Propagation law for Hermite- and Laguerre-Gaussian beams in first-order optical systems

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ABSTRACT

Starting from Hermite-Gaussian beams, we generate a general class of rotationally symmetric beams. These beams are Laguerre-Gaussian beams, parameterized by two parameters h and g, representing the curvature and the width of the beam, respectively. The Wigner distribution of each member of this class is readily derived from the Wigner distribution of the Hermite-Gaussian beam from which it is generated. If these Laguerre-Gaussian beams propagate through an isotropic *abcd*-system, they remain in their class, while the propagation of the complex beam parameter $h \pm ig$ satisfies the well-known *abcd*-law.

Keywords: Hermite-Gaussian modes, Laguerre-Gaussian modes, Wigner distribution, first-order optical systems

1. INTRODUCTION

We consider the propagation of (one-dimensional) Hermite-Gaussian beams and (rotationally symmetric) Laguerre-Gaussian beams through first-order optical systems. Extensive use is made of the Wigner distribution, in terms of which the system's input-output relationship takes the form of a mere coordinate transformation in phase space.

We start with a one-dimensional Hermite-Gaussian beam, described by its complex curvature (i.e. its curvature and its width, combined into one complex parameter). We determine the Wigner distribution of the beam and consider the propagation through a first-order optical system expressed in terms of its ray transformation (*abcd*) matrix. From the particular dependence of the Wigner distribution on the phase space coordinates, we conclude that a Hermite-Gaussian beam preserves its Hermite-Gaussian character and that its complex curvature propagates according to the bilinear *abcd*-law.

We then convert a Hermite-Gaussian beam into a Laguerre-Gaussian beam, using any appropriate Hermite-to-Laguerre mode converter. Since the mode converter is a first-order optical system, the Wigner distributions of the input and output beams are related through a mere coordinate transformation and have roughly the same form, which demonstrates once more the usefulness of the Wigner distribution.

Finally, a (rotationally symmetric) Laguerre-Gaussian beam, completely described again by a complex curvature, is considered with respect to propagation through isotropic first-order optical systems. Much like in the Hermite-Gaussian case, we observe that Laguerre-Gaussian beams keep their Laguerre-Gaussian character and that the bilinear *abcd*-law remains valid.

2. WIGNER DISTRIBUTION

In this paper we make extensive use of the Wigner distribution of a complex field amplitude $f(\mathbf{r})$, defined by[1]

$$W_f(\mathbf{r},\mathbf{q}) = \iint f(\mathbf{r}+\mathbf{r}'/2)\, f^*(\mathbf{r}-\mathbf{r}'/2)\, \exp(-i2\pi\mathbf{q}^t\mathbf{r}')\, \mathrm{d}\mathbf{r}', \tag{1}$$

where $\mathbf{r} = (x,y)^t$ denote spatial variables and $\mathbf{q} = (u,v)^t$ are their conjugates: the spatial frequency (or direction) variables. All integrations in this paper extend from $-\infty$ till $+\infty$ and, as usual, the superscripts ^{t} and * denote transposition and complex conjugation, respectively. From the Wigner distribution, the complex field amplitude can be reconstructed – up to a constant phase factor – by means of the inverse relationship

$$f(\mathbf{r}_1)f^*(\mathbf{r}_2) = \iint W_f\left(\frac{\mathbf{r}_1+\mathbf{r}_2}{2},\mathbf{q}\right)\exp[i2\pi\mathbf{q}^t(\mathbf{r}_1-\mathbf{r}_2)]\,\mathrm{d}\mathbf{q}\,; \tag{2}$$

in particular we have $|f(\mathbf{r})|^2 = \iint W_f(\mathbf{r},\mathbf{q})\,\mathrm{d}\mathbf{q}$. Note that an indeterminant constant phase factor is often not important.

A basic and important optical field is the (two-dimensional) Hermite-Gaussian beam

$$\mathcal{H}_{n,m}(\mathbf{r}; h_x+ig_x, h_y+ig_y) = \mathcal{H}_n(x; h_x+ig_x)\mathcal{H}_m(y; h_y+ig_y), \tag{3}$$

with

$$\mathcal{H}_n(x; h+ig) = (2g)^{1/4}\,(2^n n!)^{-1/2}\;H_n\left(\sqrt{2\pi g}\,x\right)\exp\left[i\pi(h+ig)x^2\right]\quad (g>0), \tag{4}$$

for which the Wigner distribution takes the form

$$W_{\mathcal{H}_{n,m}}(\mathbf{r},\mathbf{q}; h_x+ig_x, h_y+ig_y) = W_{\mathcal{H}_n}(x,u; h_x+ig_x)\,W_{\mathcal{H}_m}(y,v; h_y+ig_y), \tag{5}$$

with[2]

$$W_{\mathcal{H}_n}(x,u; h+ig) = 2\,(-1)^n\,L_n\left[4\pi\left(gx^2+g^{-1}(u-hx)^2\right)\right]\exp\left[-2\pi\left(gx^2+g^{-1}(u-hx)^2\right)\right]; \tag{6}$$

$H_n(\cdot)$ and $L_n(\cdot)$ denote the Hermite and the Laguerre polynomials, respectively.[3] We will use this two-dimensional Hermite-Gaussian beam as a starting point to generate a general class of rotationally symmetric beams by means of first-order optical systems.

3. FIRST-ORDER OPTICAL SYSTEMS

Any lossless first-order optical system can be described by the transformation[1]

$$f_o(\mathbf{r}_o) = \exp(i\phi)\sqrt{\det(\mathbf{L}_{io}/i)}\iint f_i(\mathbf{r}_i)\,\exp\left[i\pi\left(\mathbf{r}_i^t\mathbf{L}_{ii}\mathbf{r}_i - 2\mathbf{r}_i^t\mathbf{L}_{io}\mathbf{r}_o + \mathbf{r}_o^t\mathbf{L}_{oo}\mathbf{r}_o\right)\right]\,\mathrm{d}\mathbf{r}_\mathrm{i}, \tag{7}$$

where $\mathbf{L}_{ii}$, $\mathbf{L}_{oo}$, and $\mathbf{L}_{io}$ are three real 2×2 matrices, and $\mathbf{L}_{ii}$ and $\mathbf{L}_{oo}$ are symmetric. The usual (but rather irrelevant) phase factor $\exp(i\phi)$ is sometimes chosen equal to $-i\exp(i2\pi z/\lambda)$, with λ the wavelength of the light, to have an exact representation of a section z of free space in the special case $\mathbf{L}_{ii} = \mathbf{L}_{oo} = \mathbf{L}_{io} = (\lambda z)^{-1}\mathbf{I}$, where $\mathbf{I}$ represents the 2×2 identity matrix. From the submatrices $\mathbf{L}_{ii}$, $\mathbf{L}_{oo}$, and $\mathbf{L}_{io}$, we can derive the four 2×2 matrices[1]

$$\begin{aligned} &\mathbf{A} = \mathbf{L}_{io}^{-1}\mathbf{L}_{ii}, && \mathbf{B} = \mathbf{L}_{io}^{-1},\\ &\mathbf{C} = \mathbf{L}_{oo}\mathbf{L}_{io}^{-1}\mathbf{L}_{ii} - \mathbf{L}_{io}^t, && \mathbf{D} = \mathbf{L}_{oo}\mathbf{L}_{io}^{-1}, \end{aligned} \tag{8}$$

which constitute the well-known ray transformation matrix that relates the position $\mathbf{r}_i$ and direction $\mathbf{q}_i$ of an incoming ray to the position $\mathbf{r}_o$ and direction $\mathbf{q}_o$ of the outgoing ray:

$$\begin{bmatrix}\mathbf{r}_o\\ \mathbf{q}_o\end{bmatrix} = \begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}\begin{bmatrix}\mathbf{r}_i\\ \mathbf{q}_i\end{bmatrix}. \tag{9}$$

Free space in the Fresnel approximation, as mentioned before, is thus described by $\mathbf{A} = \mathbf{D} = \mathbf{I}$, $\mathbf{C} = \mathbf{0}$, and $\mathbf{B} = \lambda z\,\mathbf{I}$. A direct consequence of the symmetry of the matrices $\mathbf{L}_{ii}$ and $\mathbf{L}_{oo}$ is the symplecticity of the ray transformation matrix, yielding the relations

$$\begin{aligned} &\mathbf{A}\mathbf{B}^t = \mathbf{B}\mathbf{A}^t, \quad \mathbf{C}\mathbf{D}^t = \mathbf{D}\mathbf{C}^t, \quad \mathbf{A}\mathbf{D}^t - \mathbf{B}\mathbf{C}^t = \mathbf{I},\\ &\mathbf{A}^t\mathbf{C} = \mathbf{C}^t\mathbf{A}, \quad \mathbf{B}^t\mathbf{D} = \mathbf{D}^t\mathbf{B}, \quad \mathbf{A}^t\mathbf{D} - \mathbf{C}^t\mathbf{B} = \mathbf{I}. \end{aligned} \tag{10}$$

Using the matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{D}$ in the kernel of the transformation (7) and assuming that $\mathbf{B}$ is a non-singular matrix, we can represent the first-order system by the Collins integral[4]

$$f_o(\mathbf{r}_o) = \frac{\exp(i\phi)}{\sqrt{\det i\mathbf{B}}} \iint f_i(\mathbf{r}_i) \exp\left[i\pi\left(\mathbf{r}_i^t\mathbf{B}^{-1}\mathbf{A}\mathbf{r}_i - 2\mathbf{r}_i^t\mathbf{B}^{-1}\mathbf{r}_o + \mathbf{r}_o^t\mathbf{D}\mathbf{B}^{-1}\mathbf{r}_o\right)\right] \,\mathrm{d}\mathbf{r}_\mathrm{i}. \tag{11}$$

Collins integral leads to an elegant representation of a first-order system in terms of the Wigner distributions,

$$W_o(\mathbf{r},\mathbf{q}) = W_i(\mathbf{D}^t\mathbf{r} - \mathbf{B}^t\mathbf{q}, -\mathbf{C}^t\mathbf{r} + \mathbf{A}^t\mathbf{q}), \tag{12}$$

where the input-output relationship of the first-order system reduces to a mere coordinate transformation in $(\mathbf{r},\mathbf{q})$ space.

4. MODE CONVERTERS

4.1. One-dimensional case

First-order systems can be used as mode converters. Before we turn our attention to two-dimensional systems, we first consider a basic one-dimensional system, which is described by the one-dimensional input-output relationship, cf. Eq. (11),

$$f_o(x_o) = \frac{\exp(i\phi)}{\sqrt{ib}} \int f_i(x_i) \exp\left[i\pi\frac{ax_i^2 + dx_o^2 - 2\,x_i x_o}{b}\right] \mathrm{d}x_i.$$

with ray transformation matrix elements

$$\begin{aligned} a &= (g_i/g_o)^{1/2}\cos\varphi, \quad b = (g_i g_o)^{-1/2}\sin\varphi, \\ c &= -(g_i g_o)^{1/2}\sin\varphi, \quad d = (g_o/g_i)^{1/2}\cos\varphi. \end{aligned} \tag{13}$$

Such a system converts a (one-dimensional) Hermite-Gaussian beam $\mathcal{H}_n(x; ig_i)$ into $\mathcal{H}_n(x; ig_o)$:

$$\frac{\exp(i\varphi/2)(g_i g_o)^{1/4}}{\sqrt{i\sin\varphi}} \int \mathcal{H}_n(x_i; ig_i) \exp\left[i\pi\frac{(g_i x_i^2 + g_o x_o^2)\cos\varphi - 2(g_i g_o)^{1/2}\,x_i x_o}{\sin\varphi}\right] \mathrm{d}\mathrm{x}_\mathrm{i} = \exp(-in\varphi)\,\mathcal{H}_n(x_o; ig_o). \tag{14}$$

Note that the system (13) represents a fractional Fourier transformer with fractional angle φ, preceded and succeeded by a scaling with $(g_i/g_o)^{1/4}$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (g_i/g_o)^{1/4} & 0 \\ 0 & (g_o/g_i)^{1/4} \end{bmatrix} \begin{bmatrix} \cos\varphi & (g_i g_o)^{-1/2}\sin\varphi \\ -(g_i g_o)^{1/2}\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} (g_i/g_o)^{1/4} & 0 \\ 0 & (g_o/g_i)^{1/4} \end{bmatrix}.$$

The special choice $a = d = 0$ and $b = -1/c = (g_i g_o)^{-1/2}$ (and hence $\varphi = \pi/2$) corresponds to a Fourier transformation (with scaling), while the limit $\varphi \to 0$ corresponds to a mere scaling: $a = 1/d = (g_i/g_o)^{1/2}$ and $b = c = 0$. A two-dimensional version of this mode converter, with the two-dimensional Hermite-Gaussian beam $\mathcal{H}_{n,m}(\mathbf{r}; ig_i, ig_i)$ as its input, has been considered in Ref. [5, Eq. (4)]; the extension to the asymmetrical case, with different input and output parameters g_i in the x and y directions, is rather straightforward.

The Hermite-Gaussian beam $\mathcal{H}_n(x; ig_i)$ is a special case of the more general Hermite-Gaussian beam $\mathcal{H}_n(x; h_i + ig_i)$; the latter beam shows an additional quadratic phase dependence described by the curvature h_i, see Eq. (4). The Wigner distribution of such a beam depends only on the particular combination $g_i x^2 + g_i^{-1}(u - h_i x)^2$, see Eq. (6). If such a beam propagates through a general (one-dimensional) *abcd*-system, it remains in the class of Hermite-Gaussian beams and the output beam reads $\mathcal{H}_n(x; h_o + ig_o)$, where the input parameters h_i and g_i and output parameters h_o and g_o are related by the bilinear relationship (the *abcd*-law)

$$h_o \pm ig_o = \frac{c + d(h_i \pm ig_i)}{a + b(h_i \pm ig_i)}, \tag{15}$$

and hence

$$
\begin{aligned}
g_o &= \frac{g_i}{(a+bh_i)^2+b^2g_i^2} \\
h_o &= \frac{(a+bh_i)(c+dh_i)+bdg_i^2}{(a+bh_i)^2+b^2g_i^2} \, .
\end{aligned}
$$

Indeed: if (x_i, u_i) and (x_o, u_o) satisfy the coordinate transformations $x_o = ax_i + bu_i$ and $u_o = cx_i + du_i$, cf. Eq. (9), and if $h_i \pm ig_i$ and $h_o \pm ig_o$ satisfy the *abcd*-law (15), like u_i/x_i and u_o/x_o do as well, the relation

$$
g_i x_i^2 + g_i^{-1}(u_i - h_i x_i)^2 = g_o x_o^2 + g_o^{-1}(u_o - h_o x_o)^2
$$

holds and the particular combination $gx^2 + g^{-1}(u - hx)^2$ is thus invariant under propagation through the *abcd*-system. If we apply the bilinear relationship to the mode converter (13), we can easily verify that the Hermite-Gaussian beam $\mathcal{H}_n(x; ig_i)$ at the input of the scaled fractional Fourier transformer yields indeed $\mathcal{H}_n(x; ig_o)$ at its output: $h_o = h_i = 0$ (i.e., both the input and the output Hermite-Gaussian beam are without curvature) and $g_o = dg_i/a = -c/bg_i$.

4.2. Two-dimensional case

Let us now turn our attention to the two-dimensional case. One possible way to generate a (rotationally symmetric) Laguerre-Gaussian beam from the Hermite-Gaussian beam $\mathcal{H}_{n,m}(\mathbf{r}; ig_i, ig_i)$ is by the system

$$
\begin{aligned}
\mathbf{A} &= \frac{(g_i/g_o)^{1/2}}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, &
\mathbf{B} &= \frac{(g_i g_o)^{-1/2}}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
\mathbf{C} &= \frac{-(g_i g_o)^{1/2}}{\sqrt{2}} \begin{bmatrix} 1 & -h_o/g_o \\ -h_o/g_o & 1 \end{bmatrix}, &
\mathbf{D} &= \frac{(g_o/g_i)^{1/2}}{\sqrt{2}} \begin{bmatrix} h_o/g_o & 1 \\ 1 & h_o/g_o \end{bmatrix},
\end{aligned}
\tag{16}
$$

in which case Collins integral takes the form

$$
\frac{\exp(i\pi/8)(g_i g_o)^{1/4}}{\sqrt{\sin(\pi/4)}} \times \iint \mathcal{H}_{n,m}(\mathbf{r}_i; ig_i, ig_i) \exp\left[i\pi \left(2g_i x_i y_i + 2g_o x_o y_o + h_o(x_o^2 + y_o^2) - 2\sqrt{2}(g_i g_o)^{1/2}(x_i x_o + y_i y_o) \right) \right] \mathrm{d}\mathbf{r}_\mathrm{i}. \tag{17}
$$

The mode converter (16) is a generalization of the system described in Ref. [5, Eq. (5)], for which $h_o = 0$; the present system (16) contains an additional lens with focal distance $-1/\lambda h_o$ at its output, which yields the additional curvature term $h_o(x_o^2 + y_o^2)$ in the exponent of Collins integral (17). Note that the ray transformation matrix Ref. [5, Eq. (5)], i.e. Eq. (16) with $h_o = 0$, is comparable to the matrix $S_1(\pi/4)$ presented in Ref. [2, Eq. (14)]. The two mode converters – Eq. (16) with $h_o = 0$ and Ref. [2, Eq. (14)] – are related by a Fourier transformation,

$$
S_1(\pi/4) \begin{bmatrix} \mathbf{0} & (g_i g_o)^{-1/2}\,\mathbf{I} \\ -(g_i g_o)^{1/2}\mathbf{I} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix},
$$

where the (not explicitly mentioned) parameter α in $S_1(\pi/4)$ equals $-1/g_o$; but since a Hermite-Gaussian beam is an eigenfunction of the Fourier transformation, such a preceding Fourier transformation is rather irrelevant in the design of mode converters.

If we consider a (rotationally symmetric) Laguerre-Gaussian beam as resulting from a Hermite-Gaussian beam $\mathcal{H}_{n,m}(\mathbf{r}; ig_i, ig_i)$ after transformation through the mode converter (16), it is easy to see that its Wigner distribution takes the same form as the Wigner distribution of the Hermite-Gaussian beam, cf. Eqs. (5) and (6), but with a mere transformation of the coordinates, cf. Eq. (12):

$$
\begin{aligned}
W_{\mathcal{H}_{n,m}}(\mathbf{D}^t\mathbf{r} - \mathbf{B}^t\mathbf{q}, -\mathbf{C}^t\mathbf{r} + \mathbf{A}^t\mathbf{q}; ig_i, ig_i) &= 4\,(-1)^{n+m} \\
&\times\ L_n\left[2\pi\left(g_o\mathbf{r}^t\mathbf{r} + g_o^{-1}(\mathbf{q} - h_o\mathbf{r})^t(\mathbf{q} - h_o\mathbf{r}) + xv - yu\right)\right] \\
&\times\ L_m\left[2\pi\left(g_o\mathbf{r}^t\mathbf{r} + g_o^{-1}(\mathbf{q} - h_o\mathbf{r})^t(\mathbf{q} - h_o\mathbf{r}) - xv + yu\right)\right] \\
&\times\ \exp\left[-2\pi\left(g_o\mathbf{r}^t\mathbf{r} + g_o^{-1}(\mathbf{q} - h_o\mathbf{r})^t(\mathbf{q} - h_o\mathbf{r})\right)\right],
\end{aligned}
\tag{18}
$$

see also Ref. [2, Eq. (20)].

The latter expression depends only on the two combinations $g\mathbf{r}^t\mathbf{r} + g^{-1}(\mathbf{q} - h\mathbf{r})^t(\mathbf{q} - h\mathbf{r})$ and $xv - yu$, and thus corresponds to a rotationally symmetric beam. This can easily be seen if we represent x and y in polar coordinates, $x = r\cos\phi$ and $y = r\sin\phi$, and, with the angle ϕ as an offset, we do the same with u and v, $u = q\cos(\phi+\theta)$ and $v = q\sin(\phi+\theta)$. The two combinations then take the forms $gr^2 + g^{-1}(q^2 + h^2r^2 - 2hqr\cos\theta)$ and $qr\sin\theta$, respectively, and do not depend on the absolute angle ϕ.

The field amplitude of a Laguerre-Gaussian beam without curvature, whose Wigner distribution takes the form of Eq. (18) with $h = 0$, has been presented before; see, for instance, Ref. [2, Eq. (6)], Ref. [5, Eq. (5)], and Ref. [6, Eq. (30)]. In our general case – with curvature – the field amplitude reads

$$\mathcal{L}_{n,m}(\mathbf{r}; ig) = (2g)^{1/2} \left[\frac{(\min\{n,m\})!}{(\max\{n,m\})!}\right]^{1/2} \left(\sqrt{2\pi g}\, r\right)^{|n-m|} \exp[i(n-m)\phi]\; L^{(|n-m|)}_{\min\{n,m\}}\left(2\pi g r^2\right) \exp\left[i\pi(h+ig)r^2\right], \tag{19}$$

where $L^{(\alpha)}_n(\cdot)$ denotes the generalized Laguerre polynomial.[3] Note the vortex behavior of such a beam, represented by the phase term $\exp[i(n-m)\phi]$.

4.3. Isotropic systems

Rotationally symmetric beams remain rotationally symmetric if they propagate through isotropic systems; in the case of first-order systems this implies that the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are scalar matrices: $\mathbf{A} = a\mathbf{I}$, $\mathbf{B} = b\mathbf{I}$, $\mathbf{C} = c\mathbf{I}$, and $\mathbf{D} = d\mathbf{I}$. Using the coordinate transformation (9), it is not difficult to see that the combination $xv - yu$ is invariant: $x_iv_i - y_iu_i = x_ov_o - y_ou_o$. Because of the sole dependence of the Wigner distribution (18) on the particular combination $g\mathbf{r}^t\mathbf{r} + g^{-1}(\mathbf{q} - h\mathbf{r})^t(\mathbf{q} - h\mathbf{r})$ [and $xv - yu$], we can conclude as we did before for the Hermite-Gaussian beam: if a beam whose Wigner distribution depends on $g_i\mathbf{r}^t\mathbf{r} + g_i^{-1}(\mathbf{q} - h_i\mathbf{r})^t(\mathbf{q} - h_i\mathbf{r})$ [and $xv - yu$] is the input of a general, isotropic *abcd*-system, the Wigner distribution of the output beam depends on $g_o\mathbf{r}^t\mathbf{r} + g_o^{-1}(\mathbf{q} - h_o\mathbf{r})^t(\mathbf{q} - h_o\mathbf{r})$ [and $xv - yu$], and the input parameters h_i and g_i and output parameters h_o and g_o are related again by the bilinear relationship (15), the *abcd*-law.

From the *abcd*-law we readily derive that the condition $h_o = h_i = 0$ (i.e., both the input and the output Laguerre-Gaussian beam are without curvature) requires a system for which $dg_i/a = -c/bg_i$. This condition, together with the symplecticity condition $ad - bc = 1$, leads to a (scaled) fractional Fourier transformer again, see Eq. (13), and we get that Laguerre-Gaussian beams without curvature, i.e. $h = 0$, are eigenfunctions of a fractional Fourier transformation (with identical fractional angles $\varphi_x = \varphi_y = \varphi$). Consequently, since both the beam and the system are rotationally symmetric, such beams are also eigenfunctions of a fractional Hankel transformation (with fractional angle φ).[6]

5. CONCLUSION

We conclude that, starting from a Hermite-Gaussian beam $\mathcal{H}_{n,m}(\mathbf{r}; ig_x, ig_y)$, we can generate a class of rotationally symmetric Laguerre-Gaussian beams, parameterized by two parameters h and g. While the field amplitude of a member of this class takes the general form (19), its Wigner distribution takes the form (18). If these beams propagate through an isotropic *abcd*-system, they remain in their class, while the propagation of the complex beam parameter $h \pm ig$ satisfies the *abcd*-law (15).

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