SPECTRAL ANALYSIS OF DISCRETE SIGNALS GENERATED BY MULTIPLICATIVE AND ADDITIVE ITERATIVE PROCEDURES

T. Alieva, M.J. Bastiaans

Technische Universiteit Eindhoven
Faculteit Elektrotechniek
Postbus 513, 5600 MB Eindhoven, Netherlands
email: `t.i.alieva@ele.tue.nl`

A. Barbé

Katholieke Universiteit Leuven
Department of Electrical Engineering
ESAT, Kardinaal Mercierlaan 94
B-3001 Leuven, Belgium

ABSTRACT

The discrete Fourier transform of signals constructed through multiplicative and additive iterative procedures is determined and its specific features are considered. It is shown that – in spite of the rather different structure of multiplicative and additive signals – the Fourier transforms of both types of signals exhibit the property of self-affinity. The power spectra of additive signals produced by different generating vectors have similar forms and can be divided into similar branches. The number of branches depends on the generation level and the symmetry of the power spectrum of the generating vector.

1. INTRODUCTION

Numerous objects in science and engineering arising in many natural phenomena (e.g., turbulence, fluid mixing, aggregation) exhibit a fractal or multifractal structure [1]. The typical property of such objects is that they are infinitely complex: a small portion of it is, in a certain way, similar to the original. In signal processing, some types of noise can be treated as fractal structures [2]. Moreover, fractal signals, which keep their main characteristics under time and frequency scaling, promise to be important for communications and for many other applications [3, 4, 5].

Multifractal structures are frequently associated with cascade processes, and their models are generated through iterative procedures [6, 7]. Real-world fractal structures usually exhibit their self-affine properties only for a certain number of scales and are described in the model by the finite number of iterations n. In this paper we consider discrete signals, denoted by a row vector $\mathbf{f}_n$ and constructed from a generating vector $\mathbf{f}_1$ of size M,

$$\mathbf{f}_1 = [f_1(0), f_1(1), \ldots, f_1(m), \ldots, f_1(M-1)] \tag{1}$$

with $f_1(m) \in C$, through the following iterative procedures:

$$\mathbf{f}_n = \begin{bmatrix} [\mathbf{f}_{n-1}]^t \oplus f_1(0) \\ [\mathbf{f}_{n-1}]^t \oplus f_1(1) \\ \vdots \\ [\mathbf{f}_{n-1}]^t \oplus f_1(M-1) \end{bmatrix}^t, \tag{2}$$

where the symbol $\oplus$ stands for multiplication and summation in the multiplicative and the additive case, respectively. A signal of level n: $\mathbf{f}_n = [f_n(0), f_n(1), \ldots, f_n(m), \ldots, f_n(M^n-1)]$ has M^n elements. Its coordinates can be written as

$$f_n(k + M^{n-1}m) = f_1(m) \oplus f_{n-1}(k), \tag{3}$$

where $k = 0, 1, \ldots, M^{n-1}-1$ and $m = 0, 1, \ldots, M-1$. It can be shown (see [8] for the case of additive vectors) that

$$f_n(m + Mk) = f_{n-1}(k) \oplus f_1(m). \tag{4}$$

Equations (3) and (4) reveal the fractal structure of these signals. The affine transformation of the n^{th}-generation signal $\mathbf{f}_n$ produces the similar structure corresponding to the $(n-1)^{th}$-generation signal $\mathbf{f}_{n-1}$.

In this paper we derive the expression for the discrete Fourier transform (DFT) of multiplicative and additive signals, which reflects the hierarchial structure of such signals. The specific features of the power spectra of these signals are considered and are elucidated by some particular examples.

2. MULTIPLICATIVE SIGNALS

We start from the consideration of the signals $\mathbf{f}_n$ generated by the multiplicative iterative procedure (2). Such types of multiplicative cascades for a real positive generator are used for producing multiplicative measures appearing in a variety of natural phenomena [6]. As an examples, it is easy to see that the generator $\mathbf{f}_1 = (1, 0, 1)$ produces the well-known triadic Cantor set [1]. Figure 1 shows a graphical representation of the multiplicative vector magnitude $|f_n(l)|$ for $n = 10$, generated by the complex vector $\mathbf{f}_1 = (1, 0, 3+2i)$. Note also that a generating vector with only one nonzero element produces a multiplicative signal with also only one nonzero element.

The discrete Fourier transform (DFT) of a sequence $\mathbf{f}_n = [f_n(0), f_n(1), \ldots, f_n(m), \ldots, f_n(M^n-1)]$ is given by [9]

$$F_n(l) = \sum_{k=0}^{M^n-1} f_n(k) \exp\left(-\frac{i2\pi lk}{M^n}\right), \tag{5}$$

where l is an integer. It is well known that the DFT of a sequence which contains M^n points, is periodic with period M^n:

$$F_n(l + M^n k) = F_n(l). \tag{6}$$

Therefore we will investigate the structure of $\mathbf{F}_n$ in the region $l \in [0, M^n-1]$.

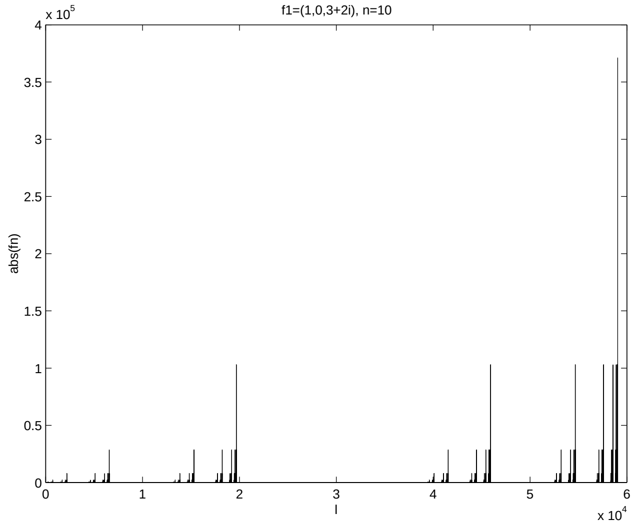

Figure 1: Magnitude of the multiplicative signal of level 10, constructed by the generator $\mathbf{f}_1 = (1, 0, 3 + 2i)$.

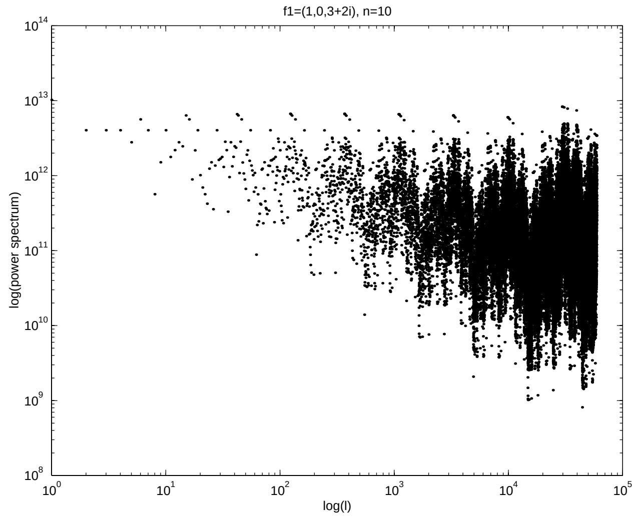

Figure 2: Power spectrum of the multiplicative signal of level 10, constructed by the generator $\mathbf{f}_1 = (1, 0, 3 + 2i)$.

Substituting from Eq. (3) into Eq. (5) we get

$$\begin{aligned} F_n(l) &= \sum_{k=0}^{M^{n-1}-1} f_{n-1}(k) \exp\left(-\frac{i2\pi lk}{M^n}\right) \\ &\quad \times \sum_{m=0}^{M-1} f_1(m) \exp\left(-\frac{i2\pi lm}{M}\right) \\ &= F_1(l) \sum_{k=0}^{M^{n-1}-1} f_{n-1}(k) \exp\left(-\frac{i2\pi lk}{M^n}\right), \quad (7) \end{aligned}$$

where $F_1(l)$ is the DFT of the generator $\mathbf{f}_1$. Since

$$F_{n-1}\left(\frac{l}{M}\right) = \sum_{k=0}^{M^{n-1}-1} f_{n-1}(k) \exp\left(-\frac{i2\pi lk}{M^n}\right)$$

corresponds to the DFT of $\mathbf{f}_{n-1}$ at the points l/M, Eq. (7) can be expressed in the form

$$F_n(l) = F_{n-1}(l/M)F_1(l). \quad (8)$$

Using the periodicity of the DFT, $F_1(l + Mk) = F_1(l)$, one finds that the DFT of $\mathbf{f}_n$ at the M-decimated point set $\{Mk \mid k = 0, 1, \ldots, M^{n-1} - 1\}$ corresponds to the DFT of $\mathbf{f}_{n-1}$ multiplied by $F_1(0)$:

$$F_n(Mk) = F_{n-1}(k)F_1(0). \quad (9)$$

This relationship, together with Eq. (4) for $m = 0$, $f_n(Mk) = f_{n-1}(k)f_1(0)$, indicates the self-affine structure of the multiplicative signals in the frequency and the time domain. Iterating Eq. (9) then yields for $p = 1, 2, \ldots, M^{n-q} - 1$:

$$F_n(M^q p) = F_1^q(0)F_{n-q}(p). \quad (10)$$

Continuing the iterative procedure (8) we derive the expression for the DFT of $\mathbf{f}_n$:

$$F_n(l) = \prod_{k=0}^{n-1} F_1(l/M^k) \quad (11)$$

with

$$F_1\left(\frac{l}{M^k}\right) = \sum_{m=0}^{M-1} f_1(m) \exp\left(-\frac{i2\pi lm}{M^{k+1}}\right).$$

In particular for $l = 0$ we get

$$F_n(0) = F_1^n(0) = \left(\sum_{m=0}^{M-1} f_1(m)\right)^n. \quad (12)$$

From Eq. (11), the hierarchial structure of the DFT of the multiplicative signal can easily be seen: the DFTs of levels n and $n-1$ are connected as

$$F_n(l) = F_{n-1}(l)F_1(l/M^{n-1}). \quad (13)$$

The power spectrum of $\mathbf{f}_n$ is a product of power spectra of $\mathbf{f}_1$, defined at the fractional points:

$$|F_n(l)|^2 = \prod_{k=0}^{n-1} \left|F_1(l/M^k)\right|^2. \quad (14)$$

If the power spectrum is represented on logarithmic scales, it takes the form of a sum of n periodic functions with periods $\log M$, $2\log M$,...,$n\log M$. This produces quasiperiodicity of the loglog graphic of the power spectrum, where the number of quasiperiods in the region corresponding to $[0, M^n - 1]$ equals $n-1$. Note also that the power spectrum of signals constructed from a generating vector with only one nonzero element is a constant vector.

Let us consider the generalized triadic Cantor set constructed from the generator $\mathbf{f}_1 = (p, 0, q)$, where $p = |p|\exp(i\varphi)$ and $q = |q|\exp(i\psi)$. For the power spectrum of $\mathbf{f}_1$ at the fractional points, which defines the power spectrum of all signals $|F_n(l)|^2$ as we can see from Eq. (14), we have

$$\left|F_1\left(\frac{l}{3^k}\right)\right|^2 = |p|^2 + |q|^2 + 2|pq|\cos\left(2\frac{2\pi l}{3^{k+1}} + \varphi - \psi\right). \quad (15)$$

Figure 2 shows the power spectrum of the multiplicative vector that was represented in Fig. 1 ($p = 1, q = 3 + 2i$), with logarithmic scales for both coordinates. One can easily extract the typical structure which repeats itself with different scale resolutions in $n - 1 = 9$ frequency regions. In the particular case $|p| = |q|$, Eq. (15) reduces to $\left|F_1(l/3^k)\right|^2 = 2^2 |p|^2 \cos^2\left(2\pi l/3^{k+1} + (\varphi - \psi)/2\right)$, and the power spectrum of the Cantor sequence has the following form:

$$|F_n(l)|^2 = 2^{2n} |p|^{2n} \prod_{k=0}^{n-1} \cos^2\left(\frac{2\pi l}{3^{k+1}} + \frac{\varphi - \psi}{2}\right). \qquad (16)$$

Thus, for the following generators $\mathbf{f}_1 = (1, 0, 1)$, $\mathbf{f}_1 = (1, 0, -1)$, and $\mathbf{f}_1 = (1, 0, i)$, which produce Cantor sequences with the same energy distributions $|\mathbf{f}_n|$, we have

$$\begin{aligned} |F_n(l)|^2 &= 2^{2n} \textstyle\prod_{k=0}^{n-1} \cos^2\left(2\pi l/3^{k+1}\right), \\ |F_n(l)|^2 &= 2^{2n} \textstyle\prod_{k=0}^{n-1} \sin^2\left(2\pi l/3^{k+1}\right), \quad \text{and} \\ |F_n(l)|^2 &= 2^{2n} \textstyle\prod_{k=0}^{n-1} \cos^2\left(2\pi l/3^{k+1} + \pi/4\right), \end{aligned}$$

respectively.

3. ADDITIVE SIGNALS

Let us consider the signals constructed through the additive procedure (2), which was used for the generation of so-called fractrices introduced recently [8]. Figure 3 shows a graphical representation of the additive vector magnitude $|f_n(l)|$ for $n = 10$, constructed from the same complex generator $\mathbf{f}_1 = (1, 0, 3 + 2i)$ which was used for the generation of the multiplicative vector in Fig. 1.

Let us derive some relationships for the DFT of the additive vector $\mathbf{f}_n$ for different iterations. It is easy to see from Eqs. (3) and (5) that

$$\begin{aligned} F_n(0) &= \sum_{k=0}^{M^{n-1}-1} \sum_{m=0}^{M-1} [f_{n-1}(k) + f_1(m)] \\ &= M F_{n-1}(0) + M^{n-1} F_1(0), \end{aligned} \qquad (17)$$

which, after n iterations, yields

$$F_n(0) = n M^{n-1} F_1(0). \qquad (18)$$

Substituting from Eq. (3) into Eq. (5) we get

$$\begin{aligned} F_n(l) &= \sum_{k=0}^{M^{n-1}-1} f_{n-1}(k) \exp\left(-\frac{i2\pi lk}{M^n}\right) S_1 \\ &\quad + \sum_{m=0}^{M-1} f_1(m) \exp\left(-\frac{i2\pi lm)}{M}\right) S_2, \end{aligned} \qquad (19)$$

where

$$S_1 = \sum_{m=0}^{M-1} \exp\left(-\frac{i2\pi lm}{M}\right) = \begin{cases} M & \text{if } l = pM \\ 0 & \text{if } l \neq pM \end{cases} \qquad (20)$$

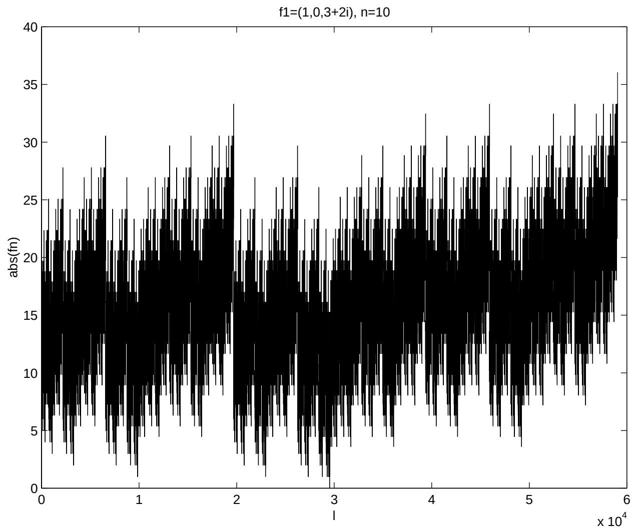

Figure 3: Magnitude of the additive signal of level 10, constructed by the generator $\mathbf{f}_1 = (1, 0, 3 + 2i)$.

and

$$\begin{aligned} S_2 &= \sum_{k=0}^{M^{n-1}-1} \exp\left(-\frac{i2\pi lk}{M^n}\right) \\ &= \begin{cases} M^{n-1} & \text{if } l = 0 \\ 0 & \text{if } l = pM, p \neq 0 \\ \dfrac{\exp(-i\pi l/M)}{\exp(-i\pi l/M^n)} \dfrac{\sin(\pi l/M)}{\sin(\pi l/M^n)} & \text{if } l \neq pM, \end{cases} \end{aligned} \qquad (21)$$

with integer p. Then one finds that the DFT of $\mathbf{f}_n$ at the M-decimated point set $\{Mp \mid p = 1, 2, \ldots, M^{n-1} - 1\}$ corresponds to the DFT of $\mathbf{f}_{n-1}$ multiplied by M:

$$\begin{aligned} F_n(Mp) &= M \sum_{k=0}^{M^{n-1}-1} f_{n-1}(k) \exp\left(-\frac{i2\pi kMp}{M^n}\right) \\ &= M F_{n-1}(p). \end{aligned} \qquad (22)$$

Therefore the DFTs of additive signals exhibit a similar property of self-affinity as the DFTs of the multiplicative ones [see Eq. (9)]. Iterating Eq. (22) then yields for $p = 1, 2, \ldots, M^{n-q} - 1$:

$$F_n(pM^q) = M^q F_{n-q}(p). \qquad (23)$$

In particular, the DFT of a fractal vector $\mathbf{f}_n$ for $l = pM^{n-1}$ produces the DFT of the generating vector f_1: $F_n(pM^{n-1}) = M^{n-1} F_1(p)$ $(p \neq 0)$ and $F_n(0) = n M^{n-1} F_1(0)$.

From Eqs. (17)-(21), it follows that the DFT of $\mathbf{f}_n$ constructed with a generator of size M is given by

$$F_n(l) = \begin{cases} M F_{n-1}(0) + M^{n-1} F_1(0) & l = 0 \\ M F_{n-1}(p) & l = pM, p \neq 0 \\ \dfrac{\exp(-i\pi l/M)}{\exp(-i\pi l/M^n)} \dfrac{\sin(\pi l/M)}{\sin(\pi l/M^n)} F_1(l) & l \neq pM. \end{cases} \qquad (24)$$

Let us now consider the power spectrum $|F_n(l)|^2$ of an additive vector. In order to analyze its structure, we define the following

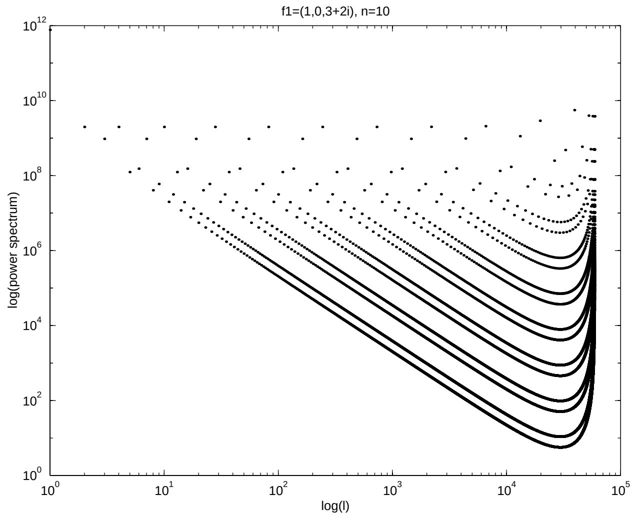

Figure 4: Power spectrum of the multiplicative signal of level 10, constructed by the generator $\mathbf{f}_1 = (1, 0, 3 + 2i)$.

subsets $\mathcal{A}_q$ of the interval $[1, M^n - 1]$ for $q \in [0, n - 2]$

$$\mathcal{A}_q = \left\{ M^q(m + kM) \;\middle|\; \begin{array}{rcl} m & \in & [0, M-1] \\ k & \in & \left[0, M^{n-1-q} - 1\right] \\ m + kM & \neq & pM^j \end{array} \right\}, \tag{25}$$

where p and j are positive integers. Then from Eqs. (23) and (24), and from the periodicity property (6) of the DFT, $F_1(m + Mk) = F_1(m)$, it follows that

$$\begin{aligned} |F_n(M^q(m + kM))|^2 &= M^{2q} \, |F_{n-q}(m + kM)|^2 \\ &= C(m) M^{2q} \sin^{-2}\left(\pi(m + kM)/M^{n-q}\right), \end{aligned} \tag{26}$$

where $C(m) = \sin^2(\pi m/M) \, |F_1(m)|^2$. For an arbitrary complex generating vector $\mathbf{f}_1$ we have $M-1$ different values of $C(m)$. If the power spectrum $|F_1(m)|^2$ of the generator is even, i.e., $|F_1(m)| = |F_1(M - m)|$, which is the case for real $\mathbf{f}_1$, then $C(m) = C(M - m)$, and we have only $\lfloor M/2 \rfloor$ different values of $C(m)$.

Let us first consider the set of points $\mathcal{A}_0$. The graph of the power spectrum defined on this point set behaves like

$$|F_n(l)|^2_{l \in \mathcal{A}_0} = C(m) \sin^{-2}(\pi l/M^n). \tag{27}$$

It is easy to see that it contains a number of affine branches that correspond to the different values of $C(m)$.

Let us now consider the set of points $\mathcal{A}_1$. The graph of the power spectrum defined on the point set $\mathcal{A}_1$ is similar to the graph defined on the point set $\mathcal{A}_0$, except for the coefficient M^2:

$$|F_n(l)|^2_{l \in \mathcal{A}_1} = M^2 C(m) \sin^{-2}(\pi l/M^{n-1}). \tag{28}$$

If we continue the procedure of subdividing the discrete frequency number into the point sets $\mathcal{A}_q$ corresponding to the different values of q, we will find out that the power spectrum consists of $(n-1) \times (M - 1)$ similar branches that behave as $\sin^{-2}(\pi l/M^{n-q})$. For the generator with an even power spectrum, the number of branches reduces to $(n-1) \times \lfloor M/2 \rfloor$. Figure 4 shows the power spectrum of the additive vector that was represented in Fig. 3, with logarithmic scales for both coordinates. The graph displays different branches whose number depends on the level n of the additive vector and the size M of the generator. Note that the number of points belonging to the set $\mathcal{A}_q$ decreases with increasing q, as can be seen in Fig. 4.

4. CONCLUSIONS

We have derived the expressions for the DFT of n^{th}-generation multiplicative and additive signals in terms of the DFT of the generating vectors and we have investigated the main features of the power spectra. In spite of the rather different form of the multiplicative and additive signals, as can be seen in Figs. 1 and 3, their DFTs exhibit a similar property of self-affinity. Thus the DFT of the n^{th}-generation signal $\mathbf{f}_n$ at the decimated points $l = pM$, where M is the size of the generator $\mathbf{f}_1$ and p is an integer, corresponds to the DFT of the $(n-1)^{th}$-generation signal $\mathbf{f}_{n-1}$ multiplied by a constant, equal to $F_1(0)$ and M for the multiplicative and the additive case, respectively. Finally we remark that the self-affinity of such types of signals makes them attractive for the different applications discussed in [5].

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