gabor expansion [ERS86], [EHF87], [Wol89] and have studied the propagation of light in homogeneous and weakly inhomogeneous media [MMTZ86], [ER88], [MF89], [FKLG91], [SHF91c], [SHF91b], [KFLG92], in layered media [MF90], and in the focal region of a parabolic reflector [DA94]. Whereas most of these papers deal with time-harmonic signals, the case of pulsed signals (where the time-pulse has again a Gaussian shape) has been considered, as well [ER87] [SHF91a] [SH91]. Note that if both the time and the space behaviour of a (temporal-spatial) elementary signal are Gaussian, Gabor's signal expansion leads to an expansion in a set of Gaussian *wave packets*, concentrated at certain time moments and certain positions, modulated by certain temporal frequencies and travelling into certain spatial directions. A description of an aperture field in terms of exponential elementary beams instead of Gaussian beams has been reported, as well [Ein88].

3.3 Critical sampling and oversampling

In the case of *critical sampling*, i.e., $UX = 1$, there exists a unique relationship between the synthesis window $g(x)$ that arises in Gabor's signal expansion (1.9) and the analysis window $\gamma(x)$ that arises in the Gabor transform (1.10). Hence, given a certain analysis window $\gamma(x)$, the corresponding synthesis window $g(x)$ can uniquely be determined (see, for instance, [Bas80], [Bas81], [Bas93]).

In the case of *oversampling*, i.e., $UX < 1$, such a unique relationship does not exist, and – given an analysis window – the synthesis window is not unique anymore; very often we choose $g(x)$ such that it has minimum L_2 norm. In that case, the synthesis window $g(x)$ resembles best (in a minimum L_2 norm sense, again) the analysis window $\gamma(x)$ [QC94]. In the case of *infinite* oversampling, i.e., $(U, X) \downarrow (0, 0)$, Gabor's signal expansion (1.9) resembles the relationship

$$\varphi(x') \int |\gamma(x)|^2 dx = \iint W_\varphi(x, u)\gamma(x' - x)e^{j2\pi ux'} dxdu, \tag{1.14}$$

which is, in fact, one possible inversion formula for the windowed Fourier transform (1.11); note that in this case the synthesis window is indeed proportional to the analysis window, see Chapter **??** in this book. Several ways are described in the literature to determine the synthesis window in the case of oversampling (see, for instance, [Dau90], [BG96], [ZZ93]).

In the case of critical sampling, $UX = 1$, Gabor's signal expansion is related to the *degrees of freedom* of a signal: each expansion coefficient a_{mk} represents one complex degree of freedom. If a spatial signal $\varphi(x)$ is, roughly, limited to the space interval $|x| < \frac{1}{2}a$, and its spatial Fourier transform $\hat{\varphi}(u)$ to the frequency interval $|u| < \frac{1}{2}b$, the number of degrees of freedom equals the number of Gabor coefficients in the space-frequency rectangle with area ab, which number is about equal to the *space-bandwidth product ab*. In Section 4 we will restrict ourselves to this case of critical sampling and study the degrees of freedom in more detail.

3.4 *Integer oversampling - Gabor transform as a product of Zak transforms*

It is well known that a correlation and a convolution can be brought into product form by means of the Fourier transform. We now try to bring the Gabor transform (1.10) in a product form, as well. We therefore introduce the Fourier transform $\hat{a}(\xi,\eta)$ of the two-dimensional array a_{mk} through the definition

$$\hat{a}(\xi,\eta) = (F\,a)(\xi,\eta) = \sum_m \sum_k a_{mk} e^{-j2\pi(m\eta - k\xi)}. \tag{1.15}$$

We substitute from the Gabor transform (1.10) and rearrange factors

$$\hat{a}(\xi,\eta) = \sum_m \left[\int \varphi(x)\gamma^*(x - mX) \left\{ \sum_k e^{-j2\pi k(Ux - \xi)} \right\} dx \right] e^{-j2\pi m\eta};$$

we will assume that here – and at other places in this chapter – such a rearranging of factors is allowed. We replace the sum of exponentials by a sum of Dirac functions and rearrange factors again

$$\hat{a}(\xi,\eta) =$$
$$\frac{1}{U} \sum_m \left[\sum_k \int \varphi(x)\gamma^*(x - mX)\delta\left(x - \frac{\xi + k}{U}\right) dx \right] e^{-j2\pi m\eta}.$$

We evaluate the integral and rearrange factors again

$$\begin{aligned} \hat{a}(\xi,\eta) &= \frac{1}{U} \sum_k \varphi\left(\frac{\xi}{U} + \frac{k}{U}\right) e^{-j2\pi k(1/U)(\eta/X)} \\ &\times \left[\sum_m \gamma^*\left(\frac{\xi}{U} + \frac{k}{U} - mX\right) e^{j2\pi(k/U - mX)(\eta/X)} \right]. \end{aligned}$$

In the case of *integer* oversampling, i.e., $1/U = pX$ with p a positive integer, the latter expression can be written as

$$\begin{aligned} \hat{a}(\xi,\eta) &= pX \sum_k \varphi(\xi pX + kpX) e^{-j2\pi k(pX)(\eta pU)} \\ &\times \left[\sum_m \gamma(\xi pX + [kp - m]X) e^{-j2\pi(kp - m)X(\eta pU)} \right]^*. \end{aligned}$$

In the right-hand side of the latter relationship, we recognize the expression

$$Z_\varphi(x, u; \Delta) = \sum_m \varphi(x + m\Delta) e^{-j2\pi m\Delta u} \tag{1.16}$$

for $\varphi(x)$ [with $x = \xi pX$, $u = \eta pU$, and $\Delta = pX$], and a similar expression $Z_\gamma(x, u; \Delta)$ for $\gamma(x)$ [with $x = \xi pX$, $u = \eta pU$, and $\Delta = X$]. The expression (1.16) is known as the *Zak transform* [Zak67], [Zak72], [Jan82], [Jan88]; note that we have explicitly stated the step size Δ, which may take different values for Z_φ and Z_γ. In terms of Zak transforms, the Fourier transform $\hat{a}(\xi, \eta)$ can thus be expressed as

$$\hat{a}(\xi, \eta) = pX Z_\varphi(\xi pX, \eta pU; pX) Z_\gamma^*(\xi pX, \eta pU; X), \tag{1.17}$$

which is the product form of the Gabor transform that we are looking for.

We remark that a Fourier transform like $\hat{a}(\xi, \eta)$ [see Eq. (1.15)] is periodic in the time variable ξ and the frequency variable η with period 1: $\hat{a}(\xi + m, \eta + k) = \hat{a}(\xi, \eta)$; hence, in considering such a Fourier transform we can restrict ourselves to the *fundamental Fourier interval* $(-\frac{1}{2} < \xi \le \frac{1}{2}, -\frac{1}{2} < \eta \le \frac{1}{2})$. We remark further that a Zak transform like $Z_\varphi(x, u; \Delta)$ [see Eq. (1.16)] is periodic in the frequency variable u with period $1/\Delta$ and *quasi-periodic* in the space variable x with quasi-period Δ:

$$Z_\varphi\left(x + m\Delta, u + \frac{k}{\Delta}; \Delta\right) = Z_\varphi(x, u; \Delta) e^{j2\pi m\Delta u}; \tag{1.18}$$

hence, in considering a Zak transform we can restrict ourselves to the *fundamental Zak interval* $(-\frac{1}{2} < x/\Delta \le \frac{1}{2}, -\frac{1}{2} < u\Delta \le \frac{1}{2})$.

The Zak transform $Z_\varphi(x, u; \Delta)$ can also be expressed in terms of the Fourier transform $\hat{\varphi}(u)$ of $\varphi(x)$ and then takes the form

$$\Delta\, Z_\varphi(x, u; \Delta) = e^{j2\pi ux} \sum_k \hat{\varphi}\left(u + \frac{k}{\Delta}\right) e^{j2\pi(k/\Delta)x}. \tag{1.19}$$

From the latter relation we conclude that for a signal that is band-limited to the frequency interval $-\frac{1}{2} < u\Delta \le \frac{1}{2}$, the Zak transform takes the form (in this fundamental frequency interval)

$$\Delta\, Z_\varphi(x, u; \Delta) = e^{j2\pi ux} \hat{\varphi}(u). \tag{1.20}$$

In Fig. 1 we have depicted the Zak transform $Z_\gamma(x, u; \Delta)$ of a Gaussian window [cf. Eq. (1.12)] for several values of Δ. Note that for small values of Δ/X the sampling frequency $1/\Delta$ is sufficiently high and the above-mentioned property (1.20) holds.

If we consider in Eq. (1.17) the domains of the functions $Z_\varphi(\xi pX, \eta pU; pX)$ and $Z_\gamma(\xi pX, \eta pU; X)$ in the fundamental Fourier interval $(-\frac{1}{2} < \xi \le \frac{1}{2}, -\frac{1}{2} < \eta \le \frac{1}{2})$ of the function $\hat{a}(\xi, \eta)$, we note that, whereas the Fourier transform $\hat{a}(\xi, \eta)$ appears only *once* in the fundamental Fourier interval, the Zak transforms

$Z_\varphi(\xi pX, \eta pU;\, pX)$ and $Z_\gamma(\xi pX, \eta pU;\, X)$ appear *p-fold*: $Z_\varphi(\xi pX, \eta pU;\, pX)$ as p identical horizontal stripes with height $1/p$ and width 1, and $Z_\gamma(\xi pX, \eta pU;\, X)$ as p vertical stripes with width $1/p$ and height 1, which stripes are identical to each other apart from the factor $\exp(j2\pi m\eta)$ [cf. the quasi-periodicity property (1.18) of the Zak transform].

Note that the product form (1.17) of the Gabor transform (1.10) enables us to determine this transform in a different way:

- we first determine the Zak transform $Z_\varphi(\xi pX, \eta pU;\, pX)$ and the Zak transform $Z_\gamma(\xi pX, \eta pU;\, X)$ of the signal $\varphi(x)$ and the window function $\gamma(x)$, respectively, by means of definition (1.16);
- we then find the Fourier transform $\hat{a}(\xi, \eta)$ by means of the product rule (1.17);
- we finally determine the Gabor transform a_{mk} via an inverse Fourier transformation.

This way of determining the Gabor transform resembles the way of calculating correlations and convolutions via the Fourier domain. Since the Zak transform is essentially a Fourier transformation, fast algorithms can be used when we are dealing with discrete-time signals, and the algorithm described above then resembles the fast convolution, well-known in digital signal processing [BG96].

The product form also suggests a coherent-optical generation of the Gabor transform, since the basic mathematical operations – Fourier transformation and multiplication – are perfectly suited to be performed by optical means. We will show in Section 5 how the Gabor transform can be generated on a rectangular lattice in the output plane of the optical system. Since the Gabor transform is a sampled version of the windowed Fourier transform, it will also be interesting to know whether the optical signal in this output plane represents the windowed Fourier transform at positions that are not on the rectangular lattice. To treat this problem, we will spend some words on the windowed Fourier transform in the next subsection.

3.5 Windowed Fourier transform - interpolation of the Gabor transform

Since the signal can be represented by its Gabor expansion, we can easily derive an interpolation procedure for the windowed Fourier transform. We start with the windowed Fourier transform (1.11) and substitute from Gabor's signal expansion (1.9)

$$W_\varphi(x, u) = \int \left[\sum_m \sum_k a_{mk} g(x' - mX) e^{j2\pi kUx'} \right] \gamma^*(x' - x) e^{-j2\pi ux'} dx'.$$

We rearrange factors and substitute $\xi = x' - mX$

$$W_\varphi(x, u) = \sum_m \sum_k a_{mk} \int g(\xi) \gamma^*(\xi - [x - mX]) e^{-j2\pi(u - kU)(\xi + mX)} d\xi.$$

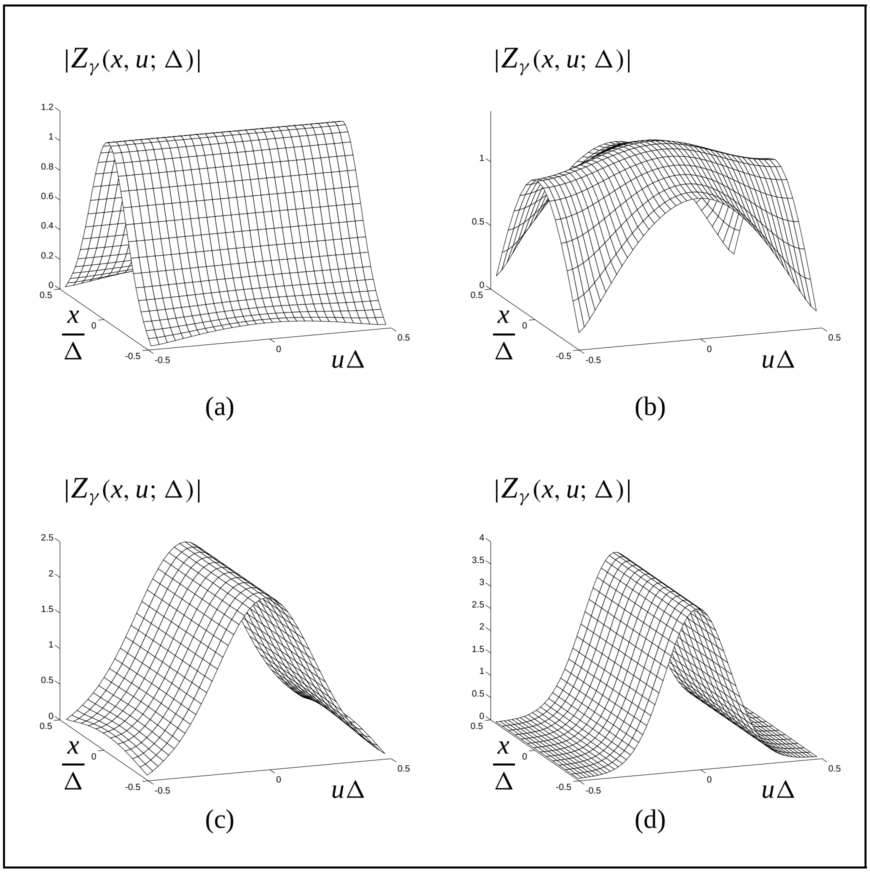

FIGURE 1. The Zak transform $Z_\gamma(x,u;\Delta)$ that corresponds to a Gaussian window function $\gamma(x) = 2^{1/4}\exp[-\pi(x/X)^2]$, for different values of Δ: (a) $\Delta = 2X$, (b) $\Delta = X$, (c) $\Delta = X/2$, and (d) $\Delta = X/3$.

We rearrange factors again and recognize the windowed Fourier transform of the synthesis window $g(x)$

$$W_\varphi(x,u) = \sum_m \sum_k a_{mk} W_g(x - mX, u - kU) e^{-j2\pi(u - kU)mX}. \qquad (1.21)$$

The latter relationship can be considered as an interpolation formula for the windowed Fourier transform in terms of the Gabor coefficients a_{mk}, where the *interpolation kernel* is in fact the windowed Fourier transform $W_g(x,u)$ of the synthesis window $g(x)$. Since the synthesis window is not unique in the case of oversampling, the same remark applies to the interpolation kernel.

We might look for the interpolation kernel $W_g(x,u)$ for which the L_2 norm takes its minimum value. From Moyal's formula [Moy49]

$$\iint |W_g(x,u)|^2 dx du = \left(\int |g(x)|^2 dx\right)\left(\int |\gamma(x)|^2 dx\right) \qquad (1.22)$$

we conclude that the minimum L_2 norm of $W_g(x,u)$ is reached when the synthesis window $g(x)$ has minimum L_2 norm itself.

In the special case $UX = 1$ (Gabor's case of critical sampling) it has been shown [Jan82] that the interpolation kernel that arises in the case of the Gaussian window [cf. Eq. (1.12)] can be expressed in the form

$$W_g(x,u)\ e^{j\pi ux} = \frac{\sigma(2K_0\zeta)}{2K_0\zeta} e^{-\frac{1}{2}\pi|\zeta|^2}, \qquad (1.23)$$

where, for convenience, we have introduced the short-hand notation

$$\zeta = uX + j\frac{x}{X}, \qquad (1.24)$$

where the constant $K_0 = \frac{1}{4}\pi^{-\frac{1}{2}}[\Gamma(\frac{1}{4})]^2 = 1.85407468$ is the complete elliptic integral for the modulus $\frac{1}{2}\sqrt{2}$ (see, for instance, [WW27], Section 22.8, The lemniscate functions), and where $\sigma(2K_0\zeta)$ represents Weierstrass' sigma function [WW27], expressible as

$$\sigma(2K_0\zeta) = \left(\frac{\pi}{K_0}\right)^{\frac{1}{2}} e^{\frac{1}{2}\pi\zeta^2}\ 2\sum_{n=0}^{\infty}(-1)^n e^{-\pi(n+\frac{1}{2})^2} \sin[2\pi(n+\tfrac{1}{2})\zeta].$$

The interpolation function has been depicted in Fig. 2a. We remark that in this case of critical sampling, the interpolation kernel has the property $W_g(mX, kU) = \delta_m \delta_k$, where δ_m represents the Kronecker delta; this property is in accordance with the fact that the two window functions should be *biorthogonal* [Bas80], [Bas81], [Bas93].

In the limiting case $(U, X) \downarrow (0, 0)$ (infinite oversampling) the optimum synthesis window $g_{opt}(x)$ – i.e., the one with minimum L_2 norm – becomes proportional to the analysis window $\gamma(x)$, and it is not difficult to show that the interpolation kernel that arises in the case of the Gaussian window [cf. Eq. (1.12)] then takes the form

$$W_g(x,u)\ e^{j\pi ux} = UXe^{-\frac{1}{2}\pi|\zeta|^2}. \tag{1.25}$$

The interpolation function has been depicted in Fig. 2b.

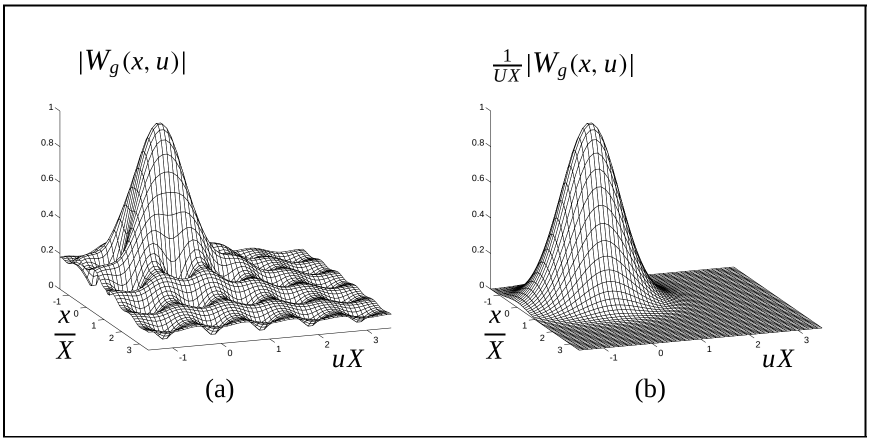

FIGURE 2. The interpolation function $W_g(x,u)$ in the case of (a) critical sampling $UX = 1$, and (b) infinite oversampling $UX \downarrow 0$.

In Section 5 we will study the interpolation of the Gabor transform in more detail.

4 Degrees of freedom of an optical signal

In his original paper [Gab46] Gabor not only chose "Gaussian elementary signals which occupy the smallest possible area in the information diagram," but he also constructed this information (or space-frequency) diagram such that "each elementary signal can be considered as conveying exactly one datum, or one quantum of information." In this case of critical sampling, Gabor's signal expansion is thus intimately related to the degrees of freedom of a signal.

In this section we consider an optical system that truncates both the space and the spatial-frequency content of the input signal. Whereas the input signal of such an optical system may have an infinite number of degrees of freedom, the number of complex degrees of freedom of the output signal is limited to the space-bandwidth product of the system. This can be proved elegantly by expanding the signal in prolate spheroidal wave functions, which are eigenfunctions of this system [Sle76], but we will present a different, more physically oriented proof, by showing that the number of non-vanishing Gabor coefficients of the output signal is equal to the space-bandwidth product of the system [Bas82a].

4.1 Representation of a linear optical system

A linear optical system that transforms an input signal φ_i into an output signal φ_o, can be described in several ways, depending on whether we describe the input and the output signal in the space or in the frequency domain. We thus have four equivalent input-output relationships,

$$\varphi_o(x_o) = \int h_{xx}(x_o, x_i)\varphi_i(x_i)dx_i, \tag{1.26}$$

$$\hat{\varphi}_o(u_o) = \int h_{ux}(u_o, x_i)\varphi_i(x_i)dx_i, \tag{1.27}$$

$$\varphi_o(x_o) = \int h_{xu}(x_o, u_i)\hat{\varphi}_i(u_i)du_i, \tag{1.28}$$

$$\hat{\varphi}_o(u_o) = \int h_{uu}(u_o, u_i)\hat{\varphi}_i(u_i)du_i, \tag{1.29}$$

in which the four *system functions* h_{xx}, h_{ux}, h_{xu}, and h_{uu} are completely determined by the system. Relation (1.26) is the usual system representation in the space domain (see, for instance [Goo96]) by means of the *impulse response* $h_{xx}(x_o, x_i)$, which is also known as the (coherent) *point spread function* in Fourier optics: the function $h_{xx}(x, x_i)$ is the space domain response of the system at point x due to the input impulse signal $\varphi_i(x) = \delta(x - x_i)$. Relation (1.29) is a similar system representation in the frequency domain: the function $h_{uu}(u, u_i)$ is the frequency domain response of the system at frequency u due to the input $\hat{\varphi}(u) = \delta(u - u_i)$, which is the Fourier transform of the harmonic input signal $\varphi(x) = \exp(j2\pi u_i x)$. In Fourier optics such a harmonic signal is a representation of the space dependence of a uniform, obliquely incident, time-harmonic plane wave; in this context we might call $h_{uu}(u_o, u_i)$ the *wave spread function* of the system. Relations (1.27) and (1.28) are hybrid system representations, since the input and the output signal are described in different domains.

We remark that there is a similarity between the four system functions h_{xx}, h_{ux}, h_{xu}, and h_{uu} and the four *Hamilton characteristics* [BW75] that can be used to describe geometric-optical systems. Indeed, for a geometric-optical system the *point characteristic* is nothing but the phase of the point spread function; similar relations hold between the *angle characteristic* and the wave spread function, and between the *mixed characteristics* and the hybrid system representations.

Unlike the *four* system representations (1.26-1.29), there is only *one* system representation when we describe the input and the output signal by their Gabor expansions. Let us therefore choose a synthesis window $g_i(x)$ [with a corresponding analysis window $\gamma_i(x)$] to represent the input signal, and a (possibly different) synthesis window $g_o(x)$ [with a corresponding analysis window $\gamma_o(x)$] to represent the output signal. The space shift X and the frequency shift $U = 1/X$ are chosen identical in the input and the output plane. Since, as an example, we will consider the hybrid system representation (1.27) later on, we describe the input signal $\varphi_i(x)$ and the Fourier transform of the output signal $\hat{\varphi}_o(u)$ of a linear system by their

Gabor expansions (1.9) with expansion coefficients a^i_{mk} and a^o_{mk}, respectively,

$$\varphi_i(x) = \sum_m \sum_k a^i_{mk} g_i(x - mX) e^{j2\pi kUx} \tag{1.30}$$

$$\hat{\varphi}_o(u) = \sum_m \sum_k a^o_{mk} \hat{g}_o(u - kU) e^{-j2\pi muX}, \tag{1.31}$$

where the expansion coefficients follow from the Gabor transforms (1.10)

$$a^i_{mk} = \int \varphi_i(x) \gamma_i^*(x - mX) e^{-j2\pi kUx} dx \tag{1.32}$$

$$a^o_{mk} = \int \hat{\varphi}_o(u) \hat{\gamma}_o^*(u - kU) e^{j2\pi muX} du. \tag{1.33}$$

Note that it is only under the condition of critical sampling $UX = 1$, that the expressions for the Gabor expansion and the Gabor transform take the forms (1.31) and (1.33), respectively.

4.2 Propagation of Gabor's expansion coefficients

It is not difficult to derive how Gabor's expansion coefficients propagate through a linear system; as an example we will choose the hybrid system representation (1.27). When we combine the input-output relation (1.27) with the Gabor expansion (1.30) and the Gabor transform (1.33), we can easily derive a relationship between the output and the input expansion coefficients a^o_{mk} and a^i_{nl}, reading

$$a^o_{mk} = \sum_n \sum_l c_{mk,nl} \, a^i_{nl}, \tag{1.34}$$

where the coefficients $c_{mk,nl}$ are completely determined by the system and by the input and output window functions through the relationship

$$c_{mk,nl} = \iint h_{ux}(u, x) \hat{\gamma}_o^*(u - kU) g_i(x - nX) e^{j2\pi(muX + lUx)} dx du. \tag{1.35}$$

Of course, similar relations can be found for the other system functions.

As an example we consider the basic coherent-optical system depicted in Fig. 3, consisting of a so-called $4f$-arrangement with rectangular apertures of width a and b in the input plane and the Fourier plane, respectively (see, for instance, [Goo96]).

The input plane is located in the front focal plane of a positive lens with focal distance f, while the output plane is located in the back focal plane of a second positive lens. The back focal plane of the first lens and the front focal plane of the second lens coincide, and form the Fourier plane, in which the Fourier transform of the input signal occurs and in which a transparency (in our case: a simple rectangular aperture) may be located. The output signal can thus be considered as a filtered version (in our case: a low-pass filtered version) of the input signal. Such a

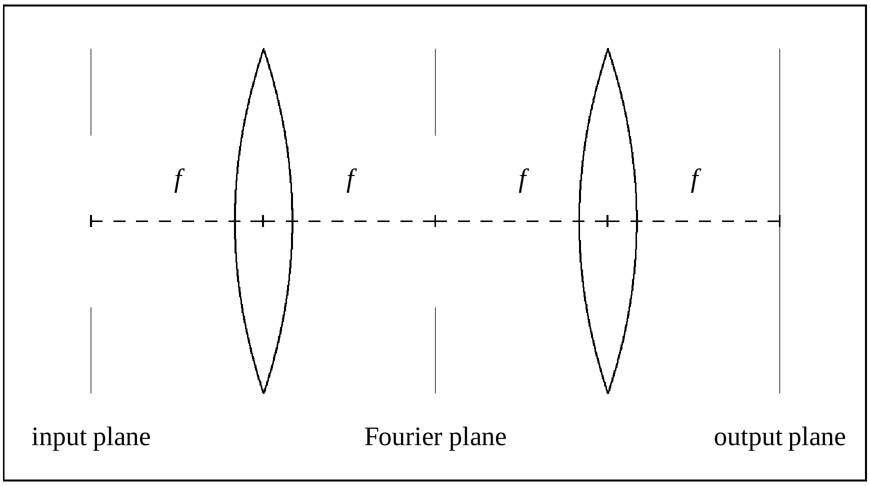

FIGURE 3. A 4-f arrangement with rectangular apertures.

system can most easily be described by a system function $h_{ux}(u, x)$, which in this case takes the form

$$h_{ux}(u, x) = \text{rect}\left(\frac{x}{a}\right) \text{rect}\left(\frac{u}{b}\right) e^{j2\pi ux}. \tag{1.36}$$

For convenience, we choose the widths of the apertures in the input and the Fourier plane equal to an odd multiple of the space and the frequency shift X and U, respectively; thus

$$a = (2M + 1)X \quad \text{and} \quad b = (2K + 1)U, \tag{1.37}$$

with M and K integers. When we substitute from relations (1.36) and (1.37) into relation (1.35), we conclude that the array of coefficients $c_{mk,nl}$ can be expressed as a 4-dimensional convolution of two arrays $d_{mk,nl}$ and $e_{mk,nl}$, where the coefficients $d_{mk,nl}$ are defined by

$$d_{mk,nl} = \begin{cases} \delta_{m-n}\delta_{k-l} & \text{for } |m| \leq M \text{ and } |k| \leq K \\ 0 & \text{elsewhere,} \end{cases} \tag{1.38}$$

and the coefficients $e_{mk,nl}$ are defined by

$$\begin{aligned} e_{mk,nl} &= \iint \text{rect}\left(\frac{x}{X}\right) \text{rect}\left(\frac{u}{U}\right) e^{j2\pi ux} \\ &\times \hat{\gamma}_o^*(u - kU) g_i(x - nX) e^{j2\pi(muX + lUx)} dx du. \end{aligned} \tag{1.39}$$

4.3 Space-bandwidth product - degrees of freedom

A system whose Gabor coefficients $c_{mk,nl}$ would have the form (1.38) is *ideal* in the sense that the Gabor coefficients of the output signal vanish outside the space-frequency rectangle with area ab. Hence, whereas the input signal of such an ideal system may have an infinite number of degrees of freedom, the number of degrees of freedom of the output signal, i.e., the number of non-vanishing Gabor coefficients, is equal to the *space-bandwidth product ab*. However, our system under consideration is not ideal: to find its Gabor coefficients $c_{mk,nl}$, the ideal array $d_{mk,nl}$ must be smeared out by convolving it with the array $e_{mk,nl}$. The latter array is, in fact, the array of Gabor coefficients of the elementary system described by the system function (1.27), with the special choice $a = X$ and $b = U$, i.e., $M = K = 0$.

Depending on the choice of the window functions in the input and the output plane, the array of coefficients $e_{mk,nl}$ can be strongly concentrated. To show this we choose a rectangular window function in the input plane and a sinc-shaped window function in the output plane, thus

$$g_i(x) = \text{rect}\left(\frac{x}{X}\right) \quad \text{and} \quad \hat{\gamma}_o(u) = \text{rect}\left(\frac{u}{U}\right). \tag{1.40}$$

We then find $e_{00,00} = 0.873$, and the strong concentration becomes apparent by noting that

$$\sum_m \sum_k \sum_n \sum_l |e_{mk,nl}|^2 = 1.$$

In general the value of $e_{00,00}$ for this elementary system is given by

$$e_{00,00} = \iint \text{rect}\left(\frac{x}{X}\right) \text{rect}\left(\frac{u}{U}\right) e^{j2\pi ux} \hat{\gamma}_o^*(u) g_i(x) dx du.$$

Furthermore, the identities

$$\sum_m \sum_k \sum_n \sum_l e_{mk,nl} = \left(\sum_k \hat{\gamma}_o(kU)\right)^* \left(\sum_n g_i(nX)\right)$$

and

$$\sum_m \sum_k \sum_n \sum_l |e_{mk,nl}|^2 = \left(\int |\hat{\gamma}_o(u)|^2 du\right) \left(\int |g_i(x)|^2 dx\right)$$

can be derived in a straightforward way, using the basic relation

$$\sum_n e^{j2\pi nUx} = X \sum_n \delta(x - nX).$$

The ratio

$$\frac{|e_{00,00}|^2}{\sum_m \sum_k \sum_n \sum_l |e_{mk,nl}|^2} \tag{1.41}$$

can be considered as a *degree of concentration* of the array $e_{mk,nl}$ around the coefficient $e_{00,00}$. By applying a variational principle to the expression (1.41), it is not difficult to show that the degree of concentration has a stationary value when $g_i(x)$ and $\hat{\gamma}_o(u)$ are chosen according to

$$g_i(x) = \psi_{2m}\left(\frac{x}{X}\right) \text{rect}\left(\frac{x}{X}\right) \quad \text{and} \quad \hat{\gamma}_o(u) = \psi_{2m}\left(\frac{u}{U}\right) \text{rect}\left(\frac{u}{U}\right), \tag{1.42}$$

where the functions $\psi_n(\xi)$ are the *prolate spheroidal wave functions* (see, for instance [Sle76]) defined by the eigenfunction equation

$$\int \psi_n(\xi)\mathrm{rect}(\xi)e^{-2\pi j\xi\eta}d\xi = j^n\sqrt{\lambda_n}\psi_n(\eta) \quad (n = 0, 1, ...) \tag{1.43}$$

and normalized according to

$$\int |\psi_n(\xi)|^2\mathrm{rect}(\xi)d\xi = 1. \tag{1.44}$$

If we choose the window functions as in relations (1.42), the corresponding stationary value of the degree of concentration is equal to λ_{2m}. An optimum value is attained for $m = 0$, for which the degree of concentration takes the value $\lambda_0 = 0.783$. This is a slightly better result than choosing the window functions as in relations (1.40), in which case the degree of concentration takes the value 0.762.

We conclude that for a proper choice of the window functions the array $e_{mk,nl}$ can be strongly concentrated. Since the Gabor coefficients $c_{mk,nl}$ of the basic optical system under consideration can be found by convolving the ideal array $d_{mk,nl}$ with the strongly concentrated array $e_{mk,nl}$, the array of system coefficients $c_{mk,nl}$ is very similar to the array $d_{mk,nl}$. Hence, the number of degrees of freedom of the output signal of this system is equal to the space-bandwidth product ab. We remark that the way in which we have proved this has a clear physical interpretation. Roughly speaking, with the Gabor expansion of the input signal in mind, only those shifted and modulated versions of the synthesis window that can pass both the input plane aperture and the Fourier plane aperture, will reach the output plane and will contribute to the output signal.

A slightly more general system than the one described by relation (1.36), is the one whose system function $h_{ux}(u, x)$ takes the form

$$h_{ux}(u, x) = \sum_m \sum_k h_{mk}\mathrm{rect}\left(\frac{x - mX}{X}\right)\mathrm{rect}\left(\frac{u - kU}{U}\right)e^{j2\pi ux}. \tag{1.45}$$

The array of system coefficients $c_{mk,nl}$ can now be expressed as a 4-dimensional convolution of the arrays $h_{mk}\delta_{m-n}\delta_{k-l}$ and $e_{mk,nl}$. In the case that the array $e_{mk,nl}$ is again strongly concentrated around the element $e_{00,00}$, the Gabor coefficients of the input and the output signal are related by the simple relation

$$a^o_{mk} \simeq h_{mk}a^i_{mk}. \tag{1.46}$$

For the special system described by relation (1.36), we easily find that the array h_{mk} equals unity in the interval ($|m| \leq M$, $|k| \leq K$) and vanishes outside that interval.

5 Coherent-optical generation of the Gabor transform via the Zak transform

The product form (1.17) of the Gabor transform suggests a generation of this transform by coherent-optical means [Bas82b], [LZ92]. Apart from being able to realize ideal imaging, coherent optics is well suited for two specific signal operations, viz. multiplication by a constant function (like in a slide projector, for instance) and Fourier transformation (between the back and the front focal plane of a lens, for instance). These two operations are exactly the ones that are needed for generation of the Gabor transform.

5.1 *Coherent-optical setup*

Let a plane wave of monochromatic laser light be normally incident upon a transparency situated in the input plane of a coherent-optical system, see Fig. 4. The transparency contains the time signal $\varphi(x)$ in a rastered format. With $X_o = pX$ being the width of this raster and $p\mu X_o$ (with $\mu > 0$) being the spacing between the raster lines, the light amplitude $\varphi_i(x_i, y_i)$ just behind the transparency reads

$$\varphi_i(x_i, y_i) = \text{rect}\left(\frac{x_i}{X_o}\right) \sum_n \varphi\left(\frac{x_i + nX_o}{X_o} pX\right) \delta(y_i - np\mu X_o). \tag{1.47}$$

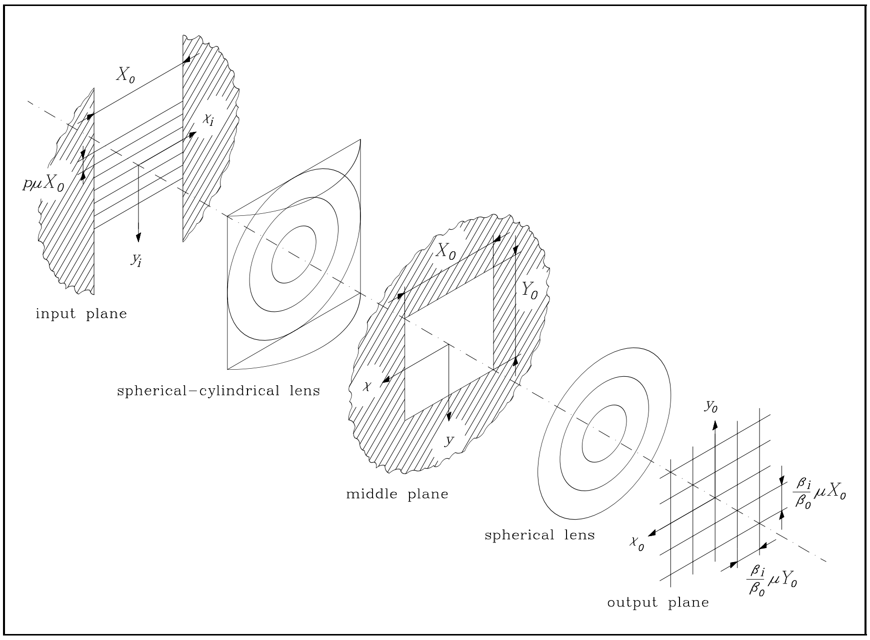

FIGURE 4. Coherent-optical setup for generation of the Gabor transform.

An anamorphic optical system between the input plane and an intermediate plane performs a Fourier transformation in the y-direction and an ideal imaging (possibly with inversion) in the x-direction. Such an anamorphic system can be realized, for instance, using a combination of a spherical and a cylindrical lens.

The anamorphic operation results in the light amplitude

$$\begin{aligned}\varphi_1(x,y) &= \iint \varphi_i(x_i,y_i)e^{-j2\pi\beta_i y y_i}\delta(x-x_i)dx_i dy_i \\ &= \operatorname{rect}\left(\frac{x}{X_o}\right) Z_\varphi\left(\frac{x}{X_o}pX,\, \frac{y}{Y_o}pU;\, pX\right) \end{aligned} \tag{1.48}$$

just in front of the intermediate plane. The parameter β_i contains the effect of the wavelength λ of the laser light and the focal length f_i of the spherical and the cylindrical lens: $\beta_i = 1/\lambda f_i$; moreover, the vertical distance Y_o is defined through $\beta_i \mu X_o Y_o = 1$ and, as usual, the frequency step U is related to the space step X through $pUX = 1$.

A transparency with amplitude transmittance

$$m(x,y) = \operatorname{rect}\left(\frac{x}{X_o}\right)\operatorname{rect}\left(\frac{y}{Y_o}\right) pXZ_\gamma^*\left(\frac{x}{X_o}pX,\, \frac{y}{Y_o}pU;\, X\right) \tag{1.49}$$

is situated in the intermediate plane. Just behind this transparency, the light amplitude takes the form

$$\varphi_2(x,y) = m(x,y)\varphi_1(x,y) = \operatorname{rect}\left(\frac{x}{X_o}\right)\operatorname{rect}\left(\frac{y}{Y_o}\right)\hat{a}\left(\frac{x}{X_o},\frac{y}{Y_o}\right), \tag{1.50}$$

where use has been made of the product form (1.17). Note that the aperture $\operatorname{rect}(x/X_o)\operatorname{rect}(y/Y_o)$ contains *one* period of the periodic Fourier transform $\hat{a}(x/X_o, y/Y_o)$, p horizontal periods of the (periodic) Zak transform

$$Z_\varphi\left(\frac{x}{X_o}pX,\, \frac{y}{Y_o}pU;\, pX\right),$$

and p vertical quasi-periods of the (quasi-periodic) Zak transform

$$Z_\gamma^*\left(\frac{x}{X_o}pX,\, \frac{y}{Y_o}pU;\, X\right).$$

Finally, a two-dimensional Fourier transformation is performed between the intermediate plane and the output plane. Such a Fourier transformation can be realized, for instance, using a spherical lens. The light amplitude in the output plane then takes the form

$$\begin{aligned}\varphi_o(x_o,y_o) &= \frac{1}{X_oY_o}\iint \varphi_2(x,y)e^{-j2\pi\beta_o(x_o x - y_o y)}dxdy \\ &= \sum_m\sum_k a_{mk}\operatorname{sinc}\left(\frac{\beta_o}{\beta_i}\frac{y_o}{\mu X_o}-m\right)\operatorname{sinc}\left(\frac{\beta_o}{\beta_i}\frac{x_o}{\mu Y_o}-k\right), \end{aligned} \tag{1.51}$$

where the sinc-function $\operatorname{sinc}(z) = \sin(\pi z)/(\pi z)$ has been introduced; the parameter β_o, again, contains the effects of the wave length λ of the laser light and the focal length f_o of the spherical lens: $\beta_o = 1/\lambda f_o$. We conclude that the Gabor transform appears on a rectangular lattice of points

$$a_{mk} = \varphi_o \left(k \frac{\beta_i}{\beta_o} \mu Y_o , m \frac{\beta_i}{\beta_o} \mu X_o \right) \tag{1.52}$$

in the output plane. Note that relationship (1.51) represents the output signal as an interpolated version of the Gabor transform, where the interpolation kernel consists of two sinc-functions, in accord with the rectangular aperture in the intermediate (Fourier) plane.

To compare the function $\varphi_o(x_o, y_o)$ with the interpolation formula (1.21), we write Eq. (1.51) in the form

$$\varphi_o \left(\frac{u}{U} \frac{\beta_i}{\beta_o} \mu Y_o , \frac{x}{X} \frac{\beta_i}{\beta_o} \mu X_o \right) = \sum_m \sum_k a_{mk} \operatorname{sinc} \left(\frac{x - mX}{X} \right) \operatorname{sinc} \left(\frac{u - kU}{U} \right) . \tag{1.53}$$

We remark that this relationship – which represents in fact a *band-limited* interpolation – can never have the form of the *exact* interpolation formula (1.21). However, if the degree of oversampling p is sufficiently high, $\varphi(x_o, y_o)$ forms a good approximation of the windowed Fourier transform, as we will show in the next section.

The technique described in this section to generate the Gabor transform (and the windowed Fourier transform), fully utilizes the two-dimensional nature of the optical system, its parallel processing features, and the large space-bandwidth product possible in optical processing. The technique exhibits a resemblance to *folded spectrum* techniques, where space-bandwidth products in the order of 300 000 are reported [Cas78].

5.2 Interpolation of the Gabor transform revisited

It is elucidating to consider the windowed Fourier transform $W_\varphi(x, u)$ as a two-dimensional function, and determine its two-dimensional Fourier transform $\hat{W}_\varphi(x, u)$, defined by

$$\hat{W}_\varphi(x, u) = (F\, W_\varphi)(x, u) = \iint W_\varphi(x', u') e^{-j2\pi(ux' - u'x)} dx' du' . \tag{1.54}$$

We substitute from Eq. (1.11) and rearrange factors

$$\hat{W}_\varphi(x, u) = \iint \varphi(x'') \gamma^*(x'' - x') e^{-j2\pi u x'} dx'' dx' \int e^{-j2\pi u'(x'' - x)} du' .$$

We evaluate the integral over u' and subsequently integrate over the variable x''

$$\hat{W}_\varphi(x, u) = \int \varphi(x) \gamma^*(x - x') e^{-j2\pi u x'} dx' .$$

After evaluating the integral over x' we get the result

$$\hat{W}_\varphi(x,u) = \varphi(x)\hat{\gamma}^*(u)e^{-j2\pi ux}. \tag{1.55}$$

Since the Gabor transform a_{mk} is a sampled version of the windowed Fourier transform $W_\varphi(x,u)$, it is worthwhile to compare $\hat{a}(\xi,\eta)$ [the Fourier transform of the two-dimensional array a_{mk}, see Eq. (1.15)] with $\hat{W}_\varphi(x,u)$ [the Fourier transform of the windowed Fourier transform $W_\varphi(x,u)$, see Eq. (1.54)]. Note that a factor $1/p = UX$ will arise from the mere fact that $a_{mk} = W_\varphi(mX, kU)$ is a *sampled* version of $W_\varphi(x,u)$, and that this factor automatically arises as the proportionality factor between the Fourier transforms $\hat{a}(\xi,\eta)$ and $\hat{W}_\varphi(x,u)$.

We start with the Gabor transform in product form [cf. Eq. (1.17)],

$$\frac{1}{p}\hat{a}\left(\frac{x}{pX},\frac{u}{pU}\right) = XZ_\varphi(x,u;\,pX)Z_\gamma^*(x,u;\,X), \tag{1.56}$$

and try to bring this expression into a form that is comparable with Eq. (1.55); in particular we want to know if the expression (1.56) may be confined to the fundamental Fourier interval. If $\gamma(x)$ is band-limited, and if X has been chosen sufficiently small in order to have the property [cf. Eq. (1.20)]

$$XZ_\gamma(x,u;\,X) = \hat{\gamma}(u)e^{j2\pi ux},$$

the right-hand side of Eq. (1.56) takes the form

$$Z_\varphi(x,u;\,pX)\hat{\gamma}^*(u)e^{-j2\pi ux}, \tag{1.57}$$

which expression shows already some resemblance to the right-hand side of Eq. (1.55). Note that indeed no aliasing will occur if $\hat{\gamma}(u)$ vanishes outside the interval $-\frac{1}{2} < uX \leq \frac{1}{2}$, in which case we can restrict ourselves to one u-period of the periodic function $\hat{a}(x/pX, u/pU)/p$.

We now substitute into the expression (1.57) from the alternate definition (1.19) for the Zak transform $Z_\varphi(x,u;\,pX)$ and get

$$\frac{1}{pX}\hat{\gamma}^*(u)\sum_k \hat{\varphi}(u+kU)e^{j2\pi kUx}. \tag{1.58}$$

To see whether the summation over k in the latter expression has the same effect as the factor $\varphi(x)\exp(-j2\pi ux)$ that appears in the right-hand side of Eq. (1.55), we apply a Fourier transformation to the expression (1.58) with respect to x over the *finite* time interval $-\frac{1}{2} < x/pX \leq \frac{1}{2}$, thus taking into account only one x-period of the periodic function $\hat{a}(x/pX, u/pU)/p$:

$$\frac{1}{pX}\hat{\gamma}^*(u)\int_{-\frac{1}{2}pX}^{\frac{1}{2}pX}\left[\sum_k \hat{\varphi}(u+kU)e^{j2\pi kUx}\right]e^{-j2\pi u'x}dx.$$

We rearrange factors

$$\frac{1}{pX}\hat{\gamma}^*(u)\sum_k \hat{\varphi}(u+kU)\int_{-\frac{1}{2}pX}^{\frac{1}{2}pX} e^{j2\pi(kU-u')x}dx$$

and evaluate the integral over x

$$\hat{\gamma}^*(u)\sum_k \hat{\varphi}(u+kU)\mathrm{sinc}\left(\frac{u'-kU}{U}\right). \tag{1.59}$$

The latter expression should now be compared with the Fourier transform of the right-hand side of Eq. (1.55) with respect to x, which reads

$$\hat{\gamma}^*(u)\hat{\varphi}(u+u'); \tag{1.60}$$

in particular, we should compare the summation over k that appears in expression (1.59) with $\hat{\varphi}(u+u')$. Now, if U is sufficiently small, the summation can be considered as an approximation of the integral

$$\int \hat{\varphi}(u+u'')\mathrm{sinc}\left(\frac{u'-u''}{U}\right)\frac{du''}{U}$$

and when the compressed sinc-function can be considered as an approximation of a Dirac function, we get the final result

$$\int \hat{\varphi}(u+u'')\delta(u'-u'')du'' = \hat{\varphi}(u+u').$$

We already concluded that for a sufficiently small value of X we were allowed to restrict ourselves to the frequency interval $-\frac{1}{2} < uX \leq \frac{1}{2}$. From the previous paragraph we also conclude that for a sufficiently small value of U we can restrict ourselves to the time interval $-\frac{1}{2} < x/pX \leq \frac{1}{2}$. These intervals form exactly the fundamental Fourier interval that we consider in the intermediate (Fourier) plane in the coherent-optical arrangement described in the previous section. We therefore conclude that the signal in the output plane of the optical system is a good approximation of the windowed Fourier transform, even if the interpolation is not in accord with the exact interpolation formula (1.21).

6 References

[Bas80] M.J. Bastiaans. The expansion of an optical signal into a discrete set of Gaussian beams. *Optik*, 57(1):95–102, 1980.

[Bas81] M.J. Bastiaans. A sampling theorem for the complex spectrogram and Gabor's expansion of a signal in Gaussian elementary signals. *Optical Eng.*, 20(4):594–598, July/Aug 1981.

[Bas82a] M.J. Bastiaans. Gabor's signal expansion and degrees of freedom of a signal. *Optica Acta*, 29(9):1223–1229, 1982.

[Bas82b] M.J. Bastiaans. Optical generation of Gabor's expansion coefficients for rastered signals. *Optica Acta*, 29(10):1349–1357, 1982.

[Bas93] M.J. Bastiaans. Gabor's signal expansion and its relation to sampling of the sliding-window spectrum. In R.J. Marks II, editor, *Advanced Topics in Shannon Sampling and Interpolation Theory*, pages 1–35. Springer Verlag, New York, 1993.

[BG96] M.J. Bastiaans and M.C.W. Geilen. On the discrete Gabor transform and the discrete Zak transform. *Signal Process.*, 49:151–166, 1996.

[BW75] M. Born and E. Wolf. *Principles of Optics*. Pergamon, Oxford, 1975.

[Cas78] D. Casasent. Optical signal processing. In D. Casasent, editor, *Optical Data Processing; Applications*, *Topics in Applied Physics*, volume 23. Springer, Berlin, 1978. Chap. 8.

[DA94] A. Dendane and J.M. Arnold. Scattered field analysis of a focused reflector using the Gabor series. *IEE Proc. Microw. Antennas Propagat.*, 141:216–222, 1994.

[Dau90] I. Daubechies. The wavelet transform, time-frequency localization and signal analysis. *IEEE Trans. Inform. Theory*, 36:961–1005, 1990.

[EHF87] P.D. Einziger, Y. Haramaty, and L.B. Felsen. Complex rays for radiation from discretized aperture distributions. *IEEE Trans. Antennas Propagat.*, AP-35:1031–1044, 1987.

[Ein88] P.D. Einziger. Gabor expansion of an aperture field in exponential elementary beams,. *IEE Electron. Lett.*, 24:665–666, 1988.

[ER87] P.D. Einziger and S. Raz. Wave solutions under complex space-time shifts. *J. Opt. Soc. Am. A*, 4:3–10, 1987.

[ER88] P.D. Einziger and S. Raz. Beam-series representation and the parabolic approximation: the frequency domain. *J. Opt. Soc. Am. A*, 5:1883–1892, 1988.

[ERS86] P.D. Einziger, S. Raz, and M. Shapira. Gabor representation and aperture theory. *J. Opt. Soc. Am. A*, 3:508–522, 1986.

[FKLG91] L.B. Felsen, J.M. Klosner, I.T. Lu, and Z. Grossfeld. Source field modeling by self-consistent Gaussian beam superposition (two-dimensional). *J. Acoust. Soc. Am.*, 89:63–72, 1991.

[Gab46] D. Gabor. Theory of communication. *J. Inst. Elec. Eng.*, 93 (III):429–457, 1946.

[Goo96] J.W. Goodman. *Introduction to Fourier Optics; Second Edition*. McGraw-Hill, 1996.

[Jan82] A.J.E.M. Janssen. Bargmann transform, Zak transform, and coherent states. *J. Math. Phys.*, 23(5):720–731, May 1982.

[Jan88] A.J.E.M. Janssen. The Zak transfrom: A signal transform for sampled time-continuous signals. *Philips J. Res.*, 43(1):23–69, 1988.

[KFLG92] J.M. Klosner, L.B. Felsen, I.T. Lu, and H. Grossfeld. Three-dimensional source field modeling by self-consistent Gaussian beam superposition. *J. Acoust. Soc. Am.*, 91:1809–1822, 1992.

[LZ92] Y. Li and Y. Zhang. Coherent optical processing of Gabor and wavelet expansions of one- and two-dimensional signals. *Opt. Eng.*, 31:1865–1885, 1992.

[MF89] J.J. Maciel and L.B. Felsen. Systematic study of fields due to extended apertures by Gaussian beam discretization. *IEEE Trans. Antennas Propagat.*, 37:884–892, 1989.

[MF90] J.J. Maciel and L.B. Felsen. Gaussian-beam analysis of propagation from an extended plane aperture distribution through dielectric layers, Part 1 – Plane layer. *IEEE Trans. Antennas Propagat.*, 38:1607–1617, 1990.

[MMTZ86] P.G. Mantica, I. Montrosset, R. Tascone, and R. Zich. Source field representation in terms of Gaussian beams. *J. Opt. Soc. Am. A*, 3:497–507, 1986.

[Moy49] P.E. Moyal. Quantum mechanics as a statistical theory. *Proc. Camb. Phil. Soc.*, 45:99–124, 1949.

[QC94] S. Qian and D. Chen. Discrete Gabor transform. *IEEE Trans. Signal Proc.*, 41(7):2429–2438, 1994.

[SH91] B.Z. Steinberg and E. Heyman. Phase-space beam summation for time-dependent radiation from large apertures: discretized parameterization. *J. Opt. Soc. Am. A*, 8:959–966, 1991.

[SHF91a] B.Z. Steinberg, E. Heyman, and L.B. Felsen. Phase-space beam summation for time-dependent radiation from large apertures: continuous parameterization. *J. Opt. Soc. Am. A*, 8:943–958, 1991.

[SHF91b] B.Z. Steinberg, E. Heyman, and L.B. Felsen. Phase-space beam summation for time-harmonic radiation from large apertures. *J. Opt. Soc. Am. A*, 8:41–59, 1991.

[SHF91c] B.Z. Steinberg, E. Heyman, and L.B. Felsen. Phase-space methods for radiation from large apertures. *Radio Sci.*, 26:219–227, 1991.

[Sie86] A.E. Siegman. *Lasers*. University Science Books, Mill Valley, CA, 1986.

[Sle76] D. Slepian. On bandwidth. *Proc. IEEE*, 64:292–300, 1976.

[Wol89] D.A. de Wolf. Gaussian decomposition of beams and other functions. *J. Appl. Phys.*, 65:5166–5169, 1989.

[WW27] E.T. Whittaker and G.N. Watson. *A Course of Modern Analysis*. Cambridge University Press, 1927. Chaps. 20–22.

[Zak67] J. Zak. Finite translations in solid state physics. *Phys. Rev. Lett.*, 19:1385–1397, 1967.

[Zak72] J. Zak. The kq-representation in the dynamics of electrons in solids. *Solid State Physics*, 27(1):1–62, 1972.

[ZZ93] M. Zibulski and Y.Y. Zeevi. Oversampling in the Gabor scheme. *IEEE Trans. Signal Proc.*, 41:2679–2687, 1993.