Wigner distribution moments measured as fractional Fourier transform intensity moments

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ABSTRACT

It is shown how all global Wigner distribution moments of arbitrary order can be measured as intensity moments in the output plane of an appropriate number of fractional Fourier transform systems (generally anamorphic ones). The minimum number of (anamorphic) fractional power spectra that are needed for the determination of these moments is derived.

1. WIGNER DISTRIBUTION

The Wigner distribution (WD) of a two-dimensional function $f(x, y)$ is defined by[1–3]

$$W_f(x,u;y,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x+\tfrac{1}{2}x', y+\tfrac{1}{2}y')\, f^*(x-\tfrac{1}{2}x', y-\tfrac{1}{2}y')\exp[-j2\pi(ux'+vy')]dx'dy'. \tag{1}$$

The WD $W_f(x,u;y,v)$ represents a space function $f(x,y)$ in a combined space/spatial-frequency domain, the so-called phase space, where u and v are the spatial-frequency variables associated to the space variables x and y, respectively. We remark that the definition of the WD – and all the results of this paper – need not be restricted to coherent light, in which case $f(x,y)$ would represent the complex field amplitude of the light, but can be extended to partially coherent light, in which case its two-point correlation function can be identified with $< f(x+\frac{1}{2}x', y+\frac{1}{2}y') f^*(x-\frac{1}{2}x', y-\frac{1}{2}y') >$.

In this paper we consider the normalized moments of the WD, where the normalization is with respect to the total energy E of the signal: $E = \iiiint W_f(x,u;y,v)dxdudydv = \iint |f(x,y)|^2 dxdy$. These normalized moments μ_{pqrs} of the WD are thus defined by

$$\mu_{pqrs}E = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(x,u;y,v)x^p u^q y^r v^s dxdudydv \quad (p,q,r,s \geq 0). \tag{2}$$

Note that for $q = s = 0$ we have intensity moments, $\mu_{p0r0}E = \iint |f(x,y)|^2 x^p y^r dxdy$, which can easily be measured.[4,5]

2. FRACTIONAL FOURIER TRANSFORM

The fractional Fourier transform (FT) of a function $f(x,y)$ is defined by[6–8]

$$F_{\alpha\beta}(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_\alpha(x,u)K_\beta(y,v)\,f(x,y)dxdy, \text{ with } K_\alpha(x,u) = \frac{\exp(j\frac{1}{2}\alpha)}{\sqrt{j\sin\alpha}}\exp\left[j\pi\frac{(x^2+u^2)\cos\alpha - 2ux}{\sin\alpha}\right]. \tag{3}$$

The fractional FT can be generated optically by a very simple, anamorphic, coherent-optical set-up, consisting only of two cylindrical lenses, whose focal lengths – in combination with some appropriate sections of free space – are related to the angles α and β.

One of the most important properties of the fractional FT is that it corresponds to a rotation of the WD in phase space:

$$W_{F_{\alpha\beta}}(x,u;y,v) = W_f(x\cos\alpha - u\sin\alpha, x\sin\alpha + u\cos\alpha; y\cos\beta - v\sin\beta, y\sin\beta + v\cos\beta). \tag{4}$$

Moreover, the fractional power spectrum, i.e., the squared modulus $|F_{\alpha\beta}(x,y)|^2$ of the fractional FT, is directly related to the WD through a projection operation: $\left|F_{\alpha\beta}(x,y)\right|^2 = \iint W_{F_{\alpha\beta}}(x,u;y,v)dudv$. Note that the fractional power spectrum is related to the intensity distribution in the output plane of an anamorphic fractional FT system and therefore can easily be measured in experiments.

We can as well define normalized moments $\mu_{pqrs}(\alpha,\beta)$ in the fractional domain, and relate these to the original moments $\mu_{pqrs} = \mu_{pqrs}(0,0)$; for the $x^p y^r$ moments (with $q = s = 0$) in particular we have

$$\mu_{p0r0}(\alpha,\beta) = \sum_{k=0}^{p}\sum_{m=0}^{r}\binom{p}{k}\binom{r}{m}\mu_{p-k,k,r-m,m}\cos^{p-k}\alpha\sin^k\alpha\cos^{r-m}\beta\sin^m\beta. \tag{5}$$

Note that these moments can easily be measured as intensity moments again: $\mu_{p0r0}(\alpha,\beta)E = \iint |F_{\alpha\beta}(x,y)|^2 x^p y^r dxdy$.

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3. RELATIONS BETWEEN MOMENTS IN THE FRACTIONAL DOMAIN

For the 2+2 = 4 first-order moments, the following 2 equations are relevant:

$$\mu_{1000}(\alpha,\beta) = \mu_{1000}\cos\alpha + \mu_{0100}\sin\alpha \quad \text{and} \quad \mu_{0010}(\alpha,\beta) = \mu_{0010}\cos\beta + \mu_{0001}\sin\beta,$$

which equations correspond to Eq. (5) with $pqrs = 1000$ and $pqrs = 0010$, respectively, and the 4 moments μ_{1000}, μ_{0100}, μ_{0010}, and μ_{0001} can be determined by measuring the intensity moments $\mu_{1000}(\alpha,\cdot)$ and $\mu_{0010}(\cdot,\beta)$ in the fractional domain for two values of α and β, for instance for 0 and $\frac{1}{2}\pi$: $\mu_{1000} = \mu_{1000}(0,\cdot)$, $\mu_{0100} = \mu_{1000}(\frac{1}{2}\pi,\cdot)$, $\mu_{0010} = \mu_{0010}(\cdot,0)$, and $\mu_{0001} = \mu_{0010}(\cdot,\frac{1}{2}\pi)$.

For the 3+4+3 = 10 second-order moments, the following 3 equations are relevant:

$$\begin{aligned}
\mu_{2000}(\alpha,\beta) &= \mu_{2000}\cos^2\alpha + 2\mu_{1100}\cos\alpha\sin\alpha + \mu_{0200}\sin^2\alpha,\\
\mu_{1010}(\alpha,\beta) &= \mu_{1010}\cos\alpha\cos\beta + \mu_{1001}\cos\alpha\sin\beta + \mu_{0110}\sin\alpha\cos\beta + \mu_{0101}\sin\alpha\sin\beta,\\
\mu_{0020}(\alpha,\beta) &= \mu_{0020}\cos^2\beta + 2\mu_{0011}\cos\beta\sin\beta + \mu_{0002}\sin^2\beta,
\end{aligned}$$

which equations correspond to Eq. (5) with $pqrs = 2000$, $pqrs = 1010$, and $pqrs = 0020$, respectively. The 3 moments μ_{2000}, μ_{1100}, and μ_{0200} can be determined by measuring the intensity moment $\mu_{2000}(\alpha,\cdot)$ in the fractional domain[9] for three values of α, for instance for 0, $\frac{1}{4}\pi$, and $\frac{1}{2}\pi$: $\mu_{2000} = \mu_{2000}(0,\cdot)$, $\mu_{0200} = \mu_{2000}(\frac{1}{2}\pi,\cdot)$, and then $\mu_{1100} = \mu_{2000}(\frac{1}{4}\pi,\cdot) - \frac{1}{2}(\mu_{2000} + \mu_{0200})$. Likewise, the 3 moments μ_{0020}, μ_{0011}, and μ_{0002} can be determined by measuring the intensity moment $\mu_{0020}(\cdot,\beta)$ for three values of β, for instance for 0, $\frac{1}{4}\pi$, and $\frac{1}{2}\pi$: $\mu_{0020} = \mu_{0020}(\cdot,0)$, $\mu_{0002} = \mu_{0020}(\cdot,\frac{1}{2}\pi)$, and then $\mu_{0011} = \mu_{0020}(\cdot,\frac{1}{4}\pi) - \frac{1}{2}(\mu_{0020} + \mu_{0002})$. The other 4 moments μ_{1010}, μ_{1001}, μ_{0110}, and μ_{0101} result from measuring the intensity moment $\mu_{1010}(\alpha,\beta)$, for instance as follows: $\mu_{1010} = \mu_{1010}(0,0)$, $\mu_{0101} = \mu_{1010}(\frac{1}{2}\pi,\frac{1}{2}\pi)$, $\mu_{0110} = \mu_{1010}(\frac{1}{2}\pi,0)$, and then $\mu_{1001} = 2\mu_{1010}(\frac{1}{4}\pi,\frac{1}{4}\pi) - \mu_{1010} - \mu_{0110} - \mu_{0101}$. We conclude that all 10 second-order moments can be determined from the knowledge of 4 fractional power spectra, where one of them has to be anamorphic (i.e., $\alpha \neq \beta$), for instance $|F_{0,0}(x,y)|^2$, $|F_{\pi/4,\pi/4}(x,y)|^2$, $|F_{\pi/2,\pi/2}(x,y)|^2$, and $|F_{\pi/2,0}(x,y)|^2$.

For higher-order moments we can proceed analogously. For the 4+6+6+4 = 20 third-order moments, the 4 relevant equations follow from Eq. (5) with $pqrs = 3000$, $pqrs = 2010$, $pqrs = 1020$, and $pqrs = 0030$, respectively, and the 20 third-order moments can be determined from the knowledge of 6 fractional power spectra, where 2 of them have to be anamorphic. For the 5+8+9+8+5 = 35 fourth-order moments, the 5 relevant equations follow from Eq. (5) with $pqrs = 4000$, $pqrs = 3010$, $pqrs = 2020$, $pqrs = 1030$, and $pqrs = 0040$, respectively, and the 35 fourth-order moments can be determined from the knowledge of 9 fractional power spectra, where 4 of them have to be anamorphic.

In general, the number of nth-order moments μ_{pqrs} (with $p+q+r+s = n$) equals $\frac{1}{6}(n+1)(n+2)(n+3)$. The total number t of fractional power spectra that we need to determine these nth-order moments reads $t = \frac{1}{4}(n+1)(n+3)$ (if $n =$ odd) or $t = \frac{1}{4}(n+2)^2$ (if $n =$ even); from these t fractional power spectra, $t-(n+1)$ are anamorphic ones.

4. CONCLUSIONS

We have shown how all global WD moments of arbitrary order can be measured as intensity moments in the output plane of an appropriate number of fractional FT systems (generally anamorphic ones, i.e., with different angles α and β), and we have derived the minimum number of (anamorphic) fractional power spectra that are needed for the determination of these moments. The results followed directly from the general relationship (5) that expresses the intensity moments in the output plane of an anamorphic fractional FT system in terms of the moments in the input plane and the angles α and β.

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