Propagation of the generating function of Hermite-Gaussian-type modes in first-order optical systems

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ABSTRACT

Based on the common Hermite-Gaussian modes, a general class of orthonormal Hermite-Gaussian-type modes is introduced. Such modes can most easily be defined by means of their generating function. A propagation law for the generating function is formulated, when these modes propagate through first-order optical systems.

Keywords: Hermite-Gaussian modes, generating function, first-order optical systems, mode converters

1. INTRODUCTION

A general class of orthonormal sets of Hermite-Gaussian type modes is introduced, by generalizing the quadratic form that arises in the generating function of the common Hermite-Gaussian modes. We study how these modes propagate through first-order optical systems and express the generating function of the set of output modes in terms of the generating function of the set of input modes.

The requirement of orthonormality yields some additional conditions for the quadratic form of the generating function. As a result of that, we will be able to express the elements of this quadratic form in terms of four matrices that can be combined into a symplectic matrix.

The main result of the paper is that this symplectic matrix propagates through a first-order optical system by a mere multiplication with the system's ray transformation matrix. From this simple propagation law we can easily derive how different members from the class of the Hermite-Gaussian-type modes (such as the common Hermite-Gaussian and Laguerre-Gaussian modes) can be converted into each other.

The propagation law reduces to the well-known bilinear $ABCD$ law in the case of Hermite-to-Hermite conversion (by means of a separable first-order system) and in the case of Laguerre-to-Laguerre conversion (by means of an isotropic first-order system). Knowledge of the generating function and in particular its propagation law may be valuable in the design of more general mode converters.

2. HERMITE-GAUSSIAN-TYPE MODES

The complex field amplitude of the common Hermite-Gaussian modes takes the form

$$\mathcal{H}_{n,m}(\mathbf{r}; w_x, w_y) = \mathcal{H}_n(x; w_x)\mathcal{H}_m(y; w_y) \tag{1}$$

with

$$\mathcal{H}_n(x; w) = 2^{1/4}\left(2^n n!\, w\right)^{-1/2} H_n\left(\sqrt{2\pi}\, x/w\right) \exp\left(-\pi\, x^2/w^2\right), \tag{2}$$

where $H_n(\cdot)$ are the Hermite polynomials[1] and where the column vector $\mathbf{r} = (x, y)^t$ is a short-hand notation for the spatial variables x and y, with the superscript ^{t} denoting transposition. Note that $\mathcal{H}_n(x; w)$ has been defined such that we have the orthonormality relationship

$$\int \mathcal{H}_n(x; w)\mathcal{H}_l(x; w)\, \mathrm{d}x = \delta_{nl} \tag{3}$$

with δ_{nl} the Kronecker delta. (All integrals in this paper extend from $-\infty$ to $+\infty$.) From the generating function of the Hermite polynomials, see Ref. [1, Eq. (22.9.17)],

$$\exp\left(-s^2+2sz\right)=\sum_{n=0}^{\infty}H_n(z)\frac{s^k}{n!}, \tag{4}$$

we can easily find the generating function of the Hermite-Gaussian modes $\mathcal{H}_{n,m}(\mathbf{r};w_x,w_y)$:

$$2^{1/2}(w_xw_y)^{-1/2}\exp\left[-(s_x^2+s_y^2)+2\sqrt{2\pi}\,(s_xx/w_x+s_yy/w_y)-\pi(x^2/w_x^2+y^2/w_y^2)\right]$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathcal{H}_{n,m}(\mathbf{r};w_x,w_y)\left(\frac{2^{n+m}}{n!m!}\right)^{1/2}s_x^n s_y^m. \tag{5}$$

The general class of sets of Hermite-Gaussian-type modes $\mathcal{H}_{n,m}(\mathbf{r};\mathbf{K},\mathbf{L})$ that we propose, is most easily defined by the generating function

$$2^{1/2}(\det\mathbf{K})^{1/2}\exp\left(-\mathbf{s}^t\mathbf{M}\mathbf{s}+2\sqrt{2\pi}\,\mathbf{s}^t\mathbf{K}\mathbf{r}-\pi\mathbf{r}^t\mathbf{L}\mathbf{r}\right)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathcal{H}_{n,m}(\mathbf{r};\mathbf{K},\mathbf{L})\left(\frac{2^{n+m}}{n!m!}\right)^{1/2}s_x^n s_y^m, \tag{6}$$

cf. Eq. (5), where we have introduced the column vector $\mathbf{s}=(s_x,s_y)^t$ and three (possibly complex) 2×2-matrices $\mathbf{K}$, $\mathbf{L}=\mathbf{L}^t$, and $\mathbf{M}=\mathbf{M}^t$. For the common Hermite-Gaussian modes $\mathcal{H}_{n,m}(\mathbf{r};w_x,w_y)$ we have, see Eq. (5),

$$\mathbf{K}=\begin{bmatrix} w_x & 0 \\ 0 & w_y \end{bmatrix}^{-1}=\mathbf{W}^{-1},\quad \mathbf{L}=\mathbf{W}^{-2},\quad \mathbf{M}=\mathbf{I}. \tag{7}$$

In most cases, the matrix $\mathbf{M}$ is completely determined by $\mathbf{K}$, for which reason we decided not to include $\mathbf{M}$ as a parameter in $\mathcal{H}_{n,m}(\mathbf{r};\mathbf{K},\mathbf{L})$. Indeed, if $\mathbf{M}=\mathbf{K}/4$, the generating function represents the two-variable Hermite-Gaussian modes defined by the matrix[2] $\mathbf{K}/2$; note that these two-variable Hermite-Gaussian modes are not orthogonal. As we will observe later, the condition of mode orthonormality also leads to a direct connection between the matrices $\mathbf{M}$ and $\mathbf{K}$, see Eq. (21).

Generating functions are useful in finding, for instance, the derivative relations for $\mathcal{H}_{n,m}(\mathbf{r};\mathbf{K},\mathbf{L})$

$$\begin{bmatrix} \dfrac{\partial\mathcal{H}_{n,m}(\mathbf{r};\mathbf{K},\mathbf{L})}{\partial x} \\ \dfrac{\partial\mathcal{H}_{n,m}(\mathbf{r};\mathbf{K},\mathbf{L})}{\partial y} \end{bmatrix}=-2\pi\,\mathcal{H}_{n,m}(\mathbf{r};\mathbf{K},\mathbf{L})\mathbf{L}\begin{bmatrix} x \\ y \end{bmatrix}+2\sqrt{\pi}\,\mathbf{K}^t\begin{bmatrix} \sqrt{n}\,\mathcal{H}_{n-1,m}(\mathbf{r};\mathbf{K},\mathbf{L}) \\ \sqrt{m}\,\mathcal{H}_{n,m-1}(\mathbf{r};\mathbf{K},\mathbf{L}) \end{bmatrix} \tag{8}$$

by differentiating the generating function with respect to $\mathbf{r}$, and the recurrence relations

$$2\sqrt{\pi}\,\mathcal{H}_{n,m}(\mathbf{r};\mathbf{K},\mathbf{L})\,\mathbf{K}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} \sqrt{n+1}\,\mathcal{H}_{n+1,m}(\mathbf{r};\mathbf{K},\mathbf{L}) \\ \sqrt{m+1}\,\mathcal{H}_{n,m+1}(\mathbf{r};\mathbf{K},\mathbf{L}) \end{bmatrix}+\mathbf{M}\begin{bmatrix} \sqrt{n}\,\mathcal{H}_{n-1,m}(\mathbf{r};\mathbf{K},\mathbf{L}) \\ \sqrt{m}\,\mathcal{H}_{n,m-1}(\mathbf{r};\mathbf{K},\mathbf{L}) \end{bmatrix} \tag{9}$$

by differentiating it with respect to $\mathbf{s}$.

3. PROPAGATION THROUGH FIRST-ORDER OPTICAL SYSTEMS

We let Hermite-Gaussian-type modes $\mathcal{H}_{n,m}(\mathbf{r};\mathbf{K},\mathbf{L})$ propagate through a lossless, first-order optical system – also called an **ABCD**-system – and determine the generating function of the set of modes to which the beam that appears at the output of this system belongs. Any lossless, first-order optical system can be described by its ray transformation matrix,[3] which relates the position $\mathbf{r}_i$ and direction $\mathbf{q}_i$ of an incoming ray to the position $\mathbf{r}_o$ and direction $\mathbf{q}_o$ of the outgoing ray:

$$\begin{bmatrix} \mathbf{r}_o \\ \mathbf{q}_o \end{bmatrix}=\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}\begin{bmatrix} \mathbf{r}_i \\ \mathbf{q}_i \end{bmatrix}. \tag{10}$$

The ray transformation matrix is real and symplectic, yielding the relations

$$\begin{array}{lll} \mathbf{A}\mathbf{B}^t = \mathbf{B}\mathbf{A}^t, & \mathbf{C}\mathbf{D}^t = \mathbf{D}\mathbf{C}^t, & \mathbf{A}\mathbf{D}^t - \mathbf{B}\mathbf{C}^t = \mathbf{I}, \\ \mathbf{A}^t\mathbf{C} = \mathbf{C}^t\mathbf{A}, & \mathbf{B}^t\mathbf{D} = \mathbf{D}^t\mathbf{B}, & \mathbf{A}^t\mathbf{D} - \mathbf{C}^t\mathbf{B} = \mathbf{I}. \end{array} \tag{11}$$

Using the matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{D}$, and assuming that $\mathbf{B}$ is a non-singular matrix, we can represent the first-order optical system by the Collins integral[4]

$$f_o(\mathbf{r}_o) = \frac{\exp(i\phi)}{\sqrt{\det i\mathbf{B}}} \iint f_i(\mathbf{r}_i) \exp\left[i\pi\left(\mathbf{r}_i^t\mathbf{B}^{-1}\mathbf{A}\mathbf{r}_i - 2\mathbf{r}_i^t\mathbf{B}^{-1}\mathbf{r}_o + \mathbf{r}_o^t\mathbf{D}\mathbf{B}^{-1}\mathbf{r}_o\right)\right] \mathrm{d}\mathbf{r}_i, \tag{12}$$

where the output amplitude $f_o(\mathbf{r})$ is expressed in terms of the input amplitude $f_i(\mathbf{r})$. The phase factor $\exp(i\phi)$ in Eq. (12) is rather irrelevant and can often be chosen arbitrarily. We remark that the signal transformation $f_i(\mathbf{r}) \to f_o(\mathbf{r})$ that corresponds to a lossless, first-order optical system, is unitary, i.e.

$$\iint f_{i,1}(\mathbf{r}) f_{i,2}^*(\mathbf{r})\, \mathrm{d}\mathbf{r} = \iint f_{o,1}(\mathbf{r}) f_{o,2}^*(\mathbf{r})\, \mathrm{d}\mathbf{r}, \tag{13}$$

where * denotes complex conjugation.

With a Hermite-Gaussian-type mode $\mathcal{H}_{n,m}(\mathbf{r};\mathbf{K}_i,\mathbf{L}_i)$ at the input of an **ABCD**-system, we denote the output beam by $\mathcal{H}_{n,m}(\mathbf{r};\mathbf{K}_o,\mathbf{L}_o)$. To find the generating function of the set of modes to which this output beam belongs, we write

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \mathcal{H}_{n,m}(\mathbf{r};\mathbf{K}_o,\mathbf{L}_o) \left(\frac{2^{n+m}}{n!m!}\right)^{1/2} s_x^n s_y^m = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \left(\frac{2^{n+m}}{n!m!}\right)^{1/2} s_x^n s_y^m \frac{\exp(i\phi)}{\sqrt{\det i\mathbf{B}}}$$
$$\times \iint \mathcal{H}_{n,m}(\mathbf{r}_i;\mathbf{K}_i,\mathbf{L}_i) \exp\left[i\pi\left(\mathbf{r}_i^t\mathbf{B}^{-1}\mathbf{A}\mathbf{r}_i - 2\mathbf{r}_i^t\mathbf{B}^{-1}\mathbf{r}_o + \mathbf{r}_o^t\mathbf{D}\mathbf{B}^{-1}\mathbf{r}_o\right)\right] \mathrm{d}\mathbf{r}_i,$$

where Collins integral (12) has been used. Upon substituting from the generating function (6) we get

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \mathcal{H}_{n,m}(\mathbf{r};\mathbf{K}_o,\mathbf{L}_o) \left(\frac{2^{n+m}}{n!m!}\right)^{1/2} s_x^n s_y^m = \frac{\exp(i\phi)}{\sqrt{\det i\mathbf{B}}} \iint 2^{1/2} \left(\det \mathbf{K}_i\right)^{1/2}$$
$$\times \exp\left(-\mathbf{s}^t\mathbf{M}_i\mathbf{s} + 2\sqrt{2\pi}\, \mathbf{s}^t\mathbf{K}_i\mathbf{r}_i - \pi\mathbf{r}_i^t\mathbf{L}_i\mathbf{r}_i\right) \exp\left[i\pi\left(\mathbf{r}_i^t\mathbf{B}^{-1}\mathbf{A}\mathbf{r}_i - 2\mathbf{r}_i^t\mathbf{B}^{-1}\mathbf{r}_o + \mathbf{r}_o^t\mathbf{D}\mathbf{B}^{-1}\mathbf{r}_o\right)\right] \mathrm{d}\mathbf{r}_i.$$

After reorganizing terms we write the right-hand side as

$$\exp(i\phi) 2^{1/2} \left(\frac{\det \mathbf{K}_i}{\det i\mathbf{B}}\right)^{1/2} \exp\left(-\mathbf{s}^t\mathbf{M}_i\mathbf{s} + \mathbf{r}_o^t\mathbf{D}\mathbf{B}^{-1}\mathbf{r}_o\right)$$
$$\times \iint \exp\left[-\pi\mathbf{r}_i^t\left(\mathbf{L}_i - i\mathbf{B}^{-1}\mathbf{A}\right)\mathbf{r}_i\right] \exp\left[-i2\pi\mathbf{r}_i^t\left(\mathbf{B}^{-1}\mathbf{r}_o + i\sqrt{2/\pi}\,\mathbf{K}_i^t\mathbf{s}\right)\right] \mathrm{d}\mathbf{r}_i$$

and it remains to calculate an integral of the form

$$\iint \exp\left(-\pi\mathbf{r}_i^t\mathbf{P}\mathbf{r}_i - i2\pi\mathbf{r}_i^t\mathbf{q}\right) \mathrm{d}\mathbf{r}_i = \frac{1}{\sqrt{\det \mathbf{P}}} \exp\left(-\pi\mathbf{q}^t\mathbf{P}^{-1}\mathbf{q}\right),$$

with $\mathbf{P} = \mathbf{L}_i - i\mathbf{B}^{-1}\mathbf{A}$ a symmetric matrix whose real part is positive definite and with $\mathbf{q} = \mathbf{B}^{-1}\mathbf{r}_o + i\sqrt{2/\pi}\,\mathbf{K}_i^t\mathbf{s}$. (The latter equality is a straightforward extension of the one-dimensional relation, see Ref. [1, Eq. (7.4.2)],

$$\int_0^{\infty} \exp\left[-(at^2 + 2bt + c)\right] \mathrm{d}t = \tfrac{1}{2}\sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2 - ac}{a}\right)\left(1 - \mathrm{erf}\frac{b}{\sqrt{a}}\right) \quad (\Re a > 0)$$

to more dimensions.) We thus get for the generating function

$$\exp(i\phi)\, 2^{1/2} \left[\frac{\det \mathbf{K}_i}{\det i\mathbf{B}\, \det(\mathbf{L}_i - i\mathbf{B}^{-1}\mathbf{A})} \right]^{1/2}$$
$$\times \exp\left(-\mathbf{s}^t \mathbf{M}_i \mathbf{s} + i\pi \mathbf{r}_o^t \mathbf{D}\mathbf{B}^{-1}\mathbf{r}_o\right) \exp\left[-\pi \left(\mathbf{B}^{-1}\mathbf{r}_o + i\sqrt{2/\pi}\,\mathbf{K}_i^t \mathbf{s} \right)^t \left(\mathbf{L}_i - i\mathbf{B}^{-1}\mathbf{A} \right)^{-1} \left(\mathbf{B}^{-1}\mathbf{r}_o + i\sqrt{2/\pi}\,\mathbf{K}_i^t \mathbf{s} \right) \right].$$

To simplify the expressions in the exponents, we first look to the factor that appears in connection with $\mathbf{r}_o^t \mathbf{r}_o$:

$$-\pi \mathbf{r}_o^t \left[-i\mathbf{D} + \mathbf{B}^{-1^t} \left(\mathbf{L}_i - i\mathbf{B}^{-1}\mathbf{A} \right)^{-1} \right] \mathbf{B}^{-1}\mathbf{r}_o = -\pi \mathbf{r}_o^t \left[-i\mathbf{D} \left(\mathbf{L}_i - i\mathbf{B}^{-1}\mathbf{A} \right) + \mathbf{B}^{-1^t} \right] \left(\mathbf{L}_i - i\mathbf{B}^{-1}\mathbf{A} \right)^{-1} \mathbf{B}^{-1}\mathbf{r}_o$$
$$= -\pi \mathbf{r}_o^t \left[-i\mathbf{D}\mathbf{L}_i - \left(\mathbf{D}\mathbf{B}^{-1}\mathbf{A} - \mathbf{B}^{-1^t} \right) \right] \left(\mathbf{B}\mathbf{L}_i - i\mathbf{A} \right)^{-1} \mathbf{r}_o = -\pi \mathbf{r}_o^t \left(-i\mathbf{D}\mathbf{L}_i - \mathbf{C} \right) \left(\mathbf{B}\mathbf{L}_i - i\mathbf{A} \right)^{-1} \mathbf{r}_o$$
$$= -\pi \mathbf{r}_o^t \left(\mathbf{D}\mathbf{L}_i - i\mathbf{C} \right) \left(\mathbf{A} + i\mathbf{B}\mathbf{L}_i \right)^{-1} \mathbf{r}_o \equiv -\pi \mathbf{r}_o^t \mathbf{L}_o \mathbf{r}_o.$$

Next we consider the factor that appears in connection with $\mathbf{s}^t\mathbf{s}$:

$$-\mathbf{s}^t \left[\mathbf{M}_i - 2\mathbf{K}_i \left(\mathbf{L}_i - i\mathbf{B}^{-1}\mathbf{A} \right)^{-1} \mathbf{K}_i \right] \mathbf{s} = -\mathbf{s}^t \left[\mathbf{M}_i - 2i\mathbf{K}_i \left(\mathbf{A} + i\mathbf{B}\mathbf{L}_i \right)^{-1} \mathbf{K}_i \right] \mathbf{s} \equiv -\mathbf{s}^t \mathbf{M}_o \mathbf{s}.$$

Finally, the factor that appears in connection with $\mathbf{s}^t\mathbf{r}_o$ reads

$$2\sqrt{2\pi}\mathbf{s}^t \left(\mathbf{A} + i\mathbf{B}\mathbf{L}_i \right)^{-1} \mathbf{r}_o \equiv 2\sqrt{2\pi}\mathbf{s}^t \mathbf{K}_o \mathbf{r}_o,$$

and the generating function of the output beam $\mathcal{H}_{n,m}(\mathbf{r};\mathbf{K},\mathbf{L})$ takes its final form

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \mathcal{H}_{n,m}(\mathbf{r};\mathbf{K}_o,\mathbf{L}_o) \left(\frac{2^{n+m}}{n!m!} \right)^{1/2} s_x^n s_y^m = \exp(i\phi)\, 2^{1/2} \left[\frac{\det \mathbf{K}_i}{\det i\mathbf{B}\, \det(\mathbf{L}_i - i\mathbf{B}^{-1}\mathbf{A})} \right]^{1/2}$$
$$\times \exp\left(-\mathbf{s}^t \mathbf{M}_i \mathbf{s} + i\pi \mathbf{r}_o^t \mathbf{D}\mathbf{B}^{-1}\mathbf{r}_o\right) \exp\left[-\pi \left(\mathbf{B}^{-1}\mathbf{r}_o + i\sqrt{2/\pi}\,\mathbf{K}_i^t \mathbf{s} \right)^t \left(\mathbf{L}_i - i\mathbf{B}^{-1}\mathbf{A} \right)^{-1} \left(\mathbf{B}^{-1}\mathbf{r}_o + i\sqrt{2/\pi}\,\mathbf{K}_i^t \mathbf{s} \right) \right]$$
$$\equiv \exp(i\phi) 2^{1/2} (\det \mathbf{K}_o)^{1/2} \exp\left(-\mathbf{s}^t \mathbf{M}_o \mathbf{s} + 2\sqrt{2\pi}\,\mathbf{s}^t \mathbf{K}_o \mathbf{r}_o - \pi \mathbf{r}_o^t \mathbf{L}_o \mathbf{r}_o \right)$$

with

$$\mathbf{K}_o = \mathbf{K}_i \left(\mathbf{A} + \mathbf{B}\, i\mathbf{L}_i \right)^{-1}, \tag{14}$$
$$i\mathbf{L}_o = (\mathbf{C} + \mathbf{D}\, i\mathbf{L}_i)(\mathbf{A} + \mathbf{B}\, i\mathbf{L}_i)^{-1}, \tag{15}$$
$$\mathbf{M}_o = \mathbf{M}_i - 2i\mathbf{K}_i \left(\mathbf{A} + \mathbf{B}\, i\mathbf{L}_i \right)^{-1} \mathbf{B}\mathbf{K}_i^t. \tag{16}$$

We conclude that the generating function (6) keeps its form when the associated Hermite-Gaussian-type modes propagate through a first-order optical system; we only have to replace the input matrices $\mathbf{K}_i$, $\mathbf{L}_i$, and $\mathbf{M}_i$ by the output matrices $\mathbf{K}_o$, $\mathbf{L}_o$, and $\mathbf{M}_o$, respectively, in accordance with the input-output relationships (14–16). Note that Eq. (15) is in fact the well-known **ABCD**-law, and that Eqs. (14) and (15) can be combined into

$$\begin{bmatrix} \mathbf{I} \\ i\mathbf{L}_o \end{bmatrix} \mathbf{K}_o^{-1} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{I} \\ i\mathbf{L}_i \end{bmatrix} \mathbf{K}_i^{-1}. \tag{17}$$

Note that the matrix $\mathbf{M}$ plays a different role than the matrices $\mathbf{K}$ and $\mathbf{L}$.

4. CONDITIONS RESULTING FROM ORTHONORMALITY

From the requirement that the Hermite-Gaussian-type modes are orthonormal,

$$\iint \mathcal{H}_{n,m}(\mathbf{r};\mathbf{K},\mathbf{L})\, \mathcal{H}_{l,k}^*(\mathbf{r};\mathbf{K},\mathbf{L})\, \mathrm{d}\mathbf{r} = \delta_{nl}\delta_{mk}, \tag{18}$$

we get additional conditions for the three matrices $\mathbf{K}$, $\mathbf{L}$, and $\mathbf{M}$. To derive these, we consider the expression

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{2^{n+m}}{n!m!}\right)^{1/2} s_x^n s_y^m \left(\frac{2^{l+k}}{l!k!}\right)^{1/2} t_x^l t_y^k \iint \mathcal{H}_{n,m}(\mathbf{r};\mathbf{K},\mathbf{L})\,\mathcal{H}^*_{l,k}(\mathbf{r};\mathbf{K},\mathbf{L})\,\mathrm{d}\mathbf{r}$$
$$= 2\,|\det\mathbf{K}|\,\exp(-\mathbf{s}^t\mathbf{M}\mathbf{s}-\mathbf{t}^t\mathbf{M}^*\mathbf{t})\iint\exp\left[2\sqrt{2\pi}\,(\mathbf{s}^t\mathbf{K}+\mathbf{t}^t\mathbf{K}^*)\mathbf{r}-\pi\mathbf{r}^t(\mathbf{L}+\mathbf{L}^*)\mathbf{r}\right]\,\mathrm{d}\mathbf{r},$$

where $\mathbf{t}=(t_x,t_y)^t$ on the analogy of $\mathbf{s}=(s_x,s_y)^t$, and where we have substituted from the generating function (6). We note that the integral in this expression equals

$$[\det(\mathbf{L}+\mathbf{L}^*)]^{-1/2}\exp\left\{\mathbf{s}^t\left[\mathbf{K}\left(\frac{\mathbf{L}+\mathbf{L}^*}{2}\right)^{-1}\mathbf{K}^t\right]\mathbf{s}+\mathbf{t}^t\left[\mathbf{K}\left(\frac{\mathbf{L}+\mathbf{L}^*}{2}\right)^{-1}\mathbf{K}^t\right]^*\mathbf{t}\right\}$$
$$\times\exp\left\{2\mathbf{s}^t\left[\mathbf{K}\left(\frac{\mathbf{L}+\mathbf{L}^*}{2}\right)^{-1}\mathbf{K}^{*t}\right]\mathbf{t}\right\}$$

and we get

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{2^{n+m}}{n!m!}\right)^{1/2} s_x^n s_y^m \left(\frac{2^{l+k}}{l!k!}\right)^{1/2} t_x^l t_y^k \iint \mathcal{H}_{n,m}(\mathbf{r};\mathbf{K},\mathbf{L})\,\mathcal{H}^*_{l,k}(\mathbf{r};\mathbf{K},\mathbf{L})\,\mathrm{d}\mathbf{r}$$
$$= 2\,|\det\mathbf{K}|[\det(\mathbf{L}+\mathbf{L}^*)]^{-1/2}\exp\left\{2\mathbf{s}^t\left[\mathbf{K}\left(\frac{\mathbf{L}+\mathbf{L}^*}{2}\right)^{-1}\mathbf{K}^{*t}\right]\mathbf{t}\right\}$$
$$\times\exp\left\{-\mathbf{s}^t\left[\mathbf{M}-\mathbf{K}\left(\frac{\mathbf{L}+\mathbf{L}^*}{2}\right)^{-1}\mathbf{K}^t\right]\mathbf{s}-\mathbf{t}^t\left[\mathbf{M}-\mathbf{K}\left(\frac{\mathbf{L}+\mathbf{L}^*}{2}\right)^{-1}\mathbf{K}^t\right]^*\mathbf{t}\right\}.$$

To get to the orthonormality condition (18), we have to require

$$\mathbf{M}-\mathbf{K}\left(\frac{\mathbf{L}+\mathbf{L}^*}{2}\right)^{-1}\mathbf{K}^t=\mathbf{0}, \tag{19}$$

$$\mathbf{K}\left(\frac{\mathbf{L}+\mathbf{L}^*}{2}\right)^{-1}\mathbf{K}^{*t}=\mathbf{I}, \tag{20}$$

leading to the conditions

$$\mathbf{M}^{-1}=\mathbf{M}^*=\mathbf{K}^*\mathbf{K}^{-1}=(\mathbf{K}^*\mathbf{K}^{-1})^t, \tag{21}$$

$$\frac{\mathbf{L}+\mathbf{L}^*}{2}=\mathbf{K}^t\mathbf{K}^*=(\mathbf{K}^t\mathbf{K}^*)^t, \tag{22}$$

where we have also expressed the symmetry of the matrices $\mathbf{L}$ and $\mathbf{M}$ once again. Note that $\mathbf{M}=\mathbf{K}\mathbf{K}^{*-1}$ is completely determined by $\mathbf{K}$, see Eq. (21), which is the reason why we did not include $\mathbf{M}$ as a parameter in $\mathcal{H}_{n,m}(\mathbf{r};\mathbf{K},\mathbf{L})$.

If we express $\mathbf{K}^{-1}$ in its real and imaginary parts, $\mathbf{K}^{-1}=\mathbf{a}+i\mathbf{b}$, we immediately get from the realness of $(\mathbf{K}^{*t}\mathbf{K})^{-1}$, see Eq. (22), that the matrix $\mathbf{a}\mathbf{b}^t$ is symmetric. If we then express $\mathbf{L}$ as $\mathbf{L}=(\mathbf{d}-i\mathbf{c})\mathbf{K}=(\mathbf{d}-i\mathbf{c})(\mathbf{a}+i\mathbf{b})^{-1}$, the symmetry of $\mathbf{L}$ leads to

$$\mathbf{a}^t\mathbf{d}+\mathbf{b}^t\mathbf{c}=\mathbf{d}^t\mathbf{a}+\mathbf{c}^t\mathbf{b}\quad\text{and}\quad\mathbf{a}^t\mathbf{c}-\mathbf{b}^t\mathbf{d}=\mathbf{c}^t\mathbf{a}-\mathbf{d}^t\mathbf{b},$$

while Eq. (22) leads to the requirements

$$\mathbf{a}^t\mathbf{d}-\mathbf{b}^t\mathbf{c}+\mathbf{d}^t\mathbf{a}-\mathbf{c}^t\mathbf{b}=2\,\mathbf{I}\quad\text{and}\quad\mathbf{a}^t\mathbf{c}+\mathbf{b}^t\mathbf{d}=\mathbf{c}^t\mathbf{a}+\mathbf{d}^t\mathbf{b}.$$

From these four conditions we conclude that the 4×4-matrix

$$\begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix}$$

is symplectic and thus satisfies relations of the form (11).

The results in this paper resemble those derived by Wünsche.[5] The main difference is that we use as the Gaussian part $\exp(-\pi \mathbf{r}^t \mathbf{L} \mathbf{r})$, with a matrix $\mathbf{L}$ that can be chosen freely if we would only require Eq. (19) and not necessarily Eq. (20), whereas Wünsche uses a fixed expression of the form $\exp(-\pi \mathbf{r}^t \mathbf{r})$. Wünsche's results arise indeed from ours for the special choice $\mathbf{L}=\mathbf{I}$, in which case Eq. (19) leads to $\mathbf{M}=\mathbf{K}\mathbf{K}^t$, yielding the generating function

$$\exp[-\mathbf{s}^t\mathbf{K}\mathbf{K}^t\mathbf{s} + 2\mathbf{s}^t\mathbf{K}(\sqrt{2\pi}\,\mathbf{r}) - (\sqrt{2\pi}\,\mathbf{r})^t(\sqrt{2\pi}\,\mathbf{r})/2],$$

which is compatible to Ref. [5, Eq. (8.4)]. Eq. (20) would yield the additional condition $\mathbf{K}\mathbf{K}^{*t}=\mathbf{I}$.

Special cases of Hermite-Gaussian-type modes can easily be recognized. We mention the (separable) Hermite-Gaussian modes (with curvatures in the x and y directions determined by γ_x and γ_y, respectively), for which the matrices $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ are given by

$$\mathbf{W}^{-1}(\mathbf{a}+i\mathbf{b}) = \begin{bmatrix} \cos\gamma_x/\exp(i\gamma_x) & 0 \\ 0 & \cos\gamma_y/\exp(i\gamma_y) \end{bmatrix} \mathbf{W}(\mathbf{d}-i\mathbf{c}) = \begin{bmatrix} \exp(i\gamma_1) & 0 \\ 0 & \exp(i\gamma_2) \end{bmatrix}; \tag{23}$$

the common Hermite-Gaussian modes with which we started this paper [see Eqs. (1-5)], arise for the special choice $\gamma_x=\gamma_y=\gamma_1=\gamma_2=0$. Note that for Hermite-Gaussian modes the matrix

$$\mathbf{L} = \begin{bmatrix} (1+i\tan\gamma_x)\,w_x^{-2} & 0 \\ 0 & (1+i\tan\gamma_y)\,w_y^{-2} \end{bmatrix}$$

is a *diagonal* matrix, and that the **ABCD**-law (15) is useful when such modes propagate through separable systems (for which $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are *diagonal* matrices). For (rotationally symmetric) Laguerre-Gaussian modes (with its curvature determined by γ) we have

$$w^{-1}(\mathbf{a}+i\mathbf{b}) = \frac{\cos\gamma}{\exp(i\gamma)}\, w(\mathbf{d}-i\mathbf{c}) = \frac{1}{\sqrt{2}} \begin{bmatrix} \exp(i\gamma_1) & -i\exp(i\gamma_2) \\ -i\exp(i\gamma_1) & \exp(i\gamma_2) \end{bmatrix}; \tag{24}$$

the special case $\gamma=\gamma_1=\gamma_2=0$ has been reported, for instance, in Ref. 6. Note that for Laguerre-Gaussian modes the matrix $\mathbf{L}=(1+i\tan\gamma)\,w^{-2}\,\mathbf{I}$ is a *scalar* matrix, and that the **ABCD**-law (15) is useful when such modes propagate through isotropic systems (for which $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are *scalar* matrices). We remark that the discriminating parameters in the above expressions are the widths (w_x, w_y, w) and the curvatures (γ_x, γ_y, γ) of the modes; the parameters γ_1 and γ_2 lead to a mere multiplication of the complex field amplitude by a phase factor that depends on the mode-number (n,m) but not on the space variables $\mathbf{r}$.

5. AN ALTERNATE PROPAGATION LAW

Now that we know that – in the orthonormal case as described in Section 4 – the input matrices $\mathbf{K}_i$, $\mathbf{L}_i$, $\mathbf{M}_i$ and the output matrices $\mathbf{K}_o$, $\mathbf{L}_o$, $\mathbf{M}_o$ can be expressed in the special forms

$$\begin{aligned} \mathbf{K}_{i,o} &= (\mathbf{a}_{i,o}+i\mathbf{b}_{i,o})^{-1}, \\ \mathbf{L}_{i,o} &= (\mathbf{d}_{i,o}-i\mathbf{c}_{i,o})(\mathbf{a}_{i,o}+i\mathbf{b}_{i,o})^{-1}, \\ \mathbf{M}_{i,o} &= (\mathbf{a}_{i,o}+i\mathbf{b}_{i,o})^{-1}(\mathbf{a}_{i,o}-i\mathbf{b}_{i,o}), \end{aligned} \tag{25}$$

where the real matrices $\mathbf{a}_i$, $\mathbf{b}_i$, $\mathbf{c}_i$, $\mathbf{d}_i$, and the real matrices $\mathbf{a}_o$, $\mathbf{b}_o$, $\mathbf{c}_o$, $\mathbf{d}_o$ constitute two real symplectic matrices, we will combine these expressions with the input-output relations (14) and (15) to find an alternate propagation law. From Eq. (14) we have $\mathbf{K}_o^{-1}=(\mathbf{A}+\mathbf{B}i\mathbf{L}_i)\mathbf{K}_i^{-1}$, and after substituting from Eqs. (25) we get

$$\begin{aligned} \mathbf{a}_o+i\mathbf{b}_o &= \left[\mathbf{A}+\mathbf{B}i(\mathbf{d}_i-i\mathbf{c}_i)(\mathbf{a}_i+i\mathbf{b}_i)^{-1}\right](\mathbf{a}_i+i\mathbf{b}_i) \\ &= \mathbf{A}(\mathbf{a}_i+i\mathbf{b}_i)+\mathbf{B}i(\mathbf{d}_i-i\mathbf{c}_i) \\ &= (\mathbf{A}\mathbf{a}_i+\mathbf{B}\mathbf{c}_i)+i(\mathbf{A}\mathbf{b}_i+\mathbf{B}\mathbf{d}_i). \end{aligned}$$

Likewise, combining Eqs. (14) and (15), we have $i\mathbf{L}_o = (\mathbf{C} + \mathbf{D}i\mathbf{L}_i)(\mathbf{A} + \mathbf{B}i\mathbf{L}_i)^{-1} = (\mathbf{C} + \mathbf{D}i\mathbf{L}_i)\mathbf{K}_i^{-1}\mathbf{K}_o$, and after substituting from Eqs. (25) we get

$$\begin{aligned} i(\mathbf{d}_o - i\mathbf{c}_o)(\mathbf{a}_o + i\mathbf{b}_o)^{-1} &= \left[\mathbf{C} + \mathbf{D}i(\mathbf{d}_i - i\mathbf{c}_i)(\mathbf{a}_i + i\mathbf{b}_i)^{-1}\right](\mathbf{a}_i + i\mathbf{b}_i)(\mathbf{a}_o + i\mathbf{b}_o)^{-1}, \\ i(\mathbf{d}_o - i\mathbf{c}_o) &= \mathbf{C}(\mathbf{a}_i + i\mathbf{b}_i) + \mathbf{D}i(\mathbf{d}_i - i\mathbf{c}_i) \\ \mathbf{c}_o + i\mathbf{d}_o &= (\mathbf{C}\mathbf{a}_i + \mathbf{D}\mathbf{c}_i) + i(\mathbf{C}\mathbf{b}_i + \mathbf{D}\mathbf{d}_i). \end{aligned}$$

We are thus led to the elegant propagation law

$$\begin{bmatrix} \mathbf{a}_o & \mathbf{b}_o \\ \mathbf{c}_o & \mathbf{d}_o \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{a}_i & \mathbf{b}_i \\ \mathbf{c}_i & \mathbf{d}_i \end{bmatrix}. \tag{26}$$

This propagation law resembles Ref. [7, Eqs. (12) and (29)], split up into their real and imaginary parts, where $i(\mathbf{a} + i\mathbf{b}), \mathbf{d} - i\mathbf{c}$ correspond to the 'matricial rays' $\mathbf{Q}, \mathbf{P}$, see Ref. [7, Eq. (11)]: $\mathbf{Q}\sqrt{\pi} = i\mathbf{K}^{-1} = i(\mathbf{a} + i\mathbf{b})$ and $\mathbf{P}\sqrt{\pi} = \lambda\mathbf{L}\mathbf{K}^{-1} = \lambda(\mathbf{d} - i\mathbf{c})$, with λ the wavelength of the light. The treatment in Ref. 7 is based on a so-called 'mode-generating system,' which is excited by an off-axis point source at its input plane; in our case of lossless, first-order optics, this system is determined by the matricial rays $\mathbf{Q}, \mathbf{P}$ that are associated with the mode that is to be generated. The modes in Ref. 7 then arise by expanding the resulting output field in a power series of the coordinates of the point source at the input. The kernel in the Collins integral that describes the mode-generating system thus plays the role of a generating function. The present treatment is directly based on the general form (6) of the generating function, and we get the additional result that the parameters $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}, \mathbf{d}$ that characterize this generating function correspond to the real and imaginary parts of $\mathbf{Q}$ and $\mathbf{P}$, and constitute a real, symplectic matrix. With $\mathbf{r} = \mathbf{x}$ and $\lambda\mathbf{s} = \pi\mathbf{y}\sqrt{2}$, there is indeed a one-to-one correspondence between our generating function (6) and the one used in Ref. 7, see in particular Ref. [7, Eq. (22)].

We remark that all sets of Hermite-Gaussian-type modes can be converted into each other by means of properly chosen first-order optical systems, and we conclude that knowledge of the generating function and in particular its propagation law (26) may be valuable in the design of mode converters. Further work is in progress, see for instance Ref. 8, and future papers may also take lossy mode converters[7] into account.

6. CONCLUSION

A general class of orthonormal sets of Hermite-Gaussian-type modes has been introduced by formulating a generalized version of the generating function that yields the common Hermite-Gaussian modes. These sets of Hermite-Gaussian-type modes remain in their class when they propagate through first-order optical systems, and a propagation law for their generating function has been formulated. The propagation law is in a form that suits itself for the design of mode converters.

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