WIGNER DISTRIBUTION FUNCTION OF A CIRCULAR LIGHT SOURCE

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Abstract

The Wigner distribution function of a time-harmonic, uniform, circular light source is determined. Analytic expressions as well as numerical and graphical results are presented.

Introduction

In recent years the Wigner distribution function [12] has proven to be a valuable tool for the description of signals and systems in optics (see, for instance, [2, 5, 6, 10, 11] and the references cited there). The Wigner distribution function is a signal representation in space and (spatial) frequency, simultaneously, and thus forms an intermediate signal description between the pure space representation and the pure frequency representation. This simultaneous space-frequency description closely resembles the ray concept in geometrical optics, in which the position and direction of a ray are also given simultaneously, and its relationship to the generalized radiance is obvious [10, 13]. In a way, the Wigner distribution function can be considered as the amplitude of an optical ray passing through a certain position with a certain direction.

One of the important properties of the Wigner distribution function is that its propagation through Luneburg's first-order optical systems [8] (described by an $ABCD$-matrix [9]) can be expressed in a very simple form [2, 11]. A simple propagation law also arises when the light propagates through weakly inhomogeneous media, where diffraction effects can be neglected [3, 4]. In such media the amplitude of an optical ray remains invariant under propagation, and to find the propagation of the light we only have to determine the optical ray paths. Propagation of the Wigner distribution function in such media thus immediately leads to ray tracing. However, as soon as diffraction effects manifest themselves in optical systems, the method of ray tracing alone is no longer appropriate to determine the propagation of light. But, since the Wigner distribution function can take into account these diffraction effects as well, it may still be a valuable tool for the analysis of such more general systems.

In this paper we consider the effect of a circular aperture on the propagation of light. The important formula we need for that is equivalent to the formula that arises when we determine the Wigner distribution function of a time-harmonic, uniform, circular light source. We will therefore focus on this latter case.

Wigner distribution function

The Wigner distribution function $F(x, y, u, v)$ that corresponds to the complex function $\varphi(x, y)$ is defined as

$$F(x, y, u, v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi\left(x + \tfrac{1}{2}x', y + \tfrac{1}{2}y'\right)$$

$$\varphi^*\left(x - \tfrac{1}{2}x', y - \tfrac{1}{2}y'\right) e^{-j(ux' + vy')} dx'dy'; \qquad (1)$$

the variables x and y denote space variables, the variables u and v denote spatial-frequency variables, the function $\varphi(x, y)$ represents the complex amplitude of the time-harmonic optical signal $\varphi(x, y)\exp(-j\omega t)$, and the asterisk * denotes complex conjugation. Obviously, with x and y considered as parameters, the Wigner distribution function is in fact the two-dimensional Fourier transform of the product

$$\varphi\left(x + \tfrac{1}{2}x', y + \tfrac{1}{2}y'\right) \varphi^*\left(x - \tfrac{1}{2}x', y - \tfrac{1}{2}y'\right) \qquad (2)$$

with respect to the difference coordinates x' and y'.

In the case of a uniform, circular light source with radius $a > 0$, the complex amplitude $\varphi(x, y)$ satisfies the conditions

$$\varphi(x, y) = \begin{cases} 1 & \text{for } 0 \le \sqrt{x^2 + y^2} \le a \\ 0 & \text{elsewhere.} \end{cases} \qquad (3)$$

We first remark that it can easily be seen from the definition (1) that the Wigner distribution function vanishes outside the source. Moreover, since the signal $\varphi(x, y)$ is in fact a function of $\sqrt{x^2 + y^2}$, we may, without loss of generality take $y = 0$ and $x \ge 0$; in that case the integrand in the definition of the Wigner distribution function [see Eq. (1)] is confined to the 'eye-shaped' interval

$$\left(\tfrac{1}{2}y'\right)^2 + \left(\tfrac{1}{2}x' \pm x\right)^2 \le a^2. \qquad (4)$$

After carrying out the integration over y' and substituting $x \pm \frac{1}{2}x' = a\xi$, the integral in Eq. (1) can be represented in the form

$$F(x, 0, u, v) = 16a^2$$

$$\Re\left\{ e^{-j2ux} \int_{x/a}^{1} \frac{\sin\left(2av\sqrt{1-\xi^2}\right)}{2av}\, e^{j2au\xi} d\xi \right\}, \qquad (5)$$

where $\Re$ denotes the real part. We note that in general the remaining integral in Eq. (5) does not seem to have a known solution.

It is the aim of this paper to present an (approximate) solution for the integral that arises in Eq. (5). Since we have the symmetry properties $F(x, 0, \pm u, \pm v) = F(x, 0, u, v)$, we may, without loss of generality, take $u \ge 0$ and $v \ge 0$ when we derive this solution. If we then use the substitutions $x = a\sin\alpha$ (with $0 \le \alpha \le \frac{1}{2}\pi$), $2au = R\sin\theta$ and $2av = R\cos\theta$ (with $R \ge 0$ and $0 \le \theta \le \frac{1}{2}\pi$), and introduce the function

$$K(\psi, R) = \int_0^{\psi} e^{jR\cos\beta} \cos\beta d\beta, \qquad (6)$$

we are able to show that Eq. (5) can be represented in the form [7]

$$F\left(a\sin\alpha, 0, \frac{R}{2a}\sin\theta, \frac{R}{2a}\cos\theta\right)$$

$$= \frac{16a^2}{2R}\,\Re\left\{ je^{-jR\sin\alpha\sin\theta}\right.$$

$$\left.[K(\alpha - \theta, R) - K^*(\alpha + \theta, R) - j\pi J_1(R)]\right\}, \qquad (7)$$

where $J_1(z)$ is a first-order Bessel function (see, for instance, [1], Eq. 9.1.21).

The function $K(\psi,\, R)$ can be calculated numerically, of course, where, due to the properties $K(\psi, R) = -K(-\psi,\, R)$ and $K(\frac{1}{2}\pi - \psi,\, R) - K^*(\frac{1}{2}\pi + \psi,\, R) = j\pi\, J_1(R)$, the variable ψ may be confined to the interval $0 \le \psi \le \frac{1}{2}\pi$. However, for large values of the parameter R, the integration may become difficult. Fortunately, we are able to derive [7] an approximation of $K(\psi,\, R)$ for large values of R (and $0 \le \psi \le \frac{1}{2}\pi$), reading – up to the order $R^{-3/2}$ –

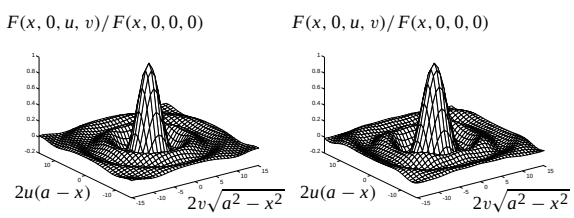
$$K(\psi,\, R) \simeq \sqrt{\frac{\pi}{R}}\left(1 + \frac{3j}{8R}\right) e^{jR}$$

$$\left[C\left(\sqrt{\frac{2R(1-\cos\psi)}{\pi}}\right) - jS\left(\sqrt{\frac{2R(1-\cos\psi)}{\pi}}\right)\right]$$

$$+ \frac{j}{R}\frac{\sqrt{2}\cos\psi - \sqrt{1+\cos\psi}}{\sqrt{2}\sin\psi}\, e^{jR\cos\psi}, \qquad (8)$$

where $C(z)$ and $S(z)$ are the two Fresnel integrals (see, for instance, [1], Sec. 7.3).

As an illustration we have depicted in Fig. 1 the Wigner distribution function $F(x, 0, u, v)$ as a function of u and v for several values of x. We remark that the maximum value of $F(x, 0, u, v)$ always occurs for $u = 0$ and $v = 0$; in the figures, we have adapted the scale of the vertical axis accordingly. We have also scaled the horizontal u and v axes inversely proportional to the widths $2(a-x)$ and $2\sqrt{a^2 - x^2}$ of the 'eye-shaped' interval in the $\frac{1}{2}x'$ and $\frac{1}{2}y'$ directions, respectively [see Eq. (4)], which leads to figures that look more or less similar.



(a) (b)

Figure 1: The Wigner distribution function $F(x, 0, u, v)$ of a circular source with radius a, for several values of the position variable x: (a) $x/a = 0$ and (b) $x/a = 0.95$.

The similarity between the various figures is indeed striking. We are thus tempted to conclude that all functions $F(x, 0, u, v)$ are similar to $F(0, 0, u, v)$ except for some scaling:

$$F(x, 0, u, v) \simeq \frac{F(x, 0, 0, 0)}{F(0, 0, 0, 0)}$$

$$F\left(0, 0, \frac{a-x}{a}u, \frac{\sqrt{a^2-x^2}}{a}v\right)$$

$$= \left[1 - \frac{2}{\pi}\arcsin\frac{x}{a} - \frac{2}{\pi}\frac{x}{a}\sqrt{1-\left(\frac{x}{a}\right)^2}\right]$$

$$4\pi a^2 \frac{J_1\left(2\sqrt{(a-x)^2u^2 + (a^2-x^2)v^2}\right)}{\sqrt{(a-x)^2u^2 + (a^2-x^2)v^2}}. \qquad (9)$$

It can easily be seen that the $\simeq$ sign in the latter equation becomes an $=$ sign for $x = 0$, for $x = a$, and for $u = v = 0$. Numerical investigation of the approximate expression (9) in relation to the exact expression (7) leads to the conclusion that the difference between these two expressions is less than 15% of the maximum value $F(x, 0, 0, 0)$.

The similarity between the Wigner distribution functions $F(x, 0, u, v)$ for different values of x can also be deduced from examination of the 'eye'-shaped interval defined by Eq. (4). The boundary of this interval is given by the relationship $(\frac{1}{2}y')^2 + (\frac{1}{2}x' \pm x)^2 = a^2$. If we scale the x' and y' variables according to $\frac{1}{2}x' = \xi(a-x)$ and $\frac{1}{2}y' = \eta\sqrt{a^2-x^2}$, the boundary (expressed in the variables ξ and η) is then given by the relationship

$$\eta^2 + \xi^2 - \frac{2x}{a+x}(\xi^2 \pm \xi) = 1. \qquad (10)$$

We remark that for $x = 0$ the boundary is a circle, $\eta^2 + \xi^2 = 1$, while for $x = a$ the boundary is given by the intersection of two parabolas, $\eta^2 \mp \xi = 1$. However, for any value of x with $0 \le x \le a$, the scaled 'eye'-shaped interval (10) is roughly the same and its two-dimensional Fourier transform will lead to roughly the same scaled Wigner distribution function.

Conclusion

We have determined the Wigner distribution function of a time-harmonic, uniform, circular light source and thus of a circular aperture, and we have presented analytic expressions as well as numerical and graphical results. Knowledge of the Wigner distribution function of a circular aperture may be valuable in determining the effect of such an aperture in optical systems that are being analyzed by ray tracing.

References

[1] M. Abramowitz and I.A. Stegun, eds., *Handbook of Mathematical Functions* (Dover, New York, 1970).

[2] M.J. Bastiaans, "The Wigner distribution function applied to optical signals and systems," *Opt. Commun.* **25**, 26-30 (1978).

[3] M.J. Bastiaans, "Transport equations for the Wigner distribution function," *Opt. Acta* **26**, 1265-1272 (1979).

[4] M.J. Bastiaans, "Transport equations for the Wigner distribution function in an inhomogeneous and dispersive medium," *Opt. Acta* **26**, 1333-1344 (1979).

[5] M.J. Bastiaans, "Wigner distribution function and its application to first-order optics," *J. Opt. Soc. Am.* **69**, 1710-1716 (1979).

[6] M.J. Bastiaans, "Application of the Wigner distribution function to partially coherent light," *J. Opt. Soc. Am. A* **3**, 1227-1238 (1986).

[7] M.J. Bastiaans and P.G.J. van de Mortel, "Wigner distribution function of a circular aperture," submitted to *J. Opt. Soc. Am. A.*

[8] R.K. Luneburg, *Mathematical Theory of Optics* (U. California Press, Berkeley, Calif., 1966).

[9] N. Marcuvitz, "Quasiparticle view of wave propagation," *Proc. IEEE* **68**, 1380-1395 (1980).

[10] A. Walther, "Radiometry and coherence," *J. Opt. Soc. Am.* **58**, 1256-1259 (1968).

[11] A. Walther, "Propagation of the generalized radiance through lenses," *J. Opt. Soc. Am.* **68**, 1606-1610 (1978).

[12] E. Wigner, "On the quantum correction for thermodynamic equilibrium," *Phys. Rev.* **40**, 749-759 (1932).

[13] E. Wolf, "Coherence and radiometry," *J. Opt. Soc. Am.* **68**, 6-17 (1978).