PROPERTIES OF LINEAR CANONICAL INTEGRAL TRANSFORMS

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Shift, scaling, derivation etc. theorems, well known for the Fourier transform, are generalized to the case of two-dimensional astigmatic canonical integral transforms used for the description of first-order optical systems.

Summary

The canonical integral transform (CT) describes – in the paraxial approximation of the scalar diffraction theory – the evolution of the complex field amplitude $f(\mathbf{r})$ during its propagation through a first-order optical system. The CT kernel is parameterized by the well-known symplectic ray transformation matrix $\mathbf{T}$, which relates the position $\mathbf{r}_i$ and direction $\mathbf{q}_i$ of an incoming ray to the position $\mathbf{r}_o$ and direction $\mathbf{q}_o$ of the outgoing ray:

$$\begin{bmatrix}\mathbf{r}_o\\ \mathbf{q}_o\end{bmatrix} = \begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}\begin{bmatrix}\mathbf{r}_i\\ \mathbf{q}_i\end{bmatrix} = \mathbf{T}\begin{bmatrix}\mathbf{r}_i\\ \mathbf{q}_i\end{bmatrix}. \tag{1}$$

In the often used case $\det\mathbf{B} \neq 0$, the CT has the form of Collins integral[1]

$$\begin{aligned} f_o(\mathbf{r}_o) &= R^{\mathbf{T}}\left[f_i(\mathbf{r}_i)\right](\mathbf{r}_o)\\ &= \frac{1}{\sqrt{\det i\mathbf{B}}}\int f_i(\mathbf{r}_i)\,\exp\left[i\pi\left(\mathbf{r}_i^t\mathbf{B}^{-1}\mathbf{A}\mathbf{r}_i - 2\mathbf{r}_i^t\mathbf{B}^{-1}\mathbf{r}_o + \mathbf{r}_o^t\mathbf{D}\mathbf{B}^{-1}\mathbf{r}_o\right)\right]\mathrm{d}\mathbf{r}_i, \end{aligned} \tag{2}$$

where $\mathbf{r} = (x, y)^t$ denote spatial dimensionless variables and $\mathbf{q} = (q_x, q_y)^t$ are their conjugates: the spatial-frequency (or direction) variables. As usual, the superscript ^{t} denotes transposition. A simple expression can also be written in the case $\mathbf{B} = \mathbf{0}$:

$$f_o(\mathbf{r}_o) = \frac{1}{\sqrt{\det\mathbf{A}}}\,\exp\left[i\pi\left(\mathbf{r}_o^t\mathbf{C}\mathbf{A}^{-1}\mathbf{r}_o\right)\right]f_i(\mathbf{A}^{-1}\mathbf{r}_o). \tag{3}$$

For the general singular case $\det\mathbf{B} = 0$, the CT can be represented by a separable fractional Fourier transform embedded in two rotator operations.[2] Since the ray transformation matrix $\mathbf{T}$ is symplectic,

$$\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}^{-1} = \begin{bmatrix}\mathbf{D}^t & -\mathbf{B}^t\\ -\mathbf{C}^t & \mathbf{A}^t\end{bmatrix} \quad \text{and thus} \quad \begin{matrix}\mathbf{A}\mathbf{B}^t = \mathbf{B}\mathbf{A}^t, & \mathbf{C}\mathbf{D}^t = \mathbf{D}\mathbf{C}^t, & \mathbf{A}\mathbf{D}^t - \mathbf{B}\mathbf{C}^t = \mathbf{I},\\ \mathbf{A}^t\mathbf{C} = \mathbf{C}^t\mathbf{A}, & \mathbf{B}^t\mathbf{D} = \mathbf{D}^t\mathbf{B}, & \mathbf{A}^t\mathbf{D} - \mathbf{C}^t\mathbf{B} = \mathbf{I},\end{matrix} \tag{4}$$

it has only 10 free parameters in the two-dimensional case that we consider here.

Canonical integral transforms are playing a crucial role in optical analogue information processing. They are responsible for the affine transformation in phase space. Moreover, linear combinations of CTs and cascades of CTs with intermediate multiplicative displays, which can be realized by Spatial Light Modulators or holograms, permit to create a variety of other useful transforms such as the (fractional) Hilbert transform, the Wavelet transform, etc.[3] To simplify the application of CTs for optical signal processing system design, it is important to know the properties of the CT, which are summarized below.

The evolution of the angular spectrum – i.e., the Fourier transform of the complex field amplitude – under the CT can be understood from the following scheme

$$\begin{array}{ccccc} f_i(\mathbf{r}_i) & \begin{matrix}\Rightarrow\\ \Leftarrow\end{matrix} & \begin{matrix}\mathbf{T}\\ \mathbf{T}^{-1}\end{matrix} & \begin{matrix}\Rightarrow\\ \Leftarrow\end{matrix} & f_o(\mathbf{r}_o)\\ \Downarrow \mathrm{FT} & & & & \Downarrow \mathrm{FT}\\ F_i(\mathbf{q}_i) & \begin{matrix}\Rightarrow\\ \Leftarrow\end{matrix} & \begin{matrix}(\mathbf{T}^{-1})^t\\ \mathbf{T}^t\end{matrix} & \begin{matrix}\Rightarrow\\ \Leftarrow\end{matrix} & F_o(\mathbf{q}_o)\end{array}, \tag{5}$$

where the Fourier transform $F(\mathbf{q})$ of $f(\mathbf{r})$ is defined as

$$F(\mathbf{q}) = \int f(\mathbf{r})\, \exp(-i2\pi \mathbf{q}^t \mathbf{r})\, \mathrm{d}\mathbf{r}. \tag{6}$$

In general, the CTs produce affine transformations in phase space, which include scaling, rotation, and shearing in the different planes, etc.[4] Thus the separable fractional Fourier transform described by the matrices

$$\mathbf{A} = \mathbf{D} = \begin{bmatrix} \cos\theta_x & 0 \\ 0 & \cos\theta_y \end{bmatrix}, \quad \mathbf{B} = -\mathbf{C} = \begin{bmatrix} \sin\theta_x & 0 \\ 0 & \sin\theta_y \end{bmatrix}, \tag{7}$$

is responsible for rotations in the (x, q_x) and (y, q_y) planes through the angles θ_x and θ_y respectively, while the rotator operation with

$$\mathbf{A} = \mathbf{D} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}, \quad \mathbf{B} = \mathbf{C} = \mathbf{0}, \tag{8}$$

produces a rotation of the image and its Fourier transform (the angular spectrum) through an angle α. A gyrator operation with

$$\mathbf{A} = \mathbf{D} = \begin{bmatrix} \cos\vartheta & 0 \\ 0 & \cos\vartheta \end{bmatrix}, \quad \mathbf{B} = -\mathbf{C} = \begin{bmatrix} 0 & \sin\vartheta \\ \sin\vartheta & 0 \end{bmatrix}, \tag{9}$$

corresponds to twisting – rotations in the (x, q_y) and (y, q_x) planes.

The system with

$$\mathbf{A} = \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ -u & 1 \end{bmatrix}, \quad \mathbf{B} = \mathbf{C} = \mathbf{0}, \tag{10}$$

is related to shearing operations in the (x, y) and (q_x, q_y) planes, while the Fresnel and lens transforms correspond to shearing in the (x, q_x) and (y, q_y) planes.[5]

The application of the CT for optical information processing requires the formulation of theorems for shift, scaling, etc., well known in the case of the Fourier transform. A shift of the input field with respect to the optical axis by a vector $\mathbf{v}$, $f_i(\mathbf{r}_i) \Rightarrow f_i(\mathbf{r}_i - \mathbf{v})$, leads to a shift of the output signal by the vector $\mathbf{A}\mathbf{v}$ and to an additional quadratic phase factor; in detail we have

$$f_o(\mathbf{r}_o) = R^{\mathbf{T}}[f_i(\mathbf{r}_i - \mathbf{v})](\mathbf{r}_o) = \exp[i\pi(2\mathbf{r}_o - \mathbf{A}\mathbf{v})^t \mathbf{C}\mathbf{v}]\, R^{\mathbf{T}}[f_i(\mathbf{r}_i)](\mathbf{r}_o - \mathbf{A}\mathbf{v}), \tag{11}$$

where we have used the symplecticity conditions (4) and the fact that $\mathbf{v}^t\mathbf{Z}\mathbf{q} = \mathbf{q}^t\mathbf{Z}^t\mathbf{v}$. The shift theorem (11) implies that the intensity distribution does not change due to a displacement by $\mathbf{v}$, but is merely shifted by $\mathbf{A}\mathbf{v}$:

$$\left|R^{\mathbf{T}}[f_i(\mathbf{r}_i - \mathbf{v})](\mathbf{r}_o)\right| = \left|R^{\mathbf{T}}[f_i(\mathbf{r}_i)](\mathbf{r}_o - \mathbf{A}\mathbf{v})\right|. \tag{12}$$

In the case $\mathbf{B} = \mathbf{0}$ we have

$$f_o(\mathbf{r}_o) = R^{\mathbf{T}}[f_i(\mathbf{r}_i - \mathbf{v})](\mathbf{r}_o) = \frac{1}{\sqrt{\det \mathbf{A}}}\, \exp\left[i\pi\left(\mathbf{r}_o^t \mathbf{C}\mathbf{A}^{-1}\mathbf{r}_o\right)\right] f_i(\mathbf{A}^{-1}\mathbf{r}_o - \mathbf{v}). \tag{13}$$

Using the shift theorem, the CT of the convolution operation can be written in the form

$$R^{\mathbf{T}}\left[\int f(\mathbf{r}_i - \mathbf{v})\, h(\mathbf{v})\, \mathrm{d}\mathbf{v}\right](\mathbf{r}_o) = \int \exp[i\pi(2\mathbf{r}_o - \mathbf{A}\mathbf{v})^t \mathbf{C}\mathbf{v}]\, F_{\mathbf{T}}(\mathbf{r}_o - \mathbf{A}\mathbf{v})\, h(\mathbf{v})\, \mathrm{d}\mathbf{v}, \tag{14}$$

where $F_{\mathbf{T}}(\mathbf{r}_o) = R^{\mathbf{T}}[f(\mathbf{r}_i)](\mathbf{r}_o)$. In the case $\mathbf{A} = \mathbf{0}$, and therefore $\mathbf{C}^t = -\mathbf{B}^{-1}$, we get

$$R^{\mathbf{T}}\left[\int f(\mathbf{r}_i - \mathbf{v})\, h(\mathbf{v})\, \mathrm{d}\mathbf{v}\right](\mathbf{r}_o) = \sqrt{\det i\mathbf{B}}\, \exp(-i\pi \mathbf{r}_o^t \mathbf{D}\mathbf{B}^{-1}\mathbf{r}_o)\, F_T(\mathbf{r}_o)\, H_T(\mathbf{r}_o), \tag{15}$$

which reduces to the well-known product $\sqrt{\det i\mathbf{B}}\, F_T(\mathbf{r}_o)\, H_T(\mathbf{r}_o)$ when also $\mathbf{D} = \mathbf{0}$, i.e., in the case of a Fourier transformation with scaling and rotation (determined by $\mathbf{B}$).

The $\mathbf{T}$-parameterized CT of the scaled function $\sqrt{\det \mathbf{W}}\, f_i(\mathbf{W}\mathbf{r}_i)$, $R^{\mathbf{T}}[\sqrt{\det \mathbf{W}}\, f_i(\mathbf{W}\mathbf{r}_i)](\mathbf{r}_o)$, leads to the CT of $f_i(\mathbf{r}_i)$, parameterized by the matrix

$$\mathbf{T}\begin{bmatrix} \mathbf{W}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}^t \end{bmatrix}. \tag{16}$$

The gradient of the CT can be written as

$$\begin{aligned} \nabla f_o(\mathbf{r}_o) &\equiv \begin{bmatrix} \partial/\partial x_o \\ \partial/\partial y_o \end{bmatrix} f_o(\mathbf{r}_o) = f_o(\mathbf{r}_o)(i\pi\, \nabla \mathbf{r}_o^t \mathbf{D}\mathbf{B}^{-1}\mathbf{r}_o) + R^{\mathbf{T}}\left[f_i(\mathbf{r}_i)(-i2\pi\, \nabla \mathbf{r}_i^t \mathbf{B}^{-1}\mathbf{r}_o)\right](\mathbf{r}_o) \\ &= i\pi f_o(\mathbf{r}_o)\left[\mathbf{D}\mathbf{B}^{-1} + (\mathbf{D}\mathbf{B}^{-1})^t\right]\mathbf{r}_o - i2\pi(\mathbf{B}^{-1})^t R^{\mathbf{T}}\left[f_i(\mathbf{r}_i)\,\mathbf{r}_i\right](\mathbf{r}_o) \\ &= i2\pi(\mathbf{B}^{-1})^t\left\{f_o(\mathbf{r}_o)\,\mathbf{D}^t\mathbf{r}_o - R^{\mathbf{T}}\left[f_i(\mathbf{r}_i)\,\mathbf{r}_i\right](\mathbf{r}_o)\right\}, \end{aligned} \tag{17}$$

where we have used the symplecticity condition $\mathbf{B}^t\mathbf{D} = \mathbf{D}^t\mathbf{B}$.

For a limited number of functions, an analytical expression of the CT can be found. Among them we mention the signal $f_i(\mathbf{r}_i) = \exp\left(2\sqrt{2\pi}\mathbf{s}^t\mathbf{K}_i\mathbf{r}_i - \pi\mathbf{r}_i^t\mathbf{L}_i\mathbf{r}_i\right)$, which reduces to a Gaussian beam for $\mathbf{K}_i = \mathbf{0}$, and to a plane wave for $\mathbf{L}_i = \mathbf{0}$ and $\mathbf{K}_i$ imaginary. The CT of this signal is given by[6]

$$f_o(\mathbf{r}_o) = R^T[f_i(\mathbf{r}_i)](\mathbf{r}_o) = (\det(\mathbf{A} + i\mathbf{B}\mathbf{L}_i))^{-1/2}\exp\left(-\mathbf{s}^t\mathbf{M}_o\mathbf{s} + 2\sqrt{2\pi}\mathbf{s}^t\mathbf{K}_o\mathbf{r}_o - \pi\mathbf{r}_o^t\mathbf{L}_o\mathbf{r}_o\right), \tag{18}$$

where

$$\mathbf{K}_o = \mathbf{K}_i\left(\mathbf{A} + i\mathbf{B}\mathbf{L}_i\right)^{-1}, \tag{19}$$

$$i\mathbf{L}_o = \left(\mathbf{C} + i\mathbf{D}\mathbf{L}_i\right)\left(\mathbf{A} + i\mathbf{B}\mathbf{L}_i\right)^{-1}, \tag{20}$$

$$\mathbf{M}_o = -2i\mathbf{K}_i\left(\mathbf{A} + i\mathbf{B}\mathbf{L}_i\right)^{-1}\mathbf{B}\mathbf{K}_i^t. \tag{21}$$

In particular for $\mathbf{K}_i = 0$ we have

$$f_o(\mathbf{r}_o) = R^T[\exp\left(-\pi\mathbf{r}_i^t\mathbf{L}_i\mathbf{r}_i\right)](\mathbf{r}_o) = (\det(\mathbf{A} + i\mathbf{B}\mathbf{L}_i))^{-1/2}\exp\left(-\pi\mathbf{r}_o^t\mathbf{L}_o\mathbf{r}_o\right), \tag{22}$$

and for $\mathbf{L}_i = 0$

$$\begin{aligned} f_o(\mathbf{r}_o) &= R^T[\exp\left(2\sqrt{2\pi}\mathbf{s}^t\mathbf{K}_i\mathbf{r}_i\right)](\mathbf{r}_o) \\ &= (\det\mathbf{A})^{-1/2}\exp\left(2i\mathbf{s}^t\mathbf{K}_i\mathbf{A}^{-1}\mathbf{B}\mathbf{K}_i^t\mathbf{s} + 2\sqrt{2\pi}\mathbf{s}^t\mathbf{K}_i\mathbf{A}^{-1}\mathbf{r}_o + i\pi\mathbf{r}_o^t\mathbf{C}\mathbf{A}^{-1}\mathbf{r}_o\right). \end{aligned} \tag{23}$$

The discussed properties of the canonical integral transform are useful for its application in optical information processing.

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1. S. A. Collins Jr., "Lens-system diffraction integral written in terms of matrix optics," J. Opt. Soc. Am. **60**, 1168–1177 (1970).
2. T. Alieva and M. J. Bastiaans, "Alternative representation of the linear canonical integral transform," Opt. Lett. **30**, 3302–3304 (2005).
3. T. Alieva, M. J. Bastiaans, and M. L. Calvo, "Fractional transforms in optical information processing," EURASIP J. Appl. Signal Process. **2005**, 1498–1519 (2005).
4. K. B. Wolf, *Geometric Optics on Phase Space* (Springer, Berlin, 2004).
5. A. W. Lohmann, "Image rotation, Wigner rotation, and the fractional Fourier transform," J. Opt. Soc. Am. A **10**, 2181–2186 (1993).
6. M. J. Bastiaans and T. Alieva, "Propagation law for the generating function of Hermite-Gaussian-type modes in first-order optical systems," Optics Express **13**, 1107–1112 (2005).