LETTER TO THE EDITOR

Generating function for Hermite-Gaussian modes propagating through first-order optical systems

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Abstract. We consider the field that appears at the output of a first-order optical system when its input field is a Hermite-Gaussian mode. The generating function for the output modes is determined. The orthonormality property of the output modes is confirmed, and derivative and recurrence relations for these modes are derived. Mode converters that generate Laguerre-Gaussian and Hermite-Laguerre-Gaussian modes are considered as examples.

PACS numbers: 02.30.Nw, 02.30.Uu, 42.30.Kq, 42.60.Jf

In the past decade there has been a growing interest in mode converters, with which existing laser modes can be converted into other ones; see, for instance, references [1] and [2], where optical systems are presented that convert Hermite-Gaussian modes into Laguerre-Gaussian modes, reference [3], where also a more general type of modes (the so-called Hermite-Laguerre-Gaussian modes) is generated from Hermite-Gaussian modes, and reference [4], where mode converters based on the fractional Fourier transform are studied. In this Letter we consider the propagation of Hermite-Gaussian modes through first-order optical systems, which systems may indeed act as mode converters. Based on the generating function for the Hermite-Gaussian modes at the input of the system, we determine the generating function of the modes that appears at the output and we show that knowledge of this generating function is useful for the design of mode converters.

We consider Hermite-Gaussian modes, whose field amplitude takes the form

$$\mathcal{H}_{n,m}(\boldsymbol{r}; w_x, w_y) = \mathcal{H}_n(x; w_x)\mathcal{H}_m(y; w_y) \tag{1}$$

with

$$\mathcal{H}_n(x; w) = 2^{1/4}\,(2^n n!\,w)^{-1/2}\,H_n\left(\sqrt{2\pi}\,\frac{x}{w}\right)\exp\left(-\pi\,\frac{x^2}{w^2}\right), \tag{2}$$

where $H_n(\cdot)$ are the Hermite polynomials [5, Section 22] and where the column vector $\boldsymbol{r} = (x, y)^{\rm t}$ is a short-hand notation for the spatial variables x and y, with the superscript $^{\rm t}$ denoting transposition. Note that $\mathcal{H}_n(x; w)$ has been defined such that we have the orthonormality relationship

$$\int \mathcal{H}_n(x; w)\mathcal{H}_l(x; w)\,{\rm d}x = \delta_{nl} \tag{3}$$

with δ_{nl} the Kronecker delta. (Unless otherwise stated, all integrals in this contribution extend from $-\infty$ to $+\infty$.) From the generating function of the Hermite polynomials [5, equation (22.9.17)],

$$\exp\left(-s^2+2sz\right)=\sum_{n=0}^{\infty}H_n(z)\frac{s^k}{n!}, \tag{4}$$

we can easily find the generating function for the (one-dimensional) Hermite-Gaussian modes $\mathcal{H}_n(x;w)$, see equation (2),

$$2^{1/4}w^{-1/2}\exp\left(-s^2+2\sqrt{2\pi}sx/w-\pi x^2/w^2\right)=\sum_{n=0}^{\infty}\mathcal{H}_n(x;w)\left(\frac{2^n}{n!}\right)^{1/2}s^n, \tag{5}$$

while for the two-dimensional Hermite-Gaussian modes $\mathcal{H}_{n,m}(\boldsymbol{r};w_x,w_y)$, see equation (1), we have

$$2^{1/2}(w_xw_y)^{-1/2}\exp\left[-(s_x^2+s_y^2)+2\sqrt{2\pi}(s_xx/w_x+s_yy/w_y)-\pi(x^2/w_x^2+y^2/w_y^2)\right]$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathcal{H}_{n,m}(\boldsymbol{r};w_x,w_y)\left(\frac{2^{n+m}}{n!m!}\right)^{1/2}s_x^ns_y^m. \tag{6}$$

We let the Hermite-Gaussian mode $\mathcal{H}_{n,m}(\boldsymbol{r};w_x,w_y)$ propagate through a first-order optical system – also called an $\boldsymbol{ABCD}$-system – and determine the generating function for the beam $\mathcal{H}_{n,m}^{ABCD}(\boldsymbol{r};w_x,w_y)$ that appears at the output of this system.

Any first-order optical system can be described by its ray transformation matrix [6], which relates the position $\boldsymbol{r}_{\rm i}$ and direction $\boldsymbol{q}_{\rm i}$ of an incoming ray to the position $\boldsymbol{r}_{\rm o}$ and direction $\boldsymbol{q}_{\rm o}$ of the outgoing ray:

$$\begin{pmatrix}\boldsymbol{r}_{\rm o}\\ \boldsymbol{q}_{\rm o}\end{pmatrix}=\begin{pmatrix}\boldsymbol{A} & \boldsymbol{B}\\ \boldsymbol{C} & \boldsymbol{D}\end{pmatrix}\begin{pmatrix}\boldsymbol{r}_{\rm i}\\ \boldsymbol{q}_{\rm i}\end{pmatrix}. \tag{7}$$

The ray transformation matrix is symplectic, yielding the relations

$$\begin{array}{lll}\boldsymbol{A}\boldsymbol{B}^{\rm t}=\boldsymbol{B}\boldsymbol{A}^{\rm t}, & \boldsymbol{C}\boldsymbol{D}^{\rm t}=\boldsymbol{D}\boldsymbol{C}^{\rm t}, & \boldsymbol{A}\boldsymbol{D}^{\rm t}-\boldsymbol{B}\boldsymbol{C}^{\rm t}=\mathbf{I},\\ \boldsymbol{A}^{\rm t}\boldsymbol{C}=\boldsymbol{C}^{\rm t}\boldsymbol{A}, & \boldsymbol{B}^{\rm t}\boldsymbol{D}=\boldsymbol{D}^{\rm t}\boldsymbol{B}, & \boldsymbol{A}^{\rm t}\boldsymbol{D}-\boldsymbol{C}^{\rm t}\boldsymbol{B}=\mathbf{I}.\end{array} \tag{8}$$

Using the matrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{D}$, and assuming that $\boldsymbol{B}$ is a non-singular matrix, we can represent the first-order optical system by the Collins integral [7]

$$f_{\rm o}(\boldsymbol{r}_{\rm o})=\frac{\exp({\rm i}\phi)}{\sqrt{\det{\rm i}\boldsymbol{B}}}\iint f_{\rm i}(\boldsymbol{r}_{\rm i})\exp\left[{\rm i}\pi(\boldsymbol{r}_{\rm i}^{\rm t}\boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{r}_{\rm i}-2\boldsymbol{r}_{\rm i}^{\rm t}\boldsymbol{B}^{-1}\boldsymbol{r}_{\rm o}+\boldsymbol{r}_{\rm o}^{\rm t}\boldsymbol{D}\boldsymbol{B}^{-1}\boldsymbol{r}_{\rm o})\right]{\rm d}\boldsymbol{r}_{\rm i}, \tag{9}$$

where the output amplitude $f_{\rm o}(\boldsymbol{r})$ is expressed in terms of the input amplitude $f_{\rm i}(\boldsymbol{r})$. The phase factor $\exp({\rm i}\phi)$ in equation (9) is rather irrelevant and can often be chosen arbitrarily. We remark that the signal transformation $f_{\rm i}(\boldsymbol{r})\to f_{\rm o}(\boldsymbol{r})$ that corresponds to a first-order optical system, is unitary:

$$\iint f_{{\rm i},1}(\boldsymbol{r})f_{{\rm i},2}^*(\boldsymbol{r})\,{\rm d}\boldsymbol{r}=\iint f_{{\rm o},1}(\boldsymbol{r})f_{{\rm o},2}^*(\boldsymbol{r})\,{\rm d}\boldsymbol{r} \tag{10}$$

with * denoting complex conjugation.

With the Hermite-Gaussian mode $\mathcal{H}_{n,m}(\boldsymbol{r};w_x,w_y)$ at the input of an $\boldsymbol{ABCD}$-system, we denote the output beam by $\mathcal{H}_{n,m}^{ABCD}(\boldsymbol{r};w_x,w_y)$. To find the generating function of this output beam, we write

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathcal{H}_{n,m}^{ABCD}(\boldsymbol{r};w_x,w_y)\left(\frac{2^{n+m}}{n!m!}\right)^{1/2}s_x^n s_y^m=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\frac{2^{n+m}}{n!m!}\right)^{1/2}s_x^n s_y^m$$
$$\times\frac{\exp(\mathrm{i}\phi)}{\sqrt{\det \mathrm{i}\boldsymbol{B}}}\iint\mathcal{H}_{n,m}(\boldsymbol{r}_\mathrm{i};w_x,w_y)$$
$$\times\exp\left[\mathrm{i}\pi(\boldsymbol{r}_\mathrm{i}^\mathrm{t}\boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{r}_\mathrm{i}-2\boldsymbol{r}_\mathrm{i}^\mathrm{t}\boldsymbol{B}^{-1}\boldsymbol{r}+\boldsymbol{r}^\mathrm{t}\boldsymbol{D}\boldsymbol{B}^{-1}\boldsymbol{r})\right]\mathrm{d}\boldsymbol{r}_\mathrm{i},$$

where Collins integral (9) has been used. After substituting from the generating function (6), reorganizing terms, and introducing the column vector $\boldsymbol{s}$ and the diagonal matrix $\boldsymbol{W}$ according to

$$\boldsymbol{s}=\begin{pmatrix}s_x\\ s_y\end{pmatrix}\quad\text{and}\quad\boldsymbol{W}=\begin{pmatrix}w_x & 0\\ 0 & w_y\end{pmatrix},$$

the generating function takes the form

$$\frac{\exp(\mathrm{i}\phi)}{\sqrt{\det \mathrm{i}\boldsymbol{B}}}\,2^{1/2}(w_xw_y)^{-1/2}\exp\left(-\boldsymbol{s}^\mathrm{t}\boldsymbol{s}+\mathrm{i}\pi\boldsymbol{r}^\mathrm{t}\boldsymbol{D}\boldsymbol{B}^{-1}\boldsymbol{r}\right)$$
$$\times\iint\exp\left[-\pi\boldsymbol{r}_\mathrm{i}^\mathrm{t}\left(\boldsymbol{W}^{-2}-\mathrm{i}\boldsymbol{B}^{-1}\boldsymbol{A}\right)\boldsymbol{r}_\mathrm{i}\right]$$
$$\times\exp\left[-\mathrm{i}2\pi\boldsymbol{r}_\mathrm{i}^\mathrm{t}\left(\boldsymbol{B}^{-1}\boldsymbol{r}+\mathrm{i}\sqrt{2/\pi}\boldsymbol{W}^{-1}\boldsymbol{s}\right)\right]\mathrm{d}\boldsymbol{r}_\mathrm{i}.$$

After introducing normalized quantities,

$$\boldsymbol{\rho}_\mathrm{i}=\boldsymbol{W}^{-1}\boldsymbol{r}_\mathrm{i}\quad\text{and}\quad\begin{pmatrix}\boldsymbol{a} & \boldsymbol{b}\\ \boldsymbol{c} & \boldsymbol{d}\end{pmatrix}=\begin{pmatrix}\boldsymbol{A} & \boldsymbol{B}\\ \boldsymbol{C} & \boldsymbol{D}\end{pmatrix}\begin{pmatrix}\boldsymbol{W} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{W}^{-1}\end{pmatrix},$$

we can write

$$\frac{\exp(\mathrm{i}\phi)}{\sqrt{\det \mathrm{i}\boldsymbol{b}}}\,2^{1/2}\exp\left(-\boldsymbol{s}^\mathrm{t}\boldsymbol{s}+\mathrm{i}\pi\boldsymbol{r}^\mathrm{t}\boldsymbol{d}\boldsymbol{b}^{-1}\boldsymbol{r}\right)$$
$$\times\iint\exp\left[-\pi\boldsymbol{\rho}_\mathrm{i}^\mathrm{t}\left(\mathbf{I}-\mathrm{i}\boldsymbol{b}^{-1}\boldsymbol{a}\right)\boldsymbol{\rho}_\mathrm{i}-\mathrm{i}2\pi\boldsymbol{\rho}_\mathrm{i}^\mathrm{t}\left(\boldsymbol{b}^{-1}\boldsymbol{r}+\mathrm{i}\sqrt{2/\pi}\,\boldsymbol{s}\right)\right]\mathrm{d}\boldsymbol{\rho}_\mathrm{i},$$

and it remains to calculate an integral of the form

$$\iint\exp\left(-\pi\boldsymbol{\rho}_\mathrm{i}^\mathrm{t}\boldsymbol{P}\boldsymbol{\rho}_\mathrm{i}-\mathrm{i}2\pi\boldsymbol{\rho}_\mathrm{i}^\mathrm{t}\boldsymbol{q}\right)\mathrm{d}\boldsymbol{\rho}_\mathrm{i}=\frac{1}{\sqrt{\det\boldsymbol{P}}}\exp\left(-\pi\boldsymbol{q}^\mathrm{t}\boldsymbol{P}^{-1}\boldsymbol{q}\right),$$

with $\boldsymbol{P}=\mathbf{I}-\mathrm{i}\boldsymbol{b}^{-1}\boldsymbol{a}$ a symmetric matrix whose real part is positive definite and with $\boldsymbol{q}=\boldsymbol{b}^{-1}\boldsymbol{r}+\mathrm{i}\sqrt{2/\pi}\,\boldsymbol{s}$. (The latter equality is a straightforward extension of the one-dimensional relation

$$\int\exp(-px^2-qx)\,\mathrm{d}x=\sqrt{\frac{\pi}{p}}\exp\left(\frac{q^2}{4p}\right)\quad(\Re p>0),$$

see, for instance, [8, equation (2.3.15.11)], to more dimensions.) We thus get for the generating function

$$\frac{\exp(\mathrm{i}\phi)}{\sqrt{\det \mathrm{i}\boldsymbol{b}}}\left[\frac{2}{\det(\mathbf{I}-\mathrm{i}\boldsymbol{b}^{-1}\boldsymbol{a})}\right]^{1/2}\exp\left(-\boldsymbol{s}^\mathrm{t}\boldsymbol{s}+\mathrm{i}\pi\boldsymbol{r}^\mathrm{t}\boldsymbol{d}\boldsymbol{b}^{-1}\boldsymbol{r}\right)$$
$$\times\exp\left[-\pi\left(\boldsymbol{b}^{-1}\boldsymbol{r}+\mathrm{i}\sqrt{2/\pi}\,\boldsymbol{s}\right)^\mathrm{t}\left(\mathbf{I}-\mathrm{i}\boldsymbol{b}^{-1}\boldsymbol{a}\right)^{-1}\left(\boldsymbol{b}^{-1}\boldsymbol{r}+\mathrm{i}\sqrt{2/\pi}\,\boldsymbol{s}\right)\right].$$

To simplify this expression we rewrite the terms in the exponent that depend on $\boldsymbol{r}^\mathrm{t}\boldsymbol{r}$ and on $\boldsymbol{s}^\mathrm{t}\boldsymbol{s}$, using the symplecticity conditions (8); the term that depends on $\boldsymbol{s}^\mathrm{t}\boldsymbol{r}$

does not need any further elaboration. The generating function of the output beam $\mathcal{H}^{ABCD}_{n,m}(\boldsymbol{r}; w_x, w_y)$ then takes its final form

$$\frac{\exp(\mathrm{i}\phi)\, 2^{1/2}}{\sqrt{\det(\boldsymbol{a}+\mathrm{i}\boldsymbol{b})}} \exp\left[-\boldsymbol{s}^{\mathrm{t}}\left(\boldsymbol{a}+\mathrm{i}\boldsymbol{b}\right)^{-1}\left(\boldsymbol{a}-\mathrm{i}\boldsymbol{b}\right)\boldsymbol{s}+2\sqrt{2\pi}\boldsymbol{s}^{\mathrm{t}}\left(\boldsymbol{a}+\mathrm{i}\boldsymbol{b}\right)^{-1}\boldsymbol{r}\right]$$
$$\times \exp\left[-\pi\boldsymbol{r}^{\mathrm{t}}\left(\boldsymbol{d}-\mathrm{i}\boldsymbol{c}\right)\left(\boldsymbol{a}+\mathrm{i}\boldsymbol{b}\right)^{-1}\boldsymbol{r}\right]$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \mathcal{H}^{ABCD}_{n,m}(\boldsymbol{r}; w_x, w_y)\left(\frac{2^{n+m}}{n!m!}\right)^{1/2} s_x^n s_y^m. \tag{11}$$

We remark that the two complex matrices $\boldsymbol{a}\pm\mathrm{i}\boldsymbol{b}$ are the most important elements in the generating function (11) of the modes $\mathcal{H}^{ABCD}_{n,m}(\boldsymbol{r}; w_x, w_y)$. Note that the complex symmetric matrix

$$\boldsymbol{M} = (\boldsymbol{a}+\mathrm{i}\boldsymbol{b})^{-1}(\boldsymbol{a}-\mathrm{i}\boldsymbol{b}) = \boldsymbol{M}^{\mathrm{t}} \tag{12}$$

is unitary and thus $\boldsymbol{M}^{-1} = \boldsymbol{M}^{*}$. Furthermore, the complex matrix $(\boldsymbol{d}-\mathrm{i}\boldsymbol{c})(\boldsymbol{a}+\mathrm{i}\boldsymbol{b})^{-1}$ is symmetric and its real part equals

$$[(\boldsymbol{a}-\mathrm{i}\boldsymbol{b})(\boldsymbol{a}^{\mathrm{t}}+\mathrm{i}\boldsymbol{b}^{\mathrm{t}})]^{-1} = (\boldsymbol{a}\boldsymbol{a}^{\mathrm{t}}+\boldsymbol{b}\boldsymbol{b}^{\mathrm{t}})^{-1} = [(\boldsymbol{a}+\mathrm{i}\boldsymbol{b})(\boldsymbol{a}^{\mathrm{t}}-\mathrm{i}\boldsymbol{b}^{\mathrm{t}})]^{-1}. \tag{13}$$

These matrix relationships are used below to confirm the orthonormality between $\mathcal{H}^{ABCD}_{n,m}(\boldsymbol{r}; w_x, w_y)$ and $\mathcal{H}^{ABCD}_{l,k}(\boldsymbol{r}; w_x, w_y)$,

$$\iint \mathcal{H}^{ABCD}_{n,m}(\boldsymbol{r}; w_x, w_y)[\mathcal{H}^{ABCD}_{l,k}(\boldsymbol{r}; w_x, w_y)]^{*}\,\mathrm{d}\boldsymbol{r} = \delta_{nl}\delta_{mk}, \tag{14}$$

which is a direct consequence of the unitarity property (10) of the first-order optical system. To see this, we consider the expression

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{2^{n+m}}{n!m!}\right)^{1/2} s_x^n s_y^m \left(\frac{2^{l+k}}{l!k!}\right)^{1/2} t_x^l t_y^k$$
$$\times \iint \mathcal{H}^{ABCD}_{n,m}(\boldsymbol{r}; w_x, w_y)[\mathcal{H}^{ABCD}_{l,k}(\boldsymbol{r}; w_x, w_y)]^{*}\,\mathrm{d}\boldsymbol{r}$$
$$= 2\left[\det(\boldsymbol{a}+\mathrm{i}\boldsymbol{b})\det(\boldsymbol{a}-\mathrm{i}\boldsymbol{b})\right]^{-1/2}\exp\left[-\boldsymbol{s}^{\mathrm{t}}\boldsymbol{M}\boldsymbol{s}-\boldsymbol{t}^{\mathrm{t}}\boldsymbol{M}^{*}\boldsymbol{t}\right]$$
$$\times \iint \exp\left[2\sqrt{2\pi}\boldsymbol{s}^{\mathrm{t}}\left(\boldsymbol{a}+\mathrm{i}\boldsymbol{b}\right)^{-1}\boldsymbol{r}+2\sqrt{2\pi}\boldsymbol{t}^{\mathrm{t}}\left(\boldsymbol{a}-\mathrm{i}\boldsymbol{b}\right)^{-1}\boldsymbol{r}\right]$$
$$\times \exp\left[-2\pi\boldsymbol{r}^{\mathrm{t}}\left(\boldsymbol{a}\boldsymbol{a}^{\mathrm{t}}+\boldsymbol{b}\boldsymbol{b}^{\mathrm{t}}\right)^{-1}\boldsymbol{r}\right]\mathrm{d}\boldsymbol{r},$$

where $\boldsymbol{t} = (t_x, t_y)^{\mathrm{t}}$ on the analogy of $\boldsymbol{s} = (s_x, s_y)^{\mathrm{t}}$, and where we have substituted from the generating function (11). We note that the integral in this expression equals

$$\tfrac{1}{2}\left[\det(\boldsymbol{a}\boldsymbol{a}^{\mathrm{t}}+\boldsymbol{b}\boldsymbol{b}^{\mathrm{t}})\right]^{1/2}\exp\left(\boldsymbol{s}^{\mathrm{t}}\boldsymbol{M}\boldsymbol{s}+\boldsymbol{t}^{\mathrm{t}}\boldsymbol{M}^{*}\boldsymbol{t}+2\boldsymbol{s}^{\mathrm{t}}\boldsymbol{t}\right)$$

and we get

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{2^{n+m}}{n!m!}\right)^{1/2} s_x^n s_y^m \left(\frac{2^{l+k}}{l!k!}\right)^{1/2} t_x^l t_y^k$$
$$\times \iint \mathcal{H}^{ABCD}_{n,m}(\boldsymbol{r}; w_x, w_y)[\mathcal{H}^{ABCD}_{l,k}(\boldsymbol{r}; w_x, w_y)]^{*}\,\mathrm{d}\boldsymbol{r} = \exp\left(2\boldsymbol{s}^{\mathrm{t}}\boldsymbol{t}\right)$$
$$= \exp(2s_x t_x)\exp(2s_y t_y) = \left[\sum_{n=0}^{\infty}\frac{(2s_x t_x)^n}{n!}\right]\left[\sum_{m=0}^{\infty}\frac{(2s_y t_y)^m}{m!}\right]$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{2^{n+m}}{n!m!}\right)^{1/2} s_x^n s_y^m \left(\frac{2^{l+k}}{l!k!}\right)^{1/2} t_x^l t_y^k \delta_{nl}\delta_{mk},$$

which leads directly to the orthonormality relationship (14).

Generating functions are useful in finding, for instance, the derivative relations for $\mathcal{H}_{n,m}^{ABCD}(\boldsymbol{r}; w_x, w_y)$

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)^{\rm t} \mathcal{H}_{n,m}^{ABCD}(\boldsymbol{r}) = -2\pi \mathcal{H}_{n,m}^{ABCD}(\boldsymbol{r})\,(\boldsymbol{d} - {\rm i}\boldsymbol{c})\,(\boldsymbol{a} + {\rm i}\boldsymbol{b})^{-1}\,(x, y)^{\rm t} + 2\sqrt{\pi}\,\left(\boldsymbol{a}^{\rm t} + {\rm i}\boldsymbol{b}^{\rm t}\right)^{-1} \left[\sqrt{n}\,\mathcal{H}_{n-1,m}^{ABCD}(\boldsymbol{r}), \sqrt{m}\,\mathcal{H}_{n,m-1}^{ABCD}(\boldsymbol{r})\right]^{\rm t} \quad (15)$$

by differentiating the generating function with respect to $\boldsymbol{r}$, and the recurrence relations

$$2\sqrt{\pi}\,(x, y)^{\rm t}\,\mathcal{H}_{n,m}^{ABCD}(\boldsymbol{r}) = (\boldsymbol{a} + {\rm i}\boldsymbol{b})\left[\sqrt{n+1}\,\mathcal{H}_{n+1,m}^{ABCD}(\boldsymbol{r}), \sqrt{m+1}\,\mathcal{H}_{n,m+1}^{ABCD}(\boldsymbol{r})\right]^{\rm t} + (\boldsymbol{a} - {\rm i}\boldsymbol{b})\left[\sqrt{n}\,\mathcal{H}_{n-1,m}^{ABCD}(\boldsymbol{r}), \sqrt{m}\,\mathcal{H}_{n,m-1}^{ABCD}(\boldsymbol{r})\right]^{\rm t} \quad (16)$$

by differentiating it with respect to $\boldsymbol{s}$. Note again the importance of the matrices $\boldsymbol{a} \pm {\rm i}\boldsymbol{b}$ in the latter expression. Similar relationships, also based on a generating function, have been derived recently [3] for the special first-order system that converts Hermite-Gaussian modes into so-called Hermite-Laguerre-Gaussian modes. The first-order system that performs this conversion is described by

$$w^{-1}\boldsymbol{a} = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} = w\,\boldsymbol{d}, \quad w^{-1}\boldsymbol{b} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -w\,\boldsymbol{c}, \quad (17)$$

and the generating function of these Hermite-Laguerre-Gaussian modes takes the form

$$\frac{\exp({\rm i}\phi)\,2^{1/2}}{{\rm i}w}\,\exp\left[-\pi(x^2+y^2)/w^2 + {\rm i}(s_x^2 - s_y^2)\cos 2\alpha + 2{\rm i}s_x s_y \sin 2\alpha\right] \times \exp\left[2\sqrt{\pi}\,s_x(x\cos 2\alpha - {\rm i}x + y\sin 2\alpha)/w\right] \times \exp\left[2\sqrt{\pi}\,s_y(x\sin 2\alpha - {\rm i}y - y\cos 2\alpha)/w\right]. \quad (18)$$

(Note that in reference [3], the output field amplitude $\mathcal{G}_{n,m}(x_o, y_o|\alpha)$ is defined with an additional rotation of the coordinate system over the angle α.) In particular, for $\alpha = \pi/4$ we get at the output of the system (17) a Laguerre-Gaussian beam whose generating function can be written in the form

$$\frac{\exp({\rm i}\phi)\,2^{1/2}}{{\rm i}w}\,\exp\left\{-\pi(x^2+y^2)/w^2 + 2{\rm i}s_x s_y - 2\sqrt{\pi}\,[{\rm i}s_x(x+{\rm i}y) - s_y(x-{\rm i}y)]/w\right\}. \quad (19)$$

We remark that the latter expression depends only on the combinations $x^2 + y^2$, $s_x(x + {\rm i}y)$ and $s_y(x - {\rm i}y)$, which clearly shows the vortex behaviour of such a beam.

To find the modes that show a vortex behaviour, we require that $\boldsymbol{s}^{\rm t}(\boldsymbol{a} + {\rm i}\boldsymbol{b})^{-1}\boldsymbol{r}$ depends only on the combinations $s_x(x+{\rm i}y)$ and $s_y(x-{\rm i}y)$, and $\boldsymbol{r}^{\rm t}(\boldsymbol{d}-{\rm i}\boldsymbol{c})(\boldsymbol{a}+{\rm i}\boldsymbol{b})^{-1}\boldsymbol{r}$ only on the combination $x^2 + y^2$. These requirements lead to the class of first-order optical systems described by

$$w^{-1}\,(\boldsymbol{a} + {\rm i}\boldsymbol{b}) = \frac{1}{\sqrt{2}}\begin{pmatrix} \exp({\rm i}\gamma_x) & -{\rm i}\exp({\rm i}\gamma_y) \\ -{\rm i}\exp({\rm i}\gamma_x) & \exp({\rm i}\gamma_y) \end{pmatrix} = w\,\frac{\cos\gamma}{\exp({\rm i}\gamma)}\,(\boldsymbol{d} - {\rm i}\boldsymbol{c}), \quad (20)$$

resulting indeed in the expressions

$$\begin{aligned} \boldsymbol{r}^{\rm t}(\boldsymbol{d} - {\rm i}\boldsymbol{c})(\boldsymbol{a} + {\rm i}\boldsymbol{b})^{-1}\boldsymbol{r} &= (1 + {\rm i}\tan\gamma)(x^2+y^2)/w^2, \\ \boldsymbol{s}^{\rm t}(\boldsymbol{a} + {\rm i}\boldsymbol{b})^{-1}(\boldsymbol{a} - {\rm i}\boldsymbol{b})\boldsymbol{s} &= 2{\rm i}\exp[-{\rm i}(\gamma_x + \gamma_y)]s_x s_y, \\ \boldsymbol{s}^{\rm t}(\boldsymbol{a} + {\rm i}\boldsymbol{b})^{-1}\boldsymbol{r} &= [s_x(x+{\rm i}y)\exp(-{\rm i}\gamma_x) + {\rm i}s_y(x-{\rm i}y)\exp(-{\rm i}\gamma_y)]/w\sqrt{2}. \end{aligned}$$

The generating functions of the beams that arise at the output of the system (20) thus have basically the same form as the generating function (19); equation (19) itself arises for the special choice $\gamma = 0$ and $\gamma_x = \gamma_y = \pi/2$ [3], while the case $\gamma_x = \gamma_y = 0$ has been reported in [2, equation (14)]. The first-order optical system (20) is in fact a mode converter that generates Laguerre-Gaussian modes from the common Hermite-Gaussian modes (1). For an actual physical realization, the $\boldsymbol{ABCD}$-matrix of this system can, for instance, be expressed in the product form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\tan\gamma/w^2 & 0 & 1 & 0 \\ 0 & -\tan\gamma/w^2 & 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -w^2 \\ 0 & 1 & -w^2 & 0 \\ 0 & 1/w^2 & 1 & 0 \\ 1/w^2 & 0 & 0 & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} (w/w_x)\cos\gamma_x & 0 & (ww_x)\sin\gamma_x & 0 \\ 0 & (w/w_y)\cos\gamma_y & 0 & (ww_y)\sin\gamma_y \\ (-1/ww_x)\sin\gamma_x & 0 & (w_x/w)\cos\gamma_x & 0 \\ 0 & (-1/ww_y)\sin\gamma_y & 0 & (w_y/w)\cos\gamma_y \end{pmatrix}.$$

The Hermite-to-Laguerre mode converter (20) is thus actually presented in the form of a cascade of three subsystems, with four degrees of freedom (w, γ, γ_x, and γ_y):

(i) an anamorphic fractional Fourier transformer (with proper scaling), with the important property that it conserves the Hermite-Gaussian character of the input beam. This subsystem shows different scalings w/w_x and w/w_y to compensate for the two different widths of the Hermite-Gaussian mode, and different fractional angles γ_x and γ_y;

(ii) the special Hermite-to-Laguerre mode converter $S_1(\pi/4)$ as reported by Simon and Agarwal, see [2, equation (14)];

(iii) a spherical lens with focal distance $w^2/\lambda\tan\gamma$ (with λ denoting the wavelength of the optical field).

We conclude that knowledge of the generating function is useful for the design of mode converters.

Acknowledgments

T. Alieva thanks the Spanish Ministry of Education and Science for financial support ('Ramon y Cajal' grant and projects TIC 2002-01846 and TIC 2002-11581-E).

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