GABOR'S DISCRETE SIGNAL EXPANSION AND THE DISCRETE GABOR TRANSFORM ON A NON-SEPARABLE LATTICE

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ABSTRACT

Gabor's discrete signal expansion and the discrete Gabor transform are formulated on a general, non-separable time-frequency lattice instead of on the traditional rectangular lattice. The representation of the general lattice is based on the rectangular lattice via a shear operation, which corresponds to a description of the general lattice by means of a lattice generator matrix that has the Hermite normal form. The shear operation on the lattice is associated with simple operations on the signal, on the synthesis and the analysis window, and on Gabor's expansion coefficients; these operations consist of multiplications by quadratic phase terms. This procedure makes it possible to reuse algorithms, which are designed for a rectangular lattice only, to calculate the analysis window, Gabor's expansion coefficients and Gabor's expansion on a general non-separable lattice.

1. INTRODUCTION

Recently new sampling lattices have been introduced [1] as a sampling geometry in the Gabor scheme, which geometry is different from the traditional rectangular sampling geometry. While the basic operations in Gabor's signal expansion and the Gabor transform – time shifting and modulation – are independent operations in the case of a rectangular time-frequency lattice, these operations are no longer independent in the case of non-separable time-frequency lattices. In this paper we show how results that hold for rectangular sampling (see, for instance, [2, 3, 4]) can be transformed to the general, non-separable case. This is achieved by describing the non-separable lattice by means of a lattice generator matrix. We continue with demonstrating how the general, non-separable lattice can be derived from a rectangular lattice by means of a simple shear operation, once the lattice generator matrix is represented in its Hermite normal form [5]. Moreover, we show how this shear operation on the rectangular lattice corresponds to multiplication by quadratic phase terms of the signal, the window functions, and the expansion coefficients. We thus arrive at a simple procedure to transform results that are well known for the rectangular sampling geometry, to the general, non-separable case. As a result, algorithms that are designed for a rectangular lattice explicitly, can be reused for the general non-separable case.

2. GABOR'S SIGNAL EXPANSION ON A RECTANGULAR LATTICE

We start with the usual Gabor expansion [2, 3, 4, 6] on a rectangular time-frequency lattice for a discrete signal $\varphi[n]$,

$$\varphi[n] = \sum_{k=0}^{K-1} \sum_{m} a_{mk} g_{mk}[n], \tag{1}$$

with

$$g_{mk}[n] = g[n - mN] e^{j2\pi kn/K}. \tag{2}$$

The array of Gabor coefficients a_{mk} is periodic in k with period K, and can be found via the Gabor transform

$$a_{mk} = \sum_{\ell} \varphi[\ell] \gamma^{*}_{mk}[\ell], \tag{3}$$

with $\gamma_{mk}[\ell]$ the shifted and modulated versions of the analysis window $\gamma[\ell]$ [cf. (2)].

3. GABOR'S SIGNAL EXPANSION ON A NON-SEPARABLE LATTICE

The rectangular (or separable) lattice that we considered in the previous section can be obtained by integer combinations of two orthogonal vectors

$$\boldsymbol{v}_0 = [N,\ 0]^T \quad \text{and} \quad \boldsymbol{v}_1 = [0,\ 1/K]^T .$$

We thus express the lattice Λ in the form

$$\Lambda = \{ n_0 \boldsymbol{v}_0 + n_1 \boldsymbol{v}_1 | n_0, n_1 \in \mathbb{Z} \}.$$

We now consider Gabor's signal expansion on a time-frequency lattice that is no longer separable. We call a time-frequency lattice non-separable, if the time-shifts and the modulations in the shifted and modulated windows $g_{mk}[n]$ and $\gamma_{mk}[n]$ are no independent operations anymore. Such a lattice is obtained by integer combinations of two linearly independent, but no longer orthogonal vectors, which we express in the forms

$$\boldsymbol{v}_0 = [aN,\ c/DK]^T, \quad \boldsymbol{v}_1 = [bN,\ d/DK]^T$$

with a, b, c and d integers, N and K integers, and $D = ad - bc$. The first component in the vectors $\boldsymbol{v}_0$ and $\boldsymbol{v}_1$ corresponds to a time-shift aN and bN, respectively, while the second component corresponds to a modulation by a frequency c/DK and d/DK, respectively.

Each point $\boldsymbol{\lambda} \in \Lambda$ in the time-frequency plane can be obtained by a matrix-vector product

$$\forall_{\boldsymbol{\lambda}\in\Lambda} \exists_{\boldsymbol{n}\in\mathbb{Z}^2} \;\; \boldsymbol{\lambda} = \boldsymbol{U}\boldsymbol{L}\boldsymbol{n},$$

with

$$\boldsymbol{U} = \frac{1}{DK}\begin{bmatrix} NDK & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{L} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The column vectors $\boldsymbol{v}_0$ and $\boldsymbol{v}_1$ are the columns of the lattice generator matrix $\boldsymbol{UL}$. Note, moreover, that D is equal to the determinant of the matrix $\boldsymbol{L}$. We assume that the integers a and b have no common divisors, and that the same holds for the integers c and d; hence $\gcd(a, b) = 1$ and $\gcd(c, d) = 1$. A possible common divisor can be unified with N and K. The area of a cell (a parallelogram) in the time-frequency plane is equal to the determinant of the lattice generator matrix $\boldsymbol{UL}$, which is equal to N/K. It is well known that the set of shifted and modulated versions of the window is not complete in the case that $N/K > 1$. The equality $N/K = 1$ corresponds to critical sampling, and $N/K < 1$ corresponds to oversampling.

It is clear that for a given matrix $\boldsymbol{U}$, there are a lot of matrices $\boldsymbol{L}$ that generate the same lattice Λ. One form, the Hermite normal form [5] is very interesting; it reads

$$\boldsymbol{L}' = \begin{bmatrix} 1 & 0 \\ -r & D \end{bmatrix},$$

where the integer $-r$ equals $h_0 c + h_1 d + C_0 D$, with the integers h_0 and h_1 such that $h_0 a + h_1 b = 1$ and C_0 an arbitrary integer. Note that these integers h_0 and h_1 exist, since $\gcd(a, b) = 1$, and that they can be obtained by the Euclidean algorithm (see, for instance, [7]). The columns of the matrix $\boldsymbol{UL}'$ are equal to $[N,\ -r/DK]^T$ and $[0,\ 1/K]^T$, respectively. Consequently, the shifted and modulated versions $g_{mk}[n]$ of the synthesis window $g[n]$ on the lattice Λ take the form

$$g_{mk}[n] = g[n - mN]e^{-j2\pi rmn/DK}e^{j2\pi kn/K}. \quad (4)$$

The shifted and modulated versions $\gamma_{mk}[n]$ of the analysis window $\gamma[n]$ are defined similarly. With this modified definition (4) of the set of shifted and modulated window functions, the original expressions for Gabor's signal expansion (1) and the Gabor transform (3) remain valid in the non-separable case. Note that in the case of a rectangular lattice, i.e., $D = 1$ and $r = 0$, Eq. (4) indeed reduces to Eq. (2).

4. FROM THE NON-SEPARABLE TO THE SEPARABLE CASE

A lot of algorithms to calculate the Gabor transform and Gabor's signal expansion for the rectangular case in an efficient way have been developed in the last two decades. Therefore it would be very interesting if these algorithms could be reused for the non-separable case. We will show that this is possible by rewriting Eq. (4) as follows

$$\begin{aligned} &g_{mk}[n] = \\ &g[n - mN]e^{j2\pi\left((n-mN)^2 - n^2 - m^2N^2\right)r/2NDK} \\ &\times e^{j2\pi kn/K}. \end{aligned}$$

The same is done for the shifted and modulated versions $\gamma_{mk}[n]$ of the analysis window $\gamma[n]$. Substitution of this into the Gabor transform yields

$$a'_{mk} = a_{mk}e^{-j2\pi m^2N^2r/2NDK} = \sum_{\ell}\varphi'[\ell]\gamma'^*_{mk}[\ell], \quad (5)$$

with $\varphi'[n] = \varphi[n]\exp(j2\pi n^2 r/2NDK)$ and $\gamma'[n] = \gamma[n]\exp(j2\pi n^2 r/2NDK)$. Substitution into the Gabor expansion yields

$$\varphi'[n] = \sum_{k=0}^{K-1}\sum_{m} a'_{mk}g'_{mk}[n], \quad (6)$$

with $g'[n] = g[n]\exp(j2\pi n^2 r/2NDK)$. The shifted and modulated versions of the synthesis window $g'[n]$ and the analysis window $\gamma'[n]$ are now on a rectangular lattice [cf. Eq. (2)], i.e.,

$$g'_{mk}[n] = g'[n - mN]e^{j2\pi kn/K}$$

and

$$\gamma'_{mk}[n] = \gamma'[n - mN]e^{j2\pi kn/K}.$$

Consequently, algorithms for the rectangular lattice can be reused, but the signal $\varphi[n]$, the windows $g[n]$ and $\gamma[n]$ have to be pre-multiplied by a quadratic phase term. After calculating the Gabor coefficients a'_{mk}, these Gabor coefficients a'_{mk} have to be post-multiplied by a quadratic phase term to obtain the Gabor coefficients a_{mk} for the non-separable lattice. A similar procedure holds for the Gabor expansion. In this case, after reconstructing the signal $\varphi'[n]$ by using an efficient algorithm for a rectangular lattice, the signal $\varphi'[n]$ has to be post-multiplied by a quadratic phase term to obtain the original signal $\varphi[n]$.

5. PERIODIZED GABOR EXPANSION

Most of the algorithms to calculate the Gabor coefficients a_{mk} and Gabor's signal expansion assume that the signals are periodic. This is due to the fact that these algorithms use the Fourier transform and the Zak transform for periodic signals. In order to reuse these algorithms, the Gabor transform (5) and Gabor's signal expansion (6) with the 'primed' signals have to be periodized.

We suppose that the window $\gamma[n]$ and the signal $\varphi[n]$ have a finite support N_γ and N_φ, respectively. Under these conditions of finite support, the array a_{mk}, which is periodic in the k-variable, has a finite support M in the m-variable, where the support M satisfies the condition

$$MN \geq N_\varphi + N_\gamma - 1.$$

Note that M thus corresponds with the number of time-shifts. For convenience we introduce the common divisor J of the integers K and N, i.e., $J = \gcd(K, N)$, and the integers $p = K/J$ and $q = N/J$.

We now introduce the periodized version A_{mk} of the array a_{mk} and the periodized signal $\Phi[n]$ of the signal $\varphi[n]$

$$A_{mk} = \sum_s a_{m+sM,k}, \quad \Phi[n] = \sum_s \varphi[n + sMN],$$

and the periodized windows $G[n]$ and $\Gamma[n]$ of the windows $g[n]$ and $\gamma[n]$, respectively,

$$G[n] = \sum_s g[n + sMN], \quad \Gamma[n] = \sum_s \gamma[n + sMN].$$

In [8] it has been shown that under the conditions that D is a divisor of M and DK is a divisor of MN (this means that M can be written as $M = pLD$ for an integer L), we have the following relationships

$$\Phi[n] = \sum_{k=0}^{K-1} \sum_{m=\langle M \rangle} A_{mk} G_{mk}[n]$$

and

$$A_{mk} = \sum_{n=\langle MN \rangle} \Phi[n] \Gamma^*_{mk}[n],$$

which are periodized versions of Gabor's signal expansion (1) and the Gabor transform (3), respectively. The expression $m = \langle M \rangle$ throughout denotes a finite interval of M successive integers m.

However, due to the quadratic phase terms, the array a'_{mk} cannot always be identified as one period of the periodized version A'_{mk}

$$A'_{mk} = \sum_s a'_{m+sM,k}.$$

Something similar holds for the periodized Gabor expansion (6). One can show that if the additional condition $2K$ is a divisor of $\gcd(2, pLDN)rpL$ is fulfilled then we have the following relationships

$$\Phi'[n] = \sum_{k=0}^{K-1} \sum_{m=\langle M \rangle} A'_{mk} G'_{mk}[n]$$

and

$$A'_{mk} = \sum_{n=\langle MN \rangle} \Phi'[n] {\Gamma'}^*_{mk}[n].$$

The integer r in the matrix $\boldsymbol{L}$ in its Hermite normal form is not unique. It has the form $r = r_0 + C_0 D$ with $0 \leq r_0 < D$. From the additional condition it follows that we have to find integers C_0 and C_1 such that

$$2KC_1 = \gcd(2, pLDN)(r_0 + C_0 D)pL$$

is fulfilled. This expression is equivalent to

$$2JC_1 / \gcd(2, pLDN) - LDC_0 = r_0 L, \tag{7}$$

which is solvable if $\gcd(2J/\gcd(2, pLDN), LD)$ is a divisor of $r_0 L$. This means that algorithms for calculating the Gabor transform and Gabor's signal expansion on a rectangular lattice for periodic signals can be reused in the non-separable case if the additional condition is fulfilled, i.e., $\gcd(2J/\gcd(2, pLDN), LD)$ is a divisor of $r_0 L$. Note that the integer r is thus determined by C_0 in the periodic case.

6. THE ANALYSIS WINDOW

Calculating the analysis window $\Gamma[n]$ (or dual window) plays an important role in Gabor analysis, since this window is used to calculate the Gabor coefficients A_{mk}. It has been shown that in the case of Gabor's signal expansion on a rectangular lattice, this analysis window should meet the following condition

$$\begin{aligned} &\sum_{m=<M>} \Gamma^*[n - \ell K - mN] G[n - mN] \\ &= \frac{1}{K} \sum_k \delta[\ell - kqL], \end{aligned} \tag{8}$$

for $\ell = \langle qL \rangle$. Following the procedure outlined above and substituting the 'primed' window functions into the condition (8) and replacing L by LD, it is not difficult to show that, in the general non-separable case this condition takes the form

$$\sum_{m=<M>} e^{-j2\pi m\ell r/D}\Gamma^*[n-\ell K-mN]G[n-mN]$$
$$=\frac{1}{K}\sum_k \delta[\ell-kqLD],$$

for $\ell = \langle qLD\rangle$, which is exactly the same condition as shown in [8]. As a consequence, algorithms to calculate the dual window for the rectangular case can be reused, as well. However, the additional condition that $\gcd(2J/\gcd(2,pLDN),LD)$ is a divisor of r_0L must be fulfilled again.

7. EXAMPLE

As an example we consider the quincunx lattice. In this case the matrix $\boldsymbol{L}$ looks, e.g., like

$$\boldsymbol{L}=\begin{bmatrix}1 & 0\\ 1 & 2\end{bmatrix}.$$

We choose $p = 4$, $q = 3$, $J = 2$ and $L = 1$, i.e., the oversampling is $p/q = 4/3$ and the length of one signal period is $MN = pLDqJ = 48$. In this case $\gcd(2J/\gcd(2,pLDN),LD)=2$ is not a divisor of $rL = -1$. As a consequence, an algorithm designed for the rectangular lattice is not reusable for this particular example if we use the method described in the previous sections.

One solution to overcome this, is extending the signals with a factor two by zero-padding the signals with 48 zeroes. The parameters remain the same, except that $L = 2$ in this case. Now $\gcd(2J/\gcd(2,pLDN),LD) = 2$ is a divisor of $rL = -2$ and as a consequence, the algorithm is reusable. With these parameters, the solution of Eq. (7) becomes $C_0 = k$ and $C_1 = -1+2k$ with k an arbitrary integer. As a result, we have to choose $r = -1+2k$.

8. DISCUSSION

As shown, the method above does not cover all possible situations in the case of periodic signals, due to the quadratic phase term. Changing parameters is a solution, however this can be undesirable in practice. In that case, algorithms designed for the non-separable case can be used (see [8]).

Other transformations could be used to transform the results in the rectangular Gabor scheme to the non-separable case. However, it is very likely, that periodizing the signals will result in similar problems. Nevertheless, it is a subject of further research.

9. CONCLUSIONS

We have shown how the discrete Gabor scheme for the rectangular (or separable) lattice can be extended to the general, non-separable lattice in a structured way; this was achieved by describing the non-separable lattice by means of a lattice generator matrix. We have written the generator matrix in the Hermite normal form to obtain a shear representation on the shifted and modulated windows, which shear representation then lead to a modification of the rectangular Gabor scheme and resulted in the Gabor scheme on a non-separable lattice. Efficient algorithms that are well known for the rectangular case, could thus easily be applied to the non-separable case: we have shown that the shear operation merely involves pre- and post-multiplications by quadratic phase terms.

10. REFERENCES

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