ON OPTIMUM OVERSAMPLING IN THE GABOR SCHEME

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ABSTRACT

The windowed Fourier transform of a time signal is considered, as well as a way to reconstruct the signal from a sufficiently densely sampled version of its windowed Fourier transform using a Gabor representation; following Gabor, sampling occurs on a two-dimensional time-frequency lattice with equidistant time intervals and equidistant frequency intervals. In the limit of infinitely dense sampling, the optimum synthesis window (which appears in Gabor's reconstruction formula) becomes similar to the analysis window (which is used in the windowed Fourier transform). It is shown that this similarity can already be reached for a rather small degree of oversampling, if the sampling distances in the time and frequency directions are properly chosen. A procedure is presented with which the optimum ratio of the sampling intervals can be determined. The theory is elucidated by finding the optimum ratio in the cases of a Gaussian and an exponential analysis window.

1. INTRODUCTION

In recent years, the sampled version of the windowed Fourier transform of an arbitrary signal has received considerable interest. The sampling values that result from this windowed Fourier transform (with a given analysis window) can be used to reconstruct the signal by means of the Gabor expansion [1] (with a proper synthesis window, to be determined from the analysis window). In sampling the windowed Fourier transform, we have some freedom in choosing the sampling distances in the time and the frequency direction, as long as the density of the sampling lattice, which is determined by the *product* of the two sampling distances, is sufficiently high. But the *ratio* between the two sampling distances appears to be important, as well. Indeed, although the shape of the optimum synthesis window will always become identical to that of the analysis window for a sufficiently small value of the product of the two sampling distances – i.e. a very large degree of oversampling – this limit will be reached for a much smaller degree of oversampling if the ratio of the sampling distances is properly chosen. The present paper presents a procedure with which the optimum value of this ratio can be determined.

2. WINDOWED FOURIER TRANSFORM

Let the *windowed Fourier transform* $S_\varphi(t,\omega)$ of a signal $\varphi(t)$ be defined as

$$S_\varphi(t,\omega) = \int \varphi(t')w^*(t'-t)e^{-j\omega t'}dt', \qquad (1)$$

where $w(t)$ is the window function. (All integrations and summations in this paper extend from $-\infty$ to $+\infty$.) We note that the windowed Fourier transform can be considered as the Fourier transform of the product of the signal $\varphi(t)$ and a shifted and complex conjugated version of the *analysis window* $w(t)$. The window function may be chosen rather arbitrarily; mostly it will be a function that is more or less concentrated around the origin. In this paper we will throughout use the Gaussian window function

$$w(t) = 2^{\frac{1}{4}}e^{-\pi(t/T)^2} \qquad (2)$$

and the exponential window function

$$w(t) = \sqrt{a}e^{-a|t|/T} \quad \text{with} \quad a > 0 \qquad (3)$$

as examples. Note that we have normalized these functions such that their L_2 norms $\int |w(t)|^2 dt$ equal T.

The signal $\varphi(t)$ can be reconstructed from its windowed Fourier transform by an inverse Fourier transformation, of course. A different and more interesting way to reconstruct the signal from its windowed Fourier transform is by means of the *inversion formula*

$$\varphi(t')\int |w(t)|^2 dt = \frac{1}{2\pi}\iint S_\varphi(t,\omega)w(t'-t)e^{j\omega t'}dt\,d\omega; \qquad (4)$$

this relationship can easily be proved by substituting from the definition (1) of the windowed Fourier transform. However, in order to reconstruct the signal we need not know the entire windowed Fourier transform; it suffices to know its values at the points of the rectangular lattice ($t = m\alpha T, \omega = k\beta\Omega$) in the time-frequency domain, where $\Omega T = 2\pi$ and $\alpha\beta \leq 1$, and where m and k take all integer values. Note that the rectangular cells of this lattice occupy an area of $2\pi\alpha\beta$ in the time-frequency domain. With the sampling values defined as

$$a_{mk} = S_\varphi(m\alpha T, k\beta\Omega), \tag{5}$$

the signal $\varphi(t)$ can then be reconstructed by considering these sampling values as the *coefficients* in *Gabor's signal expansion*, with a *synthesis window* $g(t)$ that still has to be determined; thus

$$\varphi(t) = \sum_m \sum_k a_{mk} g(t - m\alpha T) e^{jk\beta\Omega t}. \tag{6}$$

In the case of *critical sampling* (i.e., $\alpha\beta = 1$), the synthesis window $g(t)$ is uniquely determined by the analysis window $w(t)$ [2, 3], whereas in the case of *oversampling* (i.e., $\alpha\beta < 1$), the synthesis window is no longer unique. In the oversampled case, the synthesis window is very often chosen such that it has minimum L_2 norm. This so-called *optimum* synthesis window $g_{opt}(t)$ not only has minimum L_2 norm, but it also resembles best (in a minimum L_2 norm sense, again) the analysis window $w(t)$. It is easy to see that for *infinite* oversampling ($\alpha T \downarrow 0, \beta\Omega \downarrow 0$), Gabor's signal expansion (6) becomes equivalent to the inversion formula (4), and the (optimum) synthesis window is indeed proportional to the analysis window:

$$g_{opt}(t) \int |w(t)|^2 dt \simeq \alpha\beta\ w(t). \tag{7}$$

As an illustration we have depicted in Fig. 1 the optimum synthesis window $g_{opt}(t)$ that corresponds to the Gaussian analysis window (2) for different values of α and β (with $\alpha = \beta$); note that the resemblance between the synthesis and the analysis window increases with increasing degree of oversampling. Several ways are described in the literature to determine the optimum synthesis window [4, 5, 6, 7, 8].

3. OPTIMUM SAMPLING LATTICE

The optimum synthesis window depends, of course, on the choice of the sampling parameters α and β. In Fig. 1 these sampling parameters were chosen identical ($\alpha = \beta$), since this choice yields the best result in the case of a Gaussian analysis window, as we will show. As an illustration of non-identical sampling parameters we have depicted in Fig. 2 the optimum synthesis window that corresponds to the Gaussian analysis window (2)

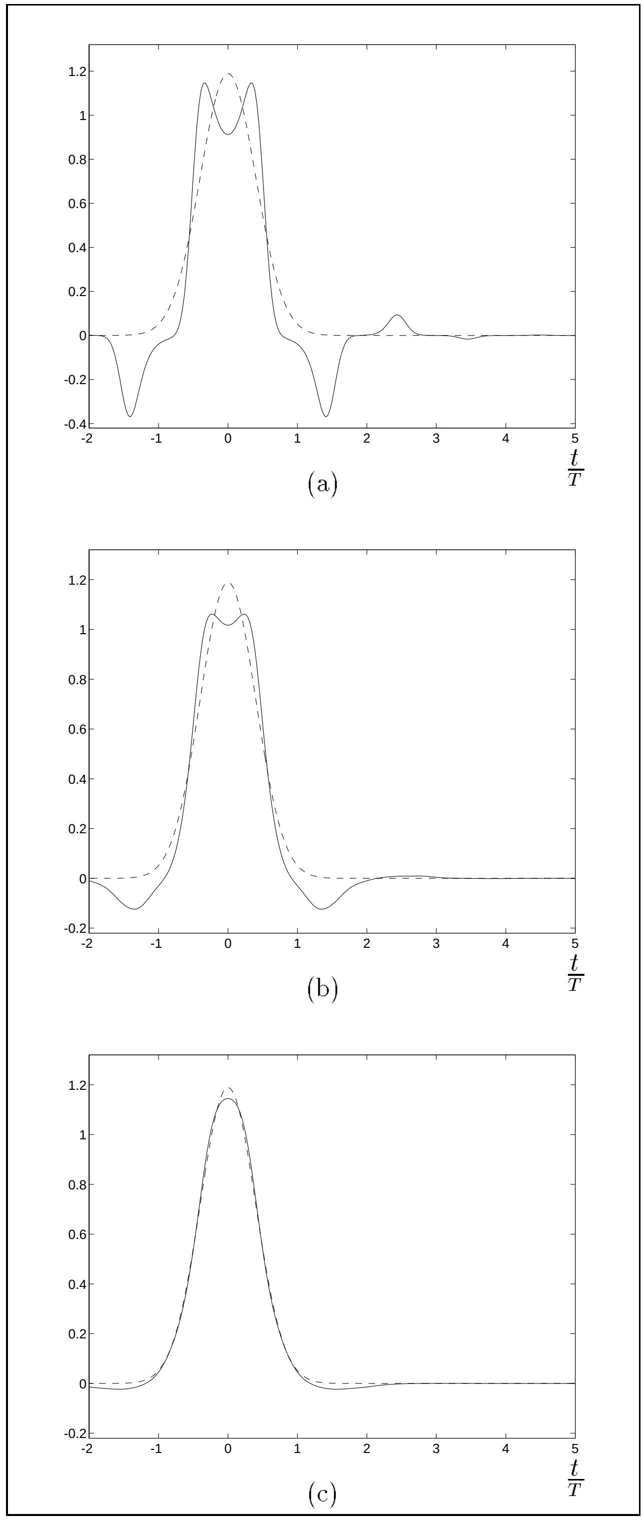

Figure 1: A Gaussian analysis window $w(t)$ (dashed line) and its corresponding optimum synthesis window $g_{opt}(t)$ (solid line) for different values of oversampling: (a) $\alpha = \beta = \sqrt{6/7}$, (b) $\alpha = \beta = \sqrt{2/3}$, and (c) $\alpha = \beta = \sqrt{2/5}$.

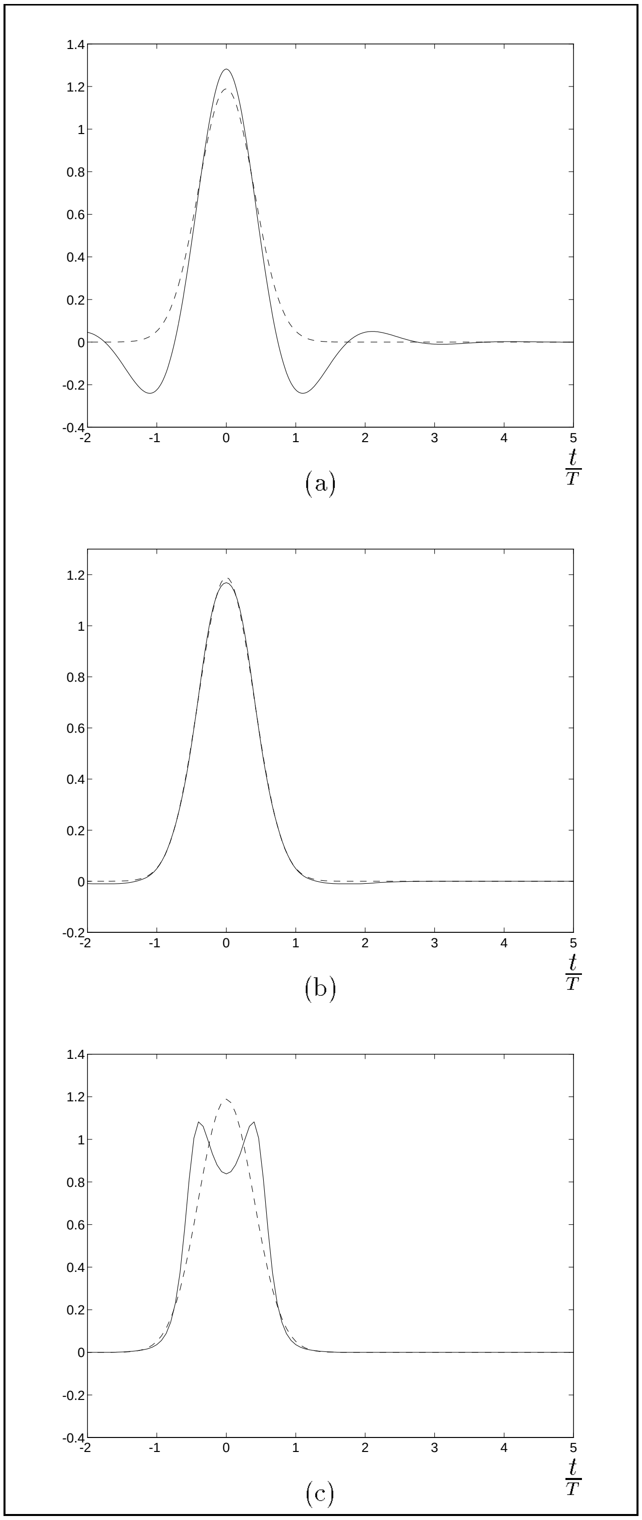

Figure 2: A Gaussian analysis window $w(t)$ (dashed line) and its corresponding optimum synthesis window $g_{opt}(t)$ (solid line) in the case of oversampling by a factor 3, for different values of α and β: (a) $\alpha = 1/3$, $\beta = 1$, (b) $\alpha = \beta = \sqrt{1/3}$, and (c) $\alpha = 1$, $\beta = 1/3$.

for several values of α, while maintaining the relation $\alpha\beta = 1/3$. It is the aim of this paper to derive the optimum values of the sampling parameters α and β for a given degree of oversampling $1/\alpha\beta$.

To find a procedure for determining the optimum parameter values we proceed as follows. Let us express the analysis window $w(t)$ by means of its Gabor expansion [cf. Eq. (6)]:

$$w(t) = \sum_m \sum_k c_{mk} g(t - m\alpha T) e^{jk\beta\Omega t}. \qquad (8)$$

The Gabor coefficients c_{mk} follow as the sampling values of the windowed Fourier transform $S_w(t,\omega)$:

$$\begin{aligned} c_{mk} &= S_w(m\alpha T, k\beta\Omega) \\ &= \int w(t') w^*(t' - m\alpha T) e^{-jk\beta\Omega t'} dt'. \qquad (9) \end{aligned}$$

If the degree of oversampling $1/\alpha\beta$ is sufficiently high, the optimum synthesis window $g_{opt}(t)$ resembles the analysis window $w(t)$ [see Eq. (7)], and the Gabor expansion (8) takes the form

$$w(t) \simeq \frac{\alpha\beta}{c_{00}} \sum_m \sum_k c_{mk} w(t - m\alpha T) e^{jk\beta\Omega t}. \qquad (10)$$

From Eq. (10) it is obvious that we would like the array of Gabor coefficients c_{mk} to be as concentrated around the origin as possible. Optimal choices for the sampling parameters α and β may now be found by considering the widths $2d_t$ and $2d_\omega$ of the function $S_w(t,\omega)$ in the t-direction and the ω-direction, respectively, and by taking the ratio of the sampling distance αT in the time direction and the sampling distance $\beta\Omega$ in the frequency direction equal to d_t/d_ω. As measures of d_t^2 and d_ω^2 we might choose, for instance, the normalized second-order moments of the one-dimensional functions $|S_w(t,0)|^2$ and $|S_w(0,\omega)|^2$, respectively:

$$d_t^2 = \frac{\int t^2 |S_w(t,0)|^2 dt}{\int |S_w(t,0)|^2 dt} \quad \text{and} \quad d_\omega^2 = \frac{\int \omega^2 |S_w(0,\omega)|^2 d\omega}{\int |S_w(0,\omega)|^2 d\omega}. \qquad (11)$$

Note that $S_w(0,\omega)$ is in fact the Fourier transform of the function $|w(t)|^2$, and that $S_w(t,0)$ is in fact the inverse Fourier transform of the squared absolute value of the Fourier transform of $w(t)$.

As an example let us determine the optimum values of the ratio α/β for the Gaussian analysis window (2) and for the exponential analysis window (3), for which windows the function $S_w(t,\omega)$ takes the form

$$S_w(t,\omega)\ e^{j\frac{1}{2}\omega t} = T\ e^{-\frac{1}{2}\pi[(t/T)^2 + (\omega/\Omega)^2]} \qquad (12)$$

and

$$S_w(t,\omega)\ e^{j\frac{1}{2}\omega t} = T\ \frac{1}{1 + (\pi/a)^2(\omega/\Omega)^2}\ e^{-a|t/T|}$$

$$\times \left(a \left| \frac{t}{T} \right| \frac{\sin[\pi(\omega/\Omega)(t/T)]}{\pi(\omega/\Omega)(t/T)} + \cos[\pi(\omega/\Omega)(t/T)] \right) , \tag{13}$$

respectively. In the case of the Gaussian analysis window, the function $|S_w(t,\omega)|^2$ is symmetrical in t/T and ω/Ω. As a consequence, we expect that the best choice for the sampling parameters is $\alpha = \beta$, which would be in accordance with the result that we showed already in Fig. 2b. For the widths $2d_t$ and $2d_\omega$ we find indeed $2d_t/T = 2d_\omega/\Omega = 2/\sqrt{\pi} \simeq 1.13$, and the ratio $\alpha/\beta = (2d_t/T)/(2d_\omega/\Omega)$ takes the value 1.

In the case of the exponential analysis window (or any other window function whose Fourier transform has a shape that is different from the shape of the window function itself), we do not have such a nice symmetry in t/T and ω/Ω. Nevertheless, we can still determine the best choices for the sampling parameters α and β. For the exponential window (3) we find $2d_t/T = 2\sqrt{7/5}/a$ and $2d_\omega/\Omega = 2a/\pi$; hence, the ratio $\alpha/\beta = (2d_t/T)/(2d_\omega/\Omega)$ should take the value $\alpha/\beta = \pi\sqrt{7/5}/a^2$. For the special value $a = \sqrt{\pi}(7/5)^{1/4} \simeq 1.928$, the ratio α/β takes the value 1. As an illustration we have depicted in Fig. 3 the optimum synthesis window that corresponds to the exponential window (3) with $a = \sqrt{\pi}(7/5)^{1/4}$ for several values of α, while maintaining the relation $\alpha\beta = 1/9$.

4. REFERENCES

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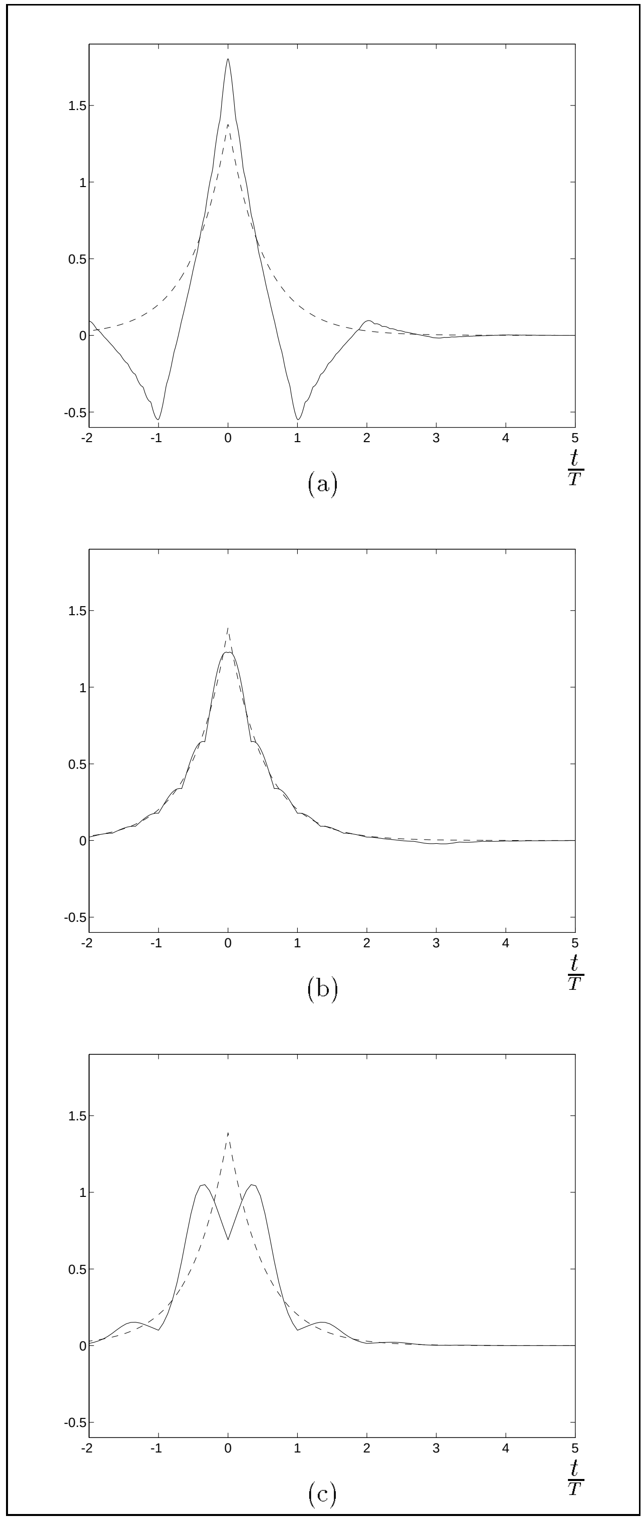

Figure 3: An exponential analysis window $w(t)$ (dashed line) and its corresponding optimum synthesis window $g_{opt}(t)$ (solid line) in the case of oversampling by a factor 9, for different values of α and β: (a) $\alpha = 1/9$, $\beta = 1$, (b) $\alpha = \beta = 1/3$, and (c) $\alpha = 1$, $\beta = 1/9$.