Powers of transfer matrices and cyclic cascades

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ABSTRACT

The parameters of the transfer matrix describing a first-order optical system that is a cascade of k identical subsystems defined by the transfer matrix M, are determined from considering the subsystem's eigenfunctions. A condition for the cascade to be cyclic is derived. Particular examples of cyclic first-order optical systems are presented.

SUMMARY

The evolution of the complex field amplitude $f(x)$ during propagation through first-order optical systems is described in the paraxial approximation of the scalar diffraction theory through the generalized Fresnel transform (GFT)[1,2] of the input field amplitude $f_i(x)$

$$f_o(u) = R^M \left[f_i(x) \right](u) = \int_{-\infty}^{\infty} f_i(x) K_M(x,u) dx, \tag{1}$$

with the kernel

$$K_M(x,u) = \begin{cases} \left(1/\sqrt{iB}\right) \exp\left(i\pi \left(Ax^2 + Du^2 - 2xu\right)/B\right) & B \neq 0 \\ \sqrt{A} \exp\left(i\pi Cu^2/A\right) \delta(x - Au) & B = 0, \end{cases} \tag{2}$$

parametrized by a real 2×2 matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3}$$

with the determinant $AD - BC$ equal to 1. The parameters A, B, C, D depend on the concrete first-order system and the wavelength. For the sake of simplicity we will consider the one-dimensional case.

Due to the additivity property of the canonical integral transform $R^{M_2} R^{M_1} = R^{M_3}$ where $M_3 = M_2 \times M_1$, the complex field amplitude at the output plane of the cascade of k identical first-order systems, each of which described by the same transfer matrix M, can be represented as the GFT for a matrix M^k of the input complex field amplitude. In Ref. 3 the authors considered the connection between the parameters of the transfer matrices M and M^k based on matrix calculus.

In this paper we derive an alternative method of determining the parameters of the cascade transfer matrix based on the analysis of the eigenfunctions of the GFT. This approach allows us to formulate a simple condition for a cascade of canonical integral transforms to be cyclic.

An input complex field amplitude $f_i(x)$ is an eigenfunction $f_M(x)$ of the canonical operator R^M corresponding to the given optical system if

$$R^M \left[f_M(x) \right](u) = a f_M(u), \tag{4}$$

where $a = \exp(i2\pi\varphi)$ is the (generally complex) eigenvalue.[1]

The structure of the eigenfunctions for the GFT (the so-called self-GFT functions) has been considered in Ref. 1 It was shown there that the functions

$$\Phi_n(x) = \left(\sqrt{\pi} 2^n \lambda n!\right)^{-\frac{1}{2}} \exp\left(-\tfrac{1}{2}(1 + i\beta)(x/\lambda)^2\right) H_n\left(x/\lambda\right) \tag{5}$$

are eigenfunctions for the operator R^M with eigenvalue $a = \exp(-i(n + \frac{1}{2})\theta)$, where $H_n(u)$ are the Hermite polynomials and where the parameters θ, λ, and β are defined from the parameters of the transfer matrix by

$$\begin{aligned} \theta &= \arccos\left(\tfrac{1}{2}(A + D)\right) \\ \lambda^2 &= 2B\left(4 - (A + D)^2\right)^{-\frac{1}{2}} \\ \beta &= (A - D)\left(4 - (A + D)^2\right)^{-\frac{1}{2}}. \end{aligned} \tag{6}$$

An eigenfunction $f_M(x)$ for the canonical integral operator R^M with eigenvalue $a = \exp(-i(n+\frac{1}{2})\theta)$, is also an eigenfunction with eigenvalue a^k for the GFT parametrized by the matrix M^k, where k is an integer. Therefore, the parameters of the k-th power M^k of the matrix M have to satisfy equations (6) where θ is replaced by $k\theta$. Then we can alternatively represent the parameters of the matrix M^k in terms of the parameters of the matrix M:

$$\begin{aligned} A^{(k)} &= \cos k\theta + \tfrac{1}{2}(A-D)\sin k\theta/\sin\theta \\ B^{(k)} &= B\sin k\theta/\sin\theta \\ C^{(k)} &= C\sin k\theta/\sin\theta \\ D^{(k)} &= \cos k\theta - \tfrac{1}{2}(A-D)\sin k\theta/\sin\theta \end{aligned} \tag{7}$$

with $\cos\theta = \frac{1}{2}(A+D)$. Equations (7) allow an easy determination of the resulting matrix M^k of the cascade of k identical first-order systems. For the fractional Fourier transform system, for instance, determined by $\beta = 0$ and $\lambda^2 = 1$, and hence by $A = D = \cos\theta$ and $B = -C = \sin\theta$, we immediately have $A^{(k)} = D^{(k)} = \cos k\theta$, and $B^{(k)} = -C^{(k)} = \sin k\theta$.

From Eqs. (7) it is easy to see that if $\theta = 2\pi m/k$, with k and m integers, we have $M^k = I$. This implies that the cascade of k optical systems described by a matrix M with parameters A and D such that

$$\tfrac{1}{2}(A+D) = \cos(2\pi m/k), \tag{8}$$

produces the identity transform, and hence $M = I^{1/k}$. We call these systems cyclic of order k. Note that, without loss of generality, we may choose $0 \le m < k$.

In the special case that $m = 1$ in Eq. (8), we denote the corresponding k-th order cyclic transfer matrix by M_k; its parameters follow from Eqs. (6) with $\theta = 2\pi/k$:

$$\begin{aligned} A_k &= \cos(2\pi/k) + \beta\sin(2\pi/k) \\ B_k &= \lambda^2\sin(2\pi/k) \\ C_k &= -\left((\beta^2+1)/\lambda^2\right)\sin(2\pi/k) \\ D_k &= \cos(2\pi/k) - \beta\sin(2\pi/k). \end{aligned} \tag{9}$$

From Eqs. (7) we conclude that the general k-th order cyclic matrix M with $\theta = 2\pi m/k$, can be expressed as the m-th power M_k^m of M_k; M_k can thus be considered as the m-th root $M^{1/m}$ of M. Moreover, the j-th power of M is equal to the l-th power of M_k, where the integers j and l are related to each other by $mj = l + Nk$,

$$M^j = M_k^{mj} = M_k^{l+Nk} = M_k^l = M^{l/m}, \tag{10}$$

and where we have used the property that M_k^k is the identity matrix. We conclude that cascade properties of any cyclic transform of order k defined by a transfer matrix M can be described by the matrix $M_k = M^{1/m}$.

We now determine an entire class of canonical systems described by the 4-th root of the identity matrix. Such systems have the property $\theta = \pi m/2$. Using Eqs. (6), we conclude that for even m one gets the inverse or the identity transformation, $A = D = \pm 1$ and $B = C = 0$, while for odd m the parameters of the transfer matrix M are connected as $D = -A$, $C = -(A^2+1)/B$. It is easy to see that the case $A = 0$ and $B = 1$ corresponds to the Fourier transforming system.

REFERENCES

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