Invariants of second-order moments of optical beams under phase-space rotations

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Abstract: The more attractive first-order optical systems such as a (fractional) Fourier transformer, a gyrator, and an image rotator [1] – used, among other applications, for optical information processing, including pattern recognition, encryption, and filtering; transversal mode conversion; and tomographic methods of coherence state estimation – correspond to different kinds of rotations in phase space. All such systems that produce phase-space rotations are described by an *orthogonal* ray transformation matrix.

Optical beams can be characterized by their ten second order moments, conveniently arranged in the symmetric 4×4 moment matrix [2,3]

$$\mathbf{M} = [m_{xx}, m_{xy}, m_{xu}, m_{xv}; m_{xy}, m_{yy}, m_{yu}, m_{yv}; m_{xu}, m_{yu}, m_{uu}, m_{uv}; m_{xv}, m_{yv}, m_{uv}, m_{vv}],$$

with (x,y) referring to space variables and (u,v) to the associated spatial-frequency variables. When a beam propagates through a first-order optical system, two independent beam invariants can be formulated, for instance: det $\mathbf{M}$ and $(m_{xx}m_{uu}-m_{xu}^2)+(m_{yy}m_{vv}-m_{yv}^2)+2(m_{xy}m_{uv}-m_{xv}m_{yu})$.

We derive new invariants, which hold for systems that produce phase-space rotations. The invariants are based on Tr $\mathbf{M} = m_{xx}+m_{yy}+m_{uu}+m_{vv}$ and on the three components of the orbital angular momentum: $Q_1 = [(m_{xx}+m_{uu})-(m_{yy}+m_{vv})]/4$, $Q_2 = (m_{xy}+m_{uv})/2$, and $Q_3 = (m_{xv}-m_{yu})/2$. In particular we show that the trace Tr $\mathbf{M}$ and the combination $Q_1^2+Q_2^2+Q_3^2$ are two new invariants.

Moreover we derive that Q_1 is an additional invariant for the antisymmetric fractional Fourier transformer $\mathsf{F}(\alpha_1,-\alpha_1)$, Q_2 is an invariant for the gyrator $\mathsf{G}(\alpha_2)$, and Q_3 for the image rotator $\mathsf{R}(\alpha_3)$. Furthermore, for $\mathsf{F}(\alpha_1,-\alpha_1)$ we have the rotation-type propagation law (with rotation angle $2\alpha_1$) for the components Q_2 and Q_3, which leads to the invariance of $Q_2^2+Q_3^2$. Similar rotation-type relations and invariants hold for $\mathsf{G}(\alpha_2)$ and $\mathsf{R}(\alpha_3)$ with cyclic permutations of the indexes. For a symmetric fractional Fourier transformer $\mathsf{F}(\alpha,-\alpha)$ all three individual components Q_1, Q_2, and Q_3 are invariant.

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