Applications of the Wigner distribution function to partially coherent light beams

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The paper presents a review of the Wigner distribution function – its definition and its most important properties – and of some of its applications to optical problems, in particular to the description of partially coherent light beams and to the propagation of such light beams through first-order optical ABCD*-systems. Special attention is paid to Gaussian beams and to second-order moments of general, partially coherent light.*

1 INTRODUCTION

In 1932 Wigner introduced a distribution function in mechanics [1] that permitted a description of mechanical phenomena in a phase space. Such a Wigner distribution function was introduced in optics by Walther [2] in 1968, to relate partial coherence to radiometry. A few years later, the Wigner distribution function was introduced in optics again [3] (especially in the area of Fourier optics), and, since then, a great number of applications of the Wigner distribution function have been reported. It is the aim of this paper to review the Wigner distribution function and some of its applications to optical problems, especially in the field of partial coherence and first-order optical *ABCD*-systems. The paper is roughly an extension to two dimensions of a previous review paper [4] on the application of the Wigner distribution function to partially coherent light, with additional material taken from some recent papers on second-order moments of the Wigner distribution function [5, 6, 7] and with some new ideas about the twist of partially coherent Gaussian light beams [8]. Some parts of this paper have already been presented in [9] and [10].

In Section 2 we describe how we represent partially coherent light. We introduce its positional power spectrum (or cross-spectral density function) and the spatial Fourier transform of that function: the directional power spectrum.

The Wigner distribution function for partially coherent light is defined in Section 3, where some of its most important properties are given and where its concept is elucidated with some simple examples. Although derived in terms of Fourier optics, we will see that the description of an optical signal by means of its Wigner distribution function closely resembles the ray concept in geometrical optics and that the properties of the Wigner distribution function have clear physical meanings.

In Section 4 we introduce a modal expansion for the cross-spectral density function of partially coherent light and derive a similar expansion for the Wigner distribution function. This modal expansion allows us to formulate more properties of the Wigner distribution function, especially in the form of inequalities.

The transformation of the Wigner distribution function in the case that the light propagates through a linear system, is described in Section 5. An optical system is treated there in two distinct forms: (1) as a black box with an input plane and an output plane, for which an input-output relationship in terms of the Wigner distribution functions is formulated, and (2) as a continuous medium, for which a transport equation for the Wigner distribution function is derived. We observe again that both the input-output relationship and the transport equation can be given a geometric-optical interpretation.

Finally, in Section 6, we study some topics related to second-order moments of the Wigner distribution function of partially coherent light. Special attention is paid to general, partially coherent Gaussian light beams, with and without twist.

We conclude this introduction with some remarks about the signals with which we are dealing and with some remarks about notation conventions. We consider scalar optical signals, which can be described by, say, $\tilde{\varphi}(x, y, z, t)$, where x, y, z denote space variables and t represents the time variable. Very often we consider signals in a plane $z =$ constant, in which case we can omit the longitudinal space variable z from the formulas. Furthermore, the transverse space variables x and y are combined into a 2-dimensional column vector $\boldsymbol{r}$. The signals with which we are dealing are thus described by a function $\tilde{\varphi}(\boldsymbol{r}, t)$.

We will throughout denote column vectors by bold-face, lower-case symbols, while matrices will be denoted by bold-face, upper-case symbols; transposition of vectors and matrices is denoted by the superscript ^{t}. Hence, for instance, the 2-dimensional column vectors $\boldsymbol{r}$ and $\boldsymbol{q}$ represent the space and spatial frequency variables $[x, y]^t$ and $[u, v]^t$, respectively, and $\boldsymbol{q}^t\boldsymbol{r}$ represents the inner product $ux + vy$. Moreover, in integral expressions, $d\boldsymbol{r}$ and $d\boldsymbol{q}$ are shorthand notations for $dxdy$ and $dudv$, respectively. The expression $\partial/\partial\boldsymbol{r}$ represents the vectorial operator $[\partial/\partial x, \partial/\partial y]^t$, whereas the expression $\partial^2/\partial\boldsymbol{r}^2 = (\partial/\partial\boldsymbol{r})^t(\partial/\partial\boldsymbol{r})$ is a shorthand notation for the scalar operator $\partial^2/\partial x^2 + \partial^2/\partial y^2$; and we have similar expressions for the frequency variable $\boldsymbol{q}$.

2 DESCRIPTION OF PARTIALLY COHERENT LIGHT

Let partially coherent light be described by a temporally stationary stochastic process $\tilde{\varphi}(\boldsymbol{r}, t)$; as far as the time dependence is concerned, the ensemble average of the product $\tilde{\varphi}(\boldsymbol{r}_1, t_1)\tilde{\varphi}^*(\boldsymbol{r}_2, t_2)$, where the asterisk denotes complex conjugation, is then only a function of the time difference $t_1 - t_2$:

$$\mathrm{E}\, \tilde{\varphi}(\boldsymbol{r}_1, t_1)\tilde{\varphi}^*(\boldsymbol{r}_2, t_2) = \tilde{\Gamma}(\boldsymbol{r}_1, \boldsymbol{r}_2, t_1 - t_2). \tag{2.1}$$

The function $\tilde{\Gamma}(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau)$ is known as the (mutual) *coherence function* [11, 12, 13, 14] of the stochastic process $\tilde{\varphi}(\boldsymbol{r}, t)$. The (mutual) *power spectrum* [13, 14] or *cross-spectral density function* [15] $\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega)$ is defined as the temporal Fourier transform of the coherence function

$$\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega) = \int \tilde{\Gamma}(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau) e^{i\omega\tau} d\tau. \tag{2.2}$$

(Unless otherwise stated, all integrations in this paper extend from $-\infty$ to $+\infty$.) The basic property [14, 15] of the power spectrum is that it is a *nonnegative definite Hermitian* function of $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, i.e.,

$$\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega) = \Gamma^*(\boldsymbol{r}_2, \boldsymbol{r}_1, \omega) \tag{2.3}$$

and

$$\iint g(\boldsymbol{r}_1, \omega)\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega) g^*(\boldsymbol{r}_2, \omega) d\boldsymbol{r}_1 d\boldsymbol{r}_2 \geq 0 \tag{2.4}$$

for any function $g(\boldsymbol{r}, \omega)$.

Instead of describing a stochastic process in a space domain by means of its power spectrum or cross-spectral density function $\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega)$, we can represent it equally well in a spatial-frequency domain by means of the spatial Fourier transform $\bar{\Gamma}(\boldsymbol{q}_1, \boldsymbol{q}_2, \omega)$ of the power spectrum

$$\bar{\Gamma}(\boldsymbol{q}_1, \boldsymbol{q}_2, \omega) = \iint \Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega) e^{-i(\boldsymbol{q}_1^t \boldsymbol{r}_1 - \boldsymbol{q}_2^t \boldsymbol{r}_2)} d\boldsymbol{r}_1 d\boldsymbol{r}_2. \tag{2.5}$$

(Throughout we represent the spatial Fourier transform of a function by the same symbol as the function itself, but marked with a bar on top of the symbol.) Unlike the power spectrum or cross-spectral density function $\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega)$, which expresses the coherence of the light at two different positions, its spatial Fourier transform $\bar{\Gamma}(\boldsymbol{q}_1, \boldsymbol{q}_2, \omega)$ expresses the coherence of the light in two different directions. Therefore we prefer to call $\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega)$ the *positional* power spectrum [4, 16] and $\bar{\Gamma}(\boldsymbol{q}_1, \boldsymbol{q}_2, \omega)$ the *directional* power spectrum [4, 16] of the light. It is evident that the directional power spectrum $\bar{\Gamma}(\boldsymbol{q}_1, \boldsymbol{q}_2, \omega)$ is a nonnegative definite Hermitian function of the two spatial-frequency (or direction) variables $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$.

Apart from the pure space representation of a stochastic process by means of its positional power spectrum or the pure spatial-frequency representation by means of its directional power spectrum, we can describe a stochastic process in space and spatial frequency *simultaneously*. In this paper we therefore use the Wigner distribution function, which is introduced in the next section. Since in the present discussion the explicit temporal-frequency dependence is of no importance, we shall, for the sake of convenience, omit the temporal-frequency variable ω from the formulas in the remainder of the paper.

3 WIGNER DISTRIBUTION FUNCTION

It is sometimes convenient to describe an optical signal not in a space domain by means of its positional power spectrum, but in a spatial-frequency domain by means of its directional power spectrum. The directional power spectrum globally shows how the energy of the signal is distributed as a function of direction (i.e., spatial frequency). However, instead of in this *global* distribution of the energy, one is often more interested in the *local* distribution of the energy as a function of spatial frequency. A similar local distribution occurs in music, for instance, in which a signal is usually described not by a time function nor by the Fourier transform of that function, but by its *musical score*.

The score is indeed a picture of the local distribution of the energy of the musical signal as a function of frequency. The horizontal axis of the score clearly represents a time axis, and the vertical one a frequency axis. When a composer writes a score, he prescribes the frequencies of the tones that should be present at a certain time. We see that the musical score is something that might be called the *local frequency spectrum* of the musical signal.

The need for a description of the signal by means of a local frequency spectrum arises in other disciplines too. Geometrical optics, for instance, is usually treated in terms of rays, and the signal is described by giving the directions of the rays that should be present at a certain position. It is not difficult to translate the concept of the musical score to geometrical optics: we simply have to consider the horizontal (time) axis as a position axis and the vertical (frequency) axis as a direction axis. A musical note then represents an optical light ray passing through a point at a certain position and having a certain direction.

Another discipline in which we can apply the idea of a local frequency spectrum is in mechanics: the position and the momentum of a particle are given in a *phase space*. It was in mechanics that Wigner introduced in 1932 a distribution function [1] that provided a description of mechanical phenomena in the phase space.

In this section we define the Wigner distribution function in optics, we given some of its properties, and we elucidate its concept by some simple examples.

3.1 Definition of the Wigner distribution function

The Wigner distribution function of a stochastic process can be defined in terms of the positional power spectrum by

$$F(\boldsymbol{r}, \boldsymbol{q}) = \int \Gamma(\boldsymbol{r} + \tfrac{1}{2}\boldsymbol{r}', \boldsymbol{r} - \tfrac{1}{2}\boldsymbol{r}') e^{-i\boldsymbol{q}^t \boldsymbol{r}'} d\boldsymbol{r}' \tag{3.1}$$

or, equivalently, in terms of the directional power spectrum by

$$F(\boldsymbol{r}, \boldsymbol{q}) = \int \bar{\Gamma}(\boldsymbol{q} + \tfrac{1}{2}\boldsymbol{q}', \boldsymbol{q} - \tfrac{1}{2}\boldsymbol{q}') e^{i\boldsymbol{r}^t \boldsymbol{q}'} d\frac{\boldsymbol{q}'}{2\pi}. \tag{3.2}$$

A distribution function according to definitions (3.1) and (3.2) was first introduced in optics by Walther [2, 17], who called it the *generalized radiance*.

The Wigner distribution function $F(\boldsymbol{r}, \boldsymbol{q})$ represents a stochastic signal in space and (spatial) frequency simultaneously and is thus a member of a wide class of phase-space distribution functions [18, 19, 20]. It forms an intermediate signal description between the pure space representation $\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2)$ and the pure frequency representation $\bar{\Gamma}(\boldsymbol{q}_1, \boldsymbol{q}_2)$. Furthermore, this simultaneous space-frequency description closely resembles the *ray concept* in geometrical optics, in which the position and direction of a ray are also given simultaneously. In a way, $F(\boldsymbol{r}, \boldsymbol{q})$ is the amplitude of a ray passing through the point $\boldsymbol{r}$ and having a frequency (i.e., direction) $\boldsymbol{q}$.

3.2 Some properties of the Wigner distribution function

Let us consider some properties of the Wigner distribution function. We consider only the most important ones; they can all be derived directly from the definitions (3.1) and (3.2). Additional properties of the Wigner distribution function, especially of the Wigner distribution function in the completely coherent case as defined by Eq. (3.24), can be found elsewhere; see, for instance, [21] and the many references cited there.

3.2.1 Fourier transformation. The definition (3.1) of the Wigner distribution function $F(\boldsymbol{r}, \boldsymbol{q})$ has the form of a *Fourier transformation* of the positional power spectrum $\Gamma(\boldsymbol{r} + \frac{1}{2}\boldsymbol{r}', \boldsymbol{r} - \frac{1}{2}\boldsymbol{r}')$ with $\boldsymbol{r}'$ and $\boldsymbol{q}$ as conjugated variables and with $\boldsymbol{r}$ as a parameter. The positional power spectrum can thus be reconstructed from the Wigner distribution function simply by applying an inverse Fourier transformation; a similar property holds for the directional power spectrum. The latter property follows from the general remark that space and frequency, or position and direction, play equivalent roles in the Wigner distribution function: if we interchange the roles of $\boldsymbol{r}$ and $\boldsymbol{q}$ in any expression containing a Wigner distribution function, we get an expression that is the dual of the original one. Thus, when the original expression describes a property in the space domain, the dual expression describes a similar property in the frequency domain, and vice versa.

3.2.2 Realness. The Wigner distribution function is *real*. Unfortunately, it is not necessarily nonnegative; this prohibits a direct interpretation of the Wigner distribution function as an energy density function (or radiance function). Friberg has shown [22] that it is not possible to define a radiance function that satisfies all the physical requirements from radiometry; in particular, as we see, the Wigner distribution function has the physically unattractive property that it may take negative values.

3.2.3 Space and frequency limitation. If the signal is *limited* to a certain space or frequency interval and vanishes outside that interval, then its Wigner distribution function is limited to the same interval. Hence, for a light source with a finite extent, the Wigner distribution function vanishes outside the source, which is surely a physically attractive property.

3.2.4 Space and frequency shift. A space or frequency *shift* of the signal yields the same shift for its Wigner distribution function. Indeed, if the Wigner distribution function $F(\boldsymbol{r}, \boldsymbol{q})$ corresponds to the positional power spectrum $\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2)$, then $F(\boldsymbol{r} - \boldsymbol{r}_o, \boldsymbol{q})$ corresponds to $\Gamma(\boldsymbol{r}_1 - \boldsymbol{r}_o, \boldsymbol{r}_2 - \boldsymbol{r}_o)$; a similar property holds for the directional power spectrum.

3.2.5 Radiometric quantities. Several integrals of the Wigner distribution function have clear physical meanings and can be interpreted as radiometric quantities. The integral over the frequency variable, for instance,

$$\int F(\boldsymbol{r}, \boldsymbol{q}) d\frac{\boldsymbol{q}}{2\pi} = \Gamma(\boldsymbol{r}, \boldsymbol{r}) \tag{3.3}$$

represents the *positional intensity* of the signal, whereas the integral over the space variable

$$\int F(\boldsymbol{r}, \boldsymbol{q}) d\boldsymbol{r} = \bar{\Gamma}(\boldsymbol{q}, \boldsymbol{q}) \tag{3.4}$$

yields the *directional intensity* of the signal, which is, apart from the usual factor $\cos^2\theta$ (where θ is the angle of observation with respect to the z-axis), proportional to the *radiant intensity* [23, 24]. The *total energy* of the signal follows from the integral over the entire space-frequency domain

$$\iint F(\boldsymbol{r}, \boldsymbol{q}) d\boldsymbol{r} d\frac{\boldsymbol{q}}{2\pi} = \int \Gamma(\boldsymbol{r}, \boldsymbol{r}) d\boldsymbol{r} = \int \bar{\Gamma}(\boldsymbol{q}, \boldsymbol{q}) d\frac{\boldsymbol{q}}{2\pi}. \tag{3.5}$$

The real symmetric 4×4 matrix $\boldsymbol{M}$ of normalized second-order moments, defined by

$$\boldsymbol{M} = \frac{\displaystyle\iint \begin{bmatrix} \boldsymbol{r}\boldsymbol{r}^t & \boldsymbol{r}\boldsymbol{q}^t \\ \boldsymbol{q}\boldsymbol{r}^t & \boldsymbol{q}\boldsymbol{q}^t \end{bmatrix} F(\boldsymbol{r}, \boldsymbol{q}) d\boldsymbol{r} d\frac{\boldsymbol{q}}{2\pi}}{\displaystyle\iint F(\boldsymbol{r}, \boldsymbol{q}) d\boldsymbol{r} d\frac{\boldsymbol{q}}{2\pi}}, \tag{3.6}$$

yields such quantities as the *effective width* d_x of the positional intensity $\Gamma(\boldsymbol{r}, \boldsymbol{r})$ in the x-direction

$$m_{xx} = \frac{\displaystyle\iint x^2 F(\boldsymbol{r}, \boldsymbol{q}) d\boldsymbol{r} d\frac{\boldsymbol{q}}{2\pi}}{\displaystyle\iint F(\boldsymbol{r}, \boldsymbol{q}) d\boldsymbol{r} d\frac{\boldsymbol{q}}{2\pi}} = \frac{\displaystyle\int x^2 \Gamma(\boldsymbol{r}, \boldsymbol{r}) d\boldsymbol{r}}{\displaystyle\int \Gamma(\boldsymbol{r}, \boldsymbol{r}) d\boldsymbol{r}} = d_x^2 \tag{3.7}$$

and the *effective width* d_u of the directional intensity $\bar{\Gamma}(\boldsymbol{q}, \boldsymbol{q})$ in the u-direction

$$m_{uu} = \frac{\displaystyle\iint u^2 F(\boldsymbol{r}, \boldsymbol{q}) d\boldsymbol{r} d\frac{\boldsymbol{q}}{2\pi}}{\displaystyle\iint F(\boldsymbol{r}, \boldsymbol{q}) d\boldsymbol{r} d\frac{\boldsymbol{q}}{2\pi}} = \frac{\displaystyle\int u^2 \bar{\Gamma}(\boldsymbol{q}, \boldsymbol{q}) d\frac{\boldsymbol{q}}{2\pi}}{\displaystyle\int \bar{\Gamma}(\boldsymbol{q}, \boldsymbol{q}) d\frac{\boldsymbol{q}}{2\pi}} = d_u^2. \tag{3.8}$$

It will be clear that the main-diagonal entries of the moment matrix $\boldsymbol{M}$, being interpretable as squares of effective widths, are positive. As a matter of fact. it can be shown that the moment matrix $\boldsymbol{M}$ is positive definite; see, for instance, [8, 9, 10].

The *radiant emittance* [23, 24] is equal to the integral

$$j_z(\boldsymbol{r}) = \int \frac{\sqrt{k^2 - \boldsymbol{q}^t\boldsymbol{q}}}{k} F(\boldsymbol{r}, \boldsymbol{q}) d\frac{\boldsymbol{q}}{2\pi}, \tag{3.9}$$

where $k = 2\pi/\lambda$ represents the usual wave number again. When we combine the radiant emittance j_z with the integral

$$\boldsymbol{j}_r(\boldsymbol{r}) = \int \frac{\boldsymbol{q}}{k} F(\boldsymbol{r}, \boldsymbol{q}) d\frac{\boldsymbol{q}}{2\pi}, \tag{3.10}$$

we can construct the 3-dimensional column vector

$$\boldsymbol{j} = [\boldsymbol{j}_r^t, j_z]^t, \tag{3.11}$$

which is known as the *geometrical vector flux* [25]. The *total radiant flux* [23] follows from integrating the radiant emittance over the space variable $\boldsymbol{r}$:

$$\int j_z(\boldsymbol{r}) d\boldsymbol{r} = \iint \frac{\sqrt{k^2 - \boldsymbol{q}^t\boldsymbol{q}}}{k} F(\boldsymbol{r}, \boldsymbol{q}) d\boldsymbol{r} d\frac{\boldsymbol{q}}{2\pi}. \tag{3.12}$$

3.2.6 Moyal's relationship. An important relationship between the Wigner distribution functions of two signals and the power spectra of these signals, which is an extension to partially coherent light of a relationship formulated by Moyal [26] for completely coherent light, reads as

$$\begin{aligned} & \iint F_1(\boldsymbol{r}, \boldsymbol{q}) F_2(\boldsymbol{r}, \boldsymbol{q}) d\boldsymbol{r} d\frac{\boldsymbol{q}}{2\pi} \\ = & \iint \Gamma_1(\boldsymbol{r}_1, \boldsymbol{r}_2) \Gamma_2^*(\boldsymbol{r}_1, \boldsymbol{r}_2) d\boldsymbol{r}_1 d\boldsymbol{r}_2 \\ = & \iint \bar{\Gamma}_1(\boldsymbol{q}_1, \boldsymbol{q}_2) \bar{\Gamma}_2^*(\boldsymbol{q}_1, \boldsymbol{q}_2) d\frac{\boldsymbol{q}_1}{2\pi} d\frac{\boldsymbol{q}_2}{2\pi}. \end{aligned} \tag{3.13}$$

This relationship has an application in averaging one Wigner distribution function with another one, which averaging always yields a nonnegative result. We show this in Section 4, after having introduced modal expansions for the power spectrum and the Wigner distribution function.

3.3 Examples of Wigner distribution functions

We illustrate the Wigner distribution function with some simple examples.

3.3.1 Incoherent light. Spatially *incoherent* light can be described by its positional power spectrum, which reads as

$$\Gamma(\boldsymbol{r} + \tfrac{1}{2}\boldsymbol{r}', \boldsymbol{r} - \tfrac{1}{2}\boldsymbol{r}') = p(\boldsymbol{r}) \delta(\boldsymbol{r}'), \tag{3.14}$$

where the intensity $p(\boldsymbol{r})$ is a nonnegative function. The corresponding Wigner distribution function takes the form

$$F(\boldsymbol{r}, \boldsymbol{q}) = p(\boldsymbol{r}); \tag{3.15}$$

note that it is a function of the space variable $\boldsymbol{r}$, only, and that it does not depend on the frequency variable $\boldsymbol{q}$.

3.3.2 Spatially stationary light. As a second example, we consider light that is *dual* to incoherent light, i.e., light whose frequency behaviour is similar to the space behaviour of incoherent light and vice versa. Such light is *spatially stationary* light. The positional power spectrum of spatially stationary light reads as

$$\Gamma(\boldsymbol{r} + \tfrac{1}{2}\boldsymbol{r}', \boldsymbol{r} - \tfrac{1}{2}\boldsymbol{r}') = s(\boldsymbol{r}'); \tag{3.16}$$

its directional power spectrum thus reads as

$$\bar{\Gamma}(\boldsymbol{q} + \tfrac{1}{2}\boldsymbol{q}', \boldsymbol{q} - \tfrac{1}{2}\boldsymbol{q}') = \bar{s}(\boldsymbol{q}) \delta\left(\frac{\boldsymbol{q}'}{2\pi}\right), \tag{3.17}$$

where the nonnegative function $\bar{s}(\boldsymbol{q})$ is the Fourier transform of $s(\boldsymbol{r}')$:

$$\bar{s}(\boldsymbol{q}) = \int s(\boldsymbol{r}') e^{-i\boldsymbol{q}^t\boldsymbol{r}'} d\boldsymbol{r}'. \tag{3.18}$$

Note that, indeed, the directional power spectrum (3.17) of spatially stationary light has a form that is similar to the positional power spectrum (3.14) of incoherent light. The duality between incoherent light and spatially stationary light is, in fact, the Van Cittert-Zernike theorem.

The Wigner distribution function of spatially stationary light reads as

$$F(\boldsymbol{r}, \boldsymbol{q}) = \bar{s}(\boldsymbol{q}); \tag{3.19}$$

note that it is a function of the frequency variable $\boldsymbol{q}$, only, and that it does not depend on the space variable $\boldsymbol{r}$. It thus has the same form as the Wigner distribution function (3.15) of incoherent light, except that it is rotated through 90° in the space-frequency domain.

3.3.3 Quasi-homogeneous light. Incoherent light and spatially stationary light are special cases of so-called *quasi-homogeneous* light [16, 23, 24]. Such quasi-homogeneous light can be locally considered as spatially stationary, having, however, a slowly varying intensity. It can be represented by a positional power spectrum such as

$$\Gamma(\boldsymbol{r} + \tfrac{1}{2}\boldsymbol{r}', \boldsymbol{r} - \tfrac{1}{2}\boldsymbol{r}') \simeq p(\boldsymbol{r}) s(\boldsymbol{r}'), \tag{3.20}$$

where p is a 'slowly' varying function compared with s. The Wigner distribution function of quasi-homogeneous light takes the form of a product:

$$F(\boldsymbol{r}, \boldsymbol{q}) \simeq p(\boldsymbol{r}) \bar{s}(\boldsymbol{q}); \tag{3.21}$$

both $p(\boldsymbol{r})$ and $\bar{s}(\boldsymbol{q})$ are nonnegative, which implies that the Wigner distribution function is nonnegative. The special case of incoherent light arises for $\bar{s}(\boldsymbol{q}) = 1$, whereas for spatially stationary light we have $p(\boldsymbol{r}) = 1$.

A *Lambertian* source [24], extending within a certain aperture along the $\boldsymbol{r}$-axis and radiating within a certain aperture along the $\boldsymbol{q}$-axis, is a well-known example of quasi-homogeneous light. For such a source we have

$$\begin{aligned} p(\boldsymbol{r}) &= \text{aperture}(\boldsymbol{r}) \\ \bar{s}(\boldsymbol{q}) &= \frac{k}{\sqrt{k^2 - \boldsymbol{q}^t\boldsymbol{q}}} \text{aperture}(\boldsymbol{q}), \end{aligned} \tag{3.22}$$

where $k = 2\pi/\lambda$ is the usual wave number, and where $\sqrt{k^2 - \boldsymbol{q}^t\boldsymbol{q}}/k = \cos\theta$ with θ the angle of observation with respect to the z-axis, again.

3.3.4 Coherent light. Completely *coherent* light is our next example. Its positional power spectrum

$$\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2) = \varphi(\boldsymbol{r}_1) \varphi^*(\boldsymbol{r}_2) \tag{3.23}$$

has the form of a product of a function with its complex-conjugate version [14], and the same holds, of course, for the directional power spectrum. The Wigner distribution function of coherent light thus takes the form

$$f(\boldsymbol{r}, \boldsymbol{q}) = \int \varphi(\boldsymbol{r} + \tfrac{1}{2}\boldsymbol{r}') \varphi^*(\boldsymbol{r} - \tfrac{1}{2}\boldsymbol{r}') e^{-i\boldsymbol{q}^t\boldsymbol{r}'} d\boldsymbol{r}'. \tag{3.24}$$

We denote the Wigner distribution function of coherent light throughout by the lower-case character f.

Let us consider one special example of coherent light, viz., a *quadratic-phase* signal

$$\varphi(\boldsymbol{r}) = e^{\frac{1}{2}i\boldsymbol{r}^t\boldsymbol{H}\boldsymbol{r}}, \tag{3.25}$$

which represents, at least for small $\boldsymbol{r}$, i.e., in the paraxial approximation, a *spherical wave* whose *curvature* is described by the real symmetric 2×2 matrix $\boldsymbol{H} = \boldsymbol{H}^t$. The Wigner distribution function of such a signal takes the simple form

$$f(\boldsymbol{r},\boldsymbol{q}) = \delta\left(\frac{\boldsymbol{q}-\boldsymbol{H}\boldsymbol{r}}{2\pi}\right), \tag{3.26}$$

and we conclude that at any point $\boldsymbol{r}$ only one frequency $\boldsymbol{q} = \boldsymbol{H}\boldsymbol{r}$ manifests itself. This corresponds exactly to the ray picture of a spherical wave.

3.3.5 Gaussian light. *Gaussian* light is our final example. The positional power spectrum of the most general partially coherent Gaussian light can be written in the form

$$\Gamma(\boldsymbol{r}_1,\boldsymbol{r}_2) = \frac{1}{\pi}\sqrt{\det\boldsymbol{G}_1}$$
$$\times \exp\left(-\tfrac{1}{4}\begin{bmatrix}\boldsymbol{r}_1+\boldsymbol{r}_2\\ \boldsymbol{r}_1-\boldsymbol{r}_2\end{bmatrix}^t\begin{bmatrix}\boldsymbol{G}_1 & -i\boldsymbol{H}\\ -i\boldsymbol{H}^t & \boldsymbol{G}_2\end{bmatrix}\begin{bmatrix}\boldsymbol{r}_1+\boldsymbol{r}_2\\ \boldsymbol{r}_1-\boldsymbol{r}_2\end{bmatrix}\right), \tag{3.27}$$

where we have chosen a representation that enables us to determine the Wigner distribution function of such light in an easy way. The exponent shows a quadratic form in which a 4-dimensional column vector $[(\boldsymbol{r}_1+\boldsymbol{r}_2)^t, (\boldsymbol{r}_1-\boldsymbol{r}_2)^t]^t$ arises, together with a symmetric 4×4 matrix. This matrix consists of four *real* 2×2 submatrices $\boldsymbol{G}_1$, $\boldsymbol{G}_2$, $\boldsymbol{H}$, and $\boldsymbol{H}^t$, where, moreover, the matrices $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ are *positive definite symmetric*. The special form of the matrix is a direct consequence of the fact that the power spectrum is a nonnegative definite *Hermitian* function. The Wigner distribution function of such Gaussian light takes the form [7, 27]

$$F(\boldsymbol{r},\boldsymbol{q}) = 4\sqrt{\frac{\det\boldsymbol{G}_1}{\det\boldsymbol{G}_2}}$$
$$\times \exp\left(-\begin{bmatrix}\boldsymbol{r}\\ \boldsymbol{q}\end{bmatrix}^t\begin{bmatrix}\boldsymbol{G}_1+\boldsymbol{H}\boldsymbol{G}_2^{-1}\boldsymbol{H}^t & -\boldsymbol{H}\boldsymbol{G}_2^{-1}\\ -\boldsymbol{G}_2^{-1}\boldsymbol{H}^t & \boldsymbol{G}_2^{-1}\end{bmatrix}\begin{bmatrix}\boldsymbol{r}\\ \boldsymbol{q}\end{bmatrix}\right). \tag{3.28}$$

In a more common way, the positional power spectrum of Gaussian light can be expressed in the form

$$\begin{aligned}\Gamma(\boldsymbol{r}_1,\boldsymbol{r}_2) \;=\; & \frac{1}{\pi}\sqrt{\det\boldsymbol{G}_1}\\ & \times\exp\{-\tfrac{1}{4}(\boldsymbol{r}_1-\boldsymbol{r}_2)^t\boldsymbol{G}_0(\boldsymbol{r}_1-\boldsymbol{r}_2)\}\\ & \times\exp\{-\tfrac{1}{2}\boldsymbol{r}_1^t[\boldsymbol{G}_1-i\tfrac{1}{2}(\boldsymbol{H}+\boldsymbol{H}^t)]\boldsymbol{r}_1\}\\ & \times\exp\{-\tfrac{1}{2}\boldsymbol{r}_2^t[\boldsymbol{G}_1+i\tfrac{1}{2}(\boldsymbol{H}+\boldsymbol{H}^t)]\boldsymbol{r}_2\}\\ & \times\exp\{-\tfrac{1}{2}\boldsymbol{r}_1^t i(\boldsymbol{H}-\boldsymbol{H}^t)\boldsymbol{r}_2\},\end{aligned} \tag{3.29}$$

where we have introduced the real, positive definite symmetric 2×2 matrix $\boldsymbol{G}_0 = \boldsymbol{G}_2 - \boldsymbol{G}_1$. Note that the asymmetry of the matrix $\boldsymbol{H}$ is a measure for the *twist* [28, 29, 30, 31, 32] of Gaussian light, and that general Gaussian light reduces to zero-twist *Gaussian Schell-model* light [33, 34], if the matrix $\boldsymbol{H}$ is symmetric, $\boldsymbol{H}-\boldsymbol{H}^t=\boldsymbol{0}$. In that case the light can be considered as spatially stationary light with a Gaussian cross-spectral density $(1/\pi)\sqrt{\det\boldsymbol{G}_1}\exp\{-\frac{1}{4}(\boldsymbol{r}_1-\boldsymbol{r}_2)^t\boldsymbol{G}_0(\boldsymbol{r}_1-\boldsymbol{r}_2)\}$, modulated by a Gaussian modulator with modulation function $\exp\{-\frac{1}{2}\boldsymbol{r}^t(\boldsymbol{G}_1-i\boldsymbol{H})\boldsymbol{r}\}$. We remark that Gaussian Schell-model light forms a large subclass of Gaussian light; it applies, for instance, in

- the completely coherent case ($\boldsymbol{H}=\boldsymbol{H}^t$, $\boldsymbol{G}_0=\boldsymbol{0}$, $\boldsymbol{G}_1=\boldsymbol{G}_2$);
- the (partially coherent) one-dimensional case ($g_0 = g_2 - g_1 \geq 0$); and
- the (partially coherent) rotationally symmetric case ($\boldsymbol{H}=h\boldsymbol{I}$, $\boldsymbol{G}_1=g_1\boldsymbol{I}$, $\boldsymbol{G}_2=g_2\boldsymbol{I}$, $\boldsymbol{G}_0=(g_2-g_1)\boldsymbol{I}$, with $\boldsymbol{I}$ the 2×2 identity matrix).

Gaussian Schell-model light reduces to so-called *symplectic* Gaussian light [7], if the matrices $\boldsymbol{G}_0$, $\boldsymbol{G}_1$, and $\boldsymbol{G}_2$ are proportional to each other, $\boldsymbol{G}_1=\sigma\boldsymbol{G}$, $\boldsymbol{G}_2=\boldsymbol{G}/\sigma$, and thus $\boldsymbol{G}_0=(1/\sigma-\sigma)\boldsymbol{G}$, with $\boldsymbol{G}$ a real, positive definite symmetric 2×2 matrix and $0<\sigma\leq1$. The positional power spectrum then takes the form

$$\Gamma(\boldsymbol{r}_1,\boldsymbol{r}_2) = \frac{\sigma}{\pi}\sqrt{\det\boldsymbol{G}}$$
$$\times \exp\left(-\tfrac{1}{4}\begin{bmatrix}\boldsymbol{r}_1+\boldsymbol{r}_2\\ \boldsymbol{r}_1-\boldsymbol{r}_2\end{bmatrix}^t\begin{bmatrix}\sigma\boldsymbol{G} & -i\boldsymbol{H}\\ -i\boldsymbol{H} & \boldsymbol{G}/\sigma\end{bmatrix}\begin{bmatrix}\boldsymbol{r}_1+\boldsymbol{r}_2\\ \boldsymbol{r}_1-\boldsymbol{r}_2\end{bmatrix}\right) \tag{3.30}$$

and the corresponding Wigner distribution function reads

$$F(\boldsymbol{r},\boldsymbol{q}) = 4\sigma^2$$
$$\times \exp\left(-\sigma\begin{bmatrix}\boldsymbol{r}\\ \boldsymbol{q}\end{bmatrix}^t\begin{bmatrix}\boldsymbol{G}+\boldsymbol{H}\boldsymbol{G}^{-1}\boldsymbol{H} & -\boldsymbol{H}\boldsymbol{G}^{-1}\\ -\boldsymbol{G}^{-1}\boldsymbol{H} & \boldsymbol{G}^{-1}\end{bmatrix}\begin{bmatrix}\boldsymbol{r}\\ \boldsymbol{q}\end{bmatrix}\right). \tag{3.31}$$

The name *symplectic* Gaussian light is given by the fact that the 4×4 matrix that arises in the exponent of the Wigner distribution function (3.31) is symplectic. We will return to symplecticity later on in this paper. We remark that symplectic Gaussian light forms a large subclass of Gaussian Schell-model light; it applies again, for instance, in the completely coherent case, in the (partially coherent) one-dimensional case, and in the (partially coherent) rotationally symmetric case. And again: symplectic Gaussian light can be considered as spatially stationary light with a Gaussian positional power spectrum modulated by a Gaussian modulator with modulation function, but now with the real parts of the quadratic forms in the two exponents described, up to a positive constant, by the *same* real, positive definite symmetric matrix $\boldsymbol{G}$.

4 MODAL EXPANSIONS

To derive more properties of the Wigner distribution function, we introduce *modal expansions* for the power spectrum and the Wigner distribution function.

4.1 Modal expansion of the positional power spectrum

We represent the positional power spectrum $\Gamma(\boldsymbol{r}_1,\boldsymbol{r}_2)$ by its modal expansion [35] (see also, for instance, [34, 36], in which a modal expansion of the (nonnegative definite Hermitian) *mutual intensity* $\tilde{\Gamma}(\boldsymbol{r}_1,\boldsymbol{r}_2,0)$ is given):

$$\Gamma(\boldsymbol{r}_1,\boldsymbol{r}_2)=\frac{1}{\rho}\sum_{m=0}^{\infty}\mu_m\varphi_m\left(\frac{\boldsymbol{r}_1}{\rho}\right)\varphi_m^*\left(\frac{\boldsymbol{r}_2}{\rho}\right); \tag{4.1}$$

a similar expansion holds for the directional power spectrum. For the mathematical subtleties of this modal expansion, we refer to the standard mathematical literature [37, 38]. In the modal expansion (4.1), the functions φ_m are the *eigenfunctions*, and the numbers μ_m are the *eigenvalues* of the integral equation

$$\int\Gamma(\boldsymbol{r}_1,\boldsymbol{r}_2)\varphi_m\left(\frac{\boldsymbol{r}_2}{\rho}\right)d\boldsymbol{r}_2=\mu_m\varphi_m\left(\frac{\boldsymbol{r}_1}{\rho}\right) \tag{4.2}$$

($m=0,1,...$); the positive factor ρ is a mere scaling factor. Since the kernel $\Gamma(\boldsymbol{r}_1,\boldsymbol{r}_2)$ is *Hermitian* and under the assumption of *discrete* eigenvalues, the eigenfunctions φ_m can be made orthonormal:

$$\int\varphi_m(\boldsymbol{\xi})\varphi_n^*(\boldsymbol{\xi})d\boldsymbol{\xi}=\delta_{m-n}=\begin{cases}1 & m=n\\ 0 & m\neq n\end{cases} \tag{4.3}$$

($m,n=0,1,...$). Moreover, since the kernel $\Gamma(\boldsymbol{r}_1,\boldsymbol{r}_2)$ is *nonnegative definite* Hermitian, the eigenvalues μ_m are nonnegative. Note that the light is completely coherent if there is only *one* nonvanishing eigenvalue. As a matter of fact, the modal expansion (4.1) expresses the partially coherent light as a superposition of coherent modes.

4.2 Modal expansion of the Wigner distribution function

When we substitute the modal expansion (4.1) into the definition (3.1), the Wigner distribution function can be expressed as

$$F(\boldsymbol{r},\boldsymbol{q})=\sum_{m=0}^{\infty}\mu_m f_m\left(\frac{\boldsymbol{r}}{\rho},\rho\boldsymbol{q}\right), \tag{4.4}$$

where

$$f_m(\boldsymbol{\xi},\boldsymbol{\eta})=\int\varphi_m(\boldsymbol{\xi}+\tfrac{1}{2}\boldsymbol{\xi}')\varphi_m^*(\boldsymbol{\xi}-\tfrac{1}{2}\boldsymbol{\xi}')e^{-i\boldsymbol{\eta}^t\boldsymbol{\xi}'}d\boldsymbol{\xi}' \tag{4.5}$$

($m=0,1,...$) are the Wigner distribution functions of the eigenfunctions φ_m, as in the completely coherent case [see definition (3.24)]. By applying relation (3.13) and using the orthonormality property (4.3), it can easily be seen that the Wigner distribution functions f_m satisfy the orthonormality relation

$$\iint f_m(\boldsymbol{\xi},\boldsymbol{\eta}\, f_n(\boldsymbol{\xi},\boldsymbol{\eta})d\boldsymbol{\xi}d\frac{\boldsymbol{\eta}}{2\pi}=\left|\int\varphi_m(\boldsymbol{\xi})\varphi_n^*(\boldsymbol{\xi})d\boldsymbol{\xi}\right|^2=\delta_{m-n} \tag{4.6}$$

($m,n=0,1,...$).

4.3 Inequalities for the Wigner distribution function

The modal expansion (4.4) allows us to formulate some interesting inequalities for the Wigner distribution function.

4.3.1 De Bruijn's inequality. Using the expansion (4.4), it is easy to see that De Bruijn's inequality [39]

$$\begin{aligned}\iint\left[g(\boldsymbol{e}^t\boldsymbol{r})^2+\frac{(\boldsymbol{e}^t\boldsymbol{q})^2}{g}\right]^m F(\boldsymbol{r},\boldsymbol{q})d\boldsymbol{r}d\frac{\boldsymbol{q}}{2\pi}\\ \geq m!\iint F(\boldsymbol{r},\boldsymbol{q})d\boldsymbol{r}d\frac{\boldsymbol{q}}{2\pi},\end{aligned} \tag{4.7}$$

with $\boldsymbol{e}$ a 2-dimensional unit column vector, g a positive scalar and m a positive integer, holds not only in the completely coherent case but also for the Wigner distribution function of partially coherent light. In the special case $m=1$, and choosing $\boldsymbol{e}=[1,0]^t$, relation (4.7) reduces to $gd_x^2+(1/g)d_u^2\geq 1$, which leads to the uncertainty relation [13] $2d_xd_u\geq 1$ by choosing $g=d_u/d_x$. A similar result holds for d_y and d_v when we choose $\boldsymbol{e}=[0,1]^t$: $2d_yd_v\geq 1$; and for $\boldsymbol{e}=[\frac{1}{2}\sqrt{2},\frac{1}{2}\sqrt{2}]^t$ we get [13] $(d_x^2+d_y^2)(d_u^2+d_v^2)\geq 1$. The equality signs in these uncertainty relations occur for completely coherent Gaussian light; for all other signals, the products of the effective widths in the space and the frequency direction are larger. We thus conclude that the coherent Gaussian Wigner distribution function occupies the smallest possible area in the space-frequency domain.

4.3.2 Positive average. Using the relationship (3.13) and expanding the power spectra $\Gamma_1(\boldsymbol{r}_1,\boldsymbol{r}_2)$ and $\Gamma_2(\boldsymbol{r}_1,\boldsymbol{r}_2)$ in the form (4.1), it can readily be shown that

$$\iint F_1(\boldsymbol{r},\boldsymbol{q})F_2(\boldsymbol{r},\boldsymbol{q})d\boldsymbol{r}d\frac{\boldsymbol{q}}{2\pi}\geq 0. \tag{4.8}$$

Thus, as we remarked before, averaging one Wigner distribution function with another one always yields a nonnegative result. In particular the averaging with the Wigner distribution function of completely coherent Gaussian light is of some practical importance [39, 40, 41], since the coherent Gaussian Wigner distribution function occupies the smallest possible area in the space-frequency domain, as we concluded before.

4.3.3 Schwarz's inequality. An upper bound for the expression that arises in relation (4.8) can be found by applying Schwarz's inequality [13]:

$$\begin{aligned}&\iint F_1(\boldsymbol{r},\boldsymbol{q})F_2(\boldsymbol{r},\boldsymbol{q})d\boldsymbol{r}d\frac{\boldsymbol{q}}{2\pi}\\ &\leq\sqrt{\iint F_1^2(\boldsymbol{r},\boldsymbol{q})d\boldsymbol{r}d\frac{\boldsymbol{q}}{2\pi}}\sqrt{\iint F_2^2(\boldsymbol{r},\boldsymbol{q})d\boldsymbol{r}d\frac{\boldsymbol{q}}{2\pi}}\\ &\leq\left(\iint F_1(\boldsymbol{r},\boldsymbol{q})d\boldsymbol{r}d\frac{\boldsymbol{q}}{2\pi}\right)\left(\iint F_2(\boldsymbol{r},\boldsymbol{q})d\boldsymbol{r}d\frac{\boldsymbol{q}}{2\pi}\right),\end{aligned} \tag{4.9}$$

where the latter expression is simply the product of the total energies of the two signals, and where we have used the important inequality

$$\iint F^2(\boldsymbol{r},\boldsymbol{q})d\boldsymbol{r}d\frac{\boldsymbol{q}}{2\pi}\leq\left(\iint F(\boldsymbol{r},\boldsymbol{q})d\boldsymbol{r}d\frac{\boldsymbol{q}}{2\pi}\right)^2. \tag{4.10}$$

To prove this important inequality, we first remark that, by using the modal expansion (4.4), the identity

$$\iint F(\boldsymbol{r},\boldsymbol{q})d\boldsymbol{r}d\frac{\boldsymbol{q}}{2\pi}=\sum_{m=0}^{\infty}\mu_m \tag{4.11}$$

holds. Secondly, we observe the identity

$$\iint F^2(\boldsymbol{r},\boldsymbol{q})d\boldsymbol{r}d\frac{\boldsymbol{q}}{2\pi}=\sum_{m=0}^{\infty}\mu_m^2, \tag{4.12}$$

which can be easily proved by applying the modal expansion (4.4) and by using the orthonormality property (4.6). Finally, we remark that, since all eigenvalues μ_m are nonnegative, the inequality

$$\sum_{m=0}^{\infty}\mu_m^2\leq\left(\sum_{m=0}^{\infty}\mu_m\right)^2 \tag{4.13}$$

holds, which completes the proof of relation (4.10). Note that the equality sign in relation (4.13), and hence in relation (4.10), holds if there is only one nonvanishing eigenvalue, i.e., in the case of complete coherence. The quotient of the two expressions that arise in relation (4.10) or relation (4.13) can therefore serve as a measure of the overall degree of coherence of the light.

5 PROPAGATION OF THE WIGNER DISTRIBUTION FUNCTION

In this section we study how the Wigner distribution function propagates through linear optical systems. In Section 5.1 we therefore consider an optical system as a *black box*, with an input plane and an output plane, whereas in Section 5.2 we consider the system as a *continuous medium*, in which the signal must satisfy a certain differential equation.

5.1 Ray-spread function of an optical system

We consider the propagation of the Wigner distribution function through linear systems. A linear system can be represented in four different ways, depending on whether we describe the input and the output signal in the space or in the frequency domain. We thus have four equivalent input-output relationships, which for completely coherent light read as

$$\varphi_o(\boldsymbol{r}_o) = \int h_{rr}(\boldsymbol{r}_o,\boldsymbol{r}_i)\varphi_i(\boldsymbol{r}_i)d\boldsymbol{r}_i, \tag{5.1}$$

$$\bar{\varphi}_o(\boldsymbol{q}_o) = \int h_{qr}(\boldsymbol{q}_o,\boldsymbol{r}_i)\varphi_i(\boldsymbol{r}_i)d\boldsymbol{r}_i, \tag{5.2}$$

$$\varphi_o(\boldsymbol{r}_o) = \int h_{rq}(\boldsymbol{r}_o,\boldsymbol{q}_i)\bar{\varphi}_i(\boldsymbol{q}_i)d\frac{\boldsymbol{q}_i}{2\pi}, \tag{5.3}$$

$$\bar{\varphi}_o(\boldsymbol{q}_o) = \int h_{qq}(\boldsymbol{q}_o,\boldsymbol{q}_i)\bar{\varphi}_i(\boldsymbol{q}_i)d\frac{\boldsymbol{q}_i}{2\pi}. \tag{5.4}$$

The first relation (5.1) is the usual system representation in the space domain by means of the coherent *point-spread function* h_{rr}; we remark that the function h_{rr} is the response of the system in the space domain when the input signal is a point source. The last relation (5.4) is a similar system representation in the frequency domain, where the function h_{qq} is the response of the system in the frequency domain when the input signal is a plane wave; therefore we can call h_{qq} the *wave-spread function* of the system. The remaining two relations (5.2) and (5.3) are hybrid system representations, since the input and the output signal are described in different domains; therefore we can call the functions h_{qr} and h_{rq} *hybrid spread functions*. The system description for partially coherent light is similar; in terms of the point-spread function it takes the form

$$\Gamma_o(\boldsymbol{r}_1,\boldsymbol{r}_2)=\iint h_{rr}(\boldsymbol{r}_1,\boldsymbol{\rho_1})\Gamma_i(\boldsymbol{\rho_1},\boldsymbol{\rho_2})h^*_{rr}(\boldsymbol{r}_2,\boldsymbol{\rho_2})d\boldsymbol{\rho_1}d\boldsymbol{\rho_2}, \tag{5.5}$$

and there are similar expressions for the other system descriptions.

Unlike the *four* system representations (5.1)-(5.4) described above, there is only *one* system representation when we describe the input and the output signal by their Wigner distribution functions. Indeed, combining the system representations (5.1)-(5.4) with the definitions (3.1) and (3.2) of the Wigner distribution function results in the relation

$$F_o(\boldsymbol{r}_o,\boldsymbol{q}_o)=\iint K(\boldsymbol{r}_o,\boldsymbol{q}_o,\boldsymbol{r}_i,\boldsymbol{q}_i)F(\boldsymbol{r}_i,\boldsymbol{q}_i)d\boldsymbol{r}_id\frac{\boldsymbol{q}_i}{2\pi}, \tag{5.6}$$

in which the Wigner distribution functions of the input and the output signal are related through a superposition integral. The function K is completely determined by the system and can be expressed in terms of the four system functions h_{rr}, h_{qr}, h_{rq}, and h_{qq}. We find

$$K(\boldsymbol{r}_o,\boldsymbol{q}_o,\boldsymbol{r}_i,\boldsymbol{q}_i)=\iint h_{rr}(\boldsymbol{r}_o+\tfrac{1}{2}\boldsymbol{r}'_o,\boldsymbol{r}_i+\tfrac{1}{2}\boldsymbol{r}'_i)$$

$$\times\, h^*_{rr}(\boldsymbol{r}_o-\tfrac{1}{2}\boldsymbol{r}'_o,\boldsymbol{r}_i-\tfrac{1}{2}\boldsymbol{r}'_i)e^{-i(\boldsymbol{q}_o^t\boldsymbol{r}'_o-\boldsymbol{q}_i^t\boldsymbol{r}'_i)}d\boldsymbol{r}'_od\boldsymbol{r}'_i \tag{5.7}$$

and similar expressions for the other system functions [42]. Relation (5.7) can be considered as the definition of a *double* Wigner distribution function; hence the function K has all the properties of a Wigner distribution function, for instance, the property of realness.

Let us think about the physical meaning of the function K. In a formal way, the function K is the response of the system in the space-frequency domain when the input signal is described by a product of two Dirac functions

$$F_i(\boldsymbol{r},\boldsymbol{q})=\delta(\boldsymbol{r}-\boldsymbol{r}_i)\,\delta\left(\frac{\boldsymbol{q}-\boldsymbol{q}_i}{2\pi}\right); \tag{5.8}$$

only in a formal way, since an actual input signal yielding such a Wigner distribution function does not exist. Nevertheless, such an input signal could be considered as a *single ray* entering the system at the position $\boldsymbol{r}_i$ with direction $\boldsymbol{q}_i$. Hence the function K might be called the *ray-spread function* of the system.

Some examples of ray-spread functions of elementary optical systems [43, 44] might elucidate the concept of the ray-spread function.

5.1.1 Thin lens. First, let us consider a (generally cylindrical) *thin lens*, whose point-spread function takes the form

$$h_{rr}(\boldsymbol{r}_o,\boldsymbol{r}_i)=e^{-\frac{1}{2}i\boldsymbol{r}_o^t\boldsymbol{C}\boldsymbol{r}_o}\;\delta(\boldsymbol{r}_o-\boldsymbol{r}_i),\tag{5.9}$$

with $\boldsymbol{C}=\boldsymbol{C}^t$ a real symmetric 2×2 matrix. For a spherical (rotationally symmetric) lens with focal distance f, the matrix $\boldsymbol{C}$ would read $\boldsymbol{C}=(k/f)\boldsymbol{I}$. Clearly, a thin lens is a modulator whose modulation function is a quadratic-phase function. The corresponding ray-spread function takes the special form of a product of two Dirac functions,

$$K(\boldsymbol{r}_o,\boldsymbol{q}_o,\boldsymbol{r}_i,\boldsymbol{q}_i)=\delta(\boldsymbol{r}_i-\boldsymbol{r}_o)\;\delta\left(\frac{\boldsymbol{q}_i-\boldsymbol{C}\boldsymbol{r}_o-\boldsymbol{q}_o}{2\pi}\right),\tag{5.10}$$

and the input-output relationship of a thin lens becomes very simple:

$$F_o(\boldsymbol{r},\boldsymbol{q})=F_i(\boldsymbol{r},\boldsymbol{C}\boldsymbol{r}+\boldsymbol{q}).\tag{5.11}$$

The ray-spread function represents exactly the geometric-optical behaviour of a thin lens: if a single ray is incident upon a thin lens, it will leave the lens from the same position, but its direction will change according to the actual position; in any event, there is only one output ray.

5.1.2 Free space in the Fresnel approximation. Our second example will be the dual of a thin lens, for which the wave-spread function reads

$$h_{qq}(\boldsymbol{q}_o,\boldsymbol{q}_i)=e^{\frac{1}{2}i\boldsymbol{q}_o^t\boldsymbol{B}\boldsymbol{q}_o}\;\delta\left(\frac{\boldsymbol{q}_o-\boldsymbol{q}_i}{2\pi}\right),\tag{5.12}$$

with $\boldsymbol{B}=\boldsymbol{B}^t$ a real symmetric 2×2 matrix. For a section of *free space* over a distance z in the Fresnel approximation, the matrix $\boldsymbol{B}$ would read $\boldsymbol{B}=-(z/k)\boldsymbol{I}$. The corresponding ray-spread function again takes the special form of the product of two Dirac functions

$$K(\boldsymbol{r}_o,\boldsymbol{q}_o,\boldsymbol{r}_i,\boldsymbol{q}_i)=\delta(\boldsymbol{r}_i-\boldsymbol{r}_o-\boldsymbol{B}\boldsymbol{q}_o)\;\delta\left(\frac{\boldsymbol{q}_i-\boldsymbol{q}_o}{2\pi}\right),\tag{5.13}$$

and the input-output relationship of such a system becomes very simple:

$$F_o(\boldsymbol{r},\boldsymbol{q})=F_i(\boldsymbol{r}+\boldsymbol{B}\boldsymbol{q},\boldsymbol{q}).\tag{5.14}$$

The ray-spread function represents exactly the geometric-optical behaviour of a section of free space: if a single ray propagates through free space, its direction will remain the same, but its position will change according to the actual direction; in any event, there is again only one output ray.

5.1.3 Fourier transformer. For a *Fourier transformer*, the input-output relationship takes the form

$$F_o(\boldsymbol{r},\boldsymbol{q})=F_i(\boldsymbol{B}\boldsymbol{q},\boldsymbol{C}\boldsymbol{r}),\tag{5.15}$$

where $\boldsymbol{B}$ and $\boldsymbol{C}$ (with $\boldsymbol{B}\boldsymbol{C}^t=\boldsymbol{C}^t\boldsymbol{B}=-\boldsymbol{I}$) are real 2×2 matrices. If we realize a Fourier transformer between the front and the back focal plane of a spherical lens with focal distance f, the matrices $\boldsymbol{B}$ and $\boldsymbol{C}$ would read $\boldsymbol{B}=-(f/k)\boldsymbol{I}$ and $\boldsymbol{C}=(k/f)\boldsymbol{I}$, respectively. We conclude that the space and the frequency domain are interchanged, as can be expected for a Fourier transformer.

5.1.4 Magnifier. For a *magnifier*, the input-output relationship takes the form

$$F_o(\boldsymbol{r},\boldsymbol{q})=F_i(\boldsymbol{A}\boldsymbol{r},\boldsymbol{D}\boldsymbol{q}),\tag{5.16}$$

where $\boldsymbol{A}$ and $\boldsymbol{D}$ (with $\boldsymbol{A}\boldsymbol{D}^t=\boldsymbol{A}^t\boldsymbol{D}=\boldsymbol{I}$) are real 2×2 matrices. Using spherical lenses and sections of free space to realize a magnifier, the matrices $\boldsymbol{A}$ and $\boldsymbol{D}$ would read $\boldsymbol{A}=m\boldsymbol{I}$ and $\boldsymbol{D}=(1/m)\boldsymbol{I}$, respectively. We note that the space and the frequency domain are merely scaled, as can be expected for a magnifier.

5.1.5 Luneburg's first-order optical system. A thin lens, a section of free space in the Fresnel approximation, a Fourier transformer, and a magnifier are special cases of Luneburg's *first-order optical systems* [45], which will be our final example. A first-order optical system can, of course, be characterized by its system functions h_{rr}, h_{qr}, h_{rq}, and h_{qq}: they are all quadratic-phase functions [42]. (Note that a Dirac function can be considered as a limiting case of such a quadratically varying function.) A system representation in terms of Wigner distribution functions, however, is far more elegant. The ray-spread function of a first-order system takes the form of a product of two Dirac functions,

$$K(\boldsymbol{r}_o,\boldsymbol{q}_o,\boldsymbol{r}_i,\boldsymbol{q}_i)=\delta(\boldsymbol{r}_i-\boldsymbol{A}\boldsymbol{r}_o-\boldsymbol{B}\boldsymbol{q}_o)\;\delta\left(\frac{\boldsymbol{q}_i-\boldsymbol{C}\boldsymbol{r}_o-\boldsymbol{D}\boldsymbol{q}_o}{2\pi}\right),\tag{5.17}$$

where $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ denote real 2×2 matrices, and the input-output relationship reads very simply as

$$F_o(\boldsymbol{r},\boldsymbol{q})=F_i(\boldsymbol{A}\boldsymbol{r}+\boldsymbol{B}\boldsymbol{q},\boldsymbol{C}\boldsymbol{r}+\boldsymbol{D}\boldsymbol{q}).\tag{5.18}$$

From the ray-spread function (5.17) we conclude that a single input ray, entering the system at the position $\boldsymbol{r}_i$ with direction $\boldsymbol{q}_i$, will yield a single output ray, leaving the system at the position $\boldsymbol{r}_o$ with the direction $\boldsymbol{q}_o$. The input and output positions and directions are related by the matrix relationship

$$\begin{bmatrix}\boldsymbol{r}_i\\ \boldsymbol{q}_i\end{bmatrix}=\begin{bmatrix}\boldsymbol{A}&\boldsymbol{B}\\ \boldsymbol{C}&\boldsymbol{D}\end{bmatrix}\begin{bmatrix}\boldsymbol{r}_o\\ \boldsymbol{q}_o\end{bmatrix}.\tag{5.19}$$

Relation (5.19) is a well-known geometric-optical matrix description of a first-order optical system [45]; the 4×4 $\boldsymbol{ABCD}$-matrix in this relationship is known as the *ray transformation matrix* [46]. We observe again a perfect resemblance to the geometric-optical behaviour of a first-order system (see also, for instance, [17]).

We remark that the ray transformation matrix is *symplectic* [45, 46, 47]. To express symplecticity in an easy way, we introduce the 4×4 matrix $\boldsymbol{J}$ according to

$$\boldsymbol{J}=i\begin{bmatrix}\boldsymbol{0}&-\boldsymbol{I}\\ \boldsymbol{I}&\boldsymbol{0}\end{bmatrix},\tag{5.20}$$

where $\boldsymbol{I}$ denotes the 2×2 identity matrix. The matrix $\boldsymbol{J}$ has the properties $\boldsymbol{J}=\boldsymbol{J}^{-1}=\boldsymbol{J}^{\dagger}=-\boldsymbol{J}^t$, where $\boldsymbol{J}^{-1}$, $\boldsymbol{J}^{\dagger}$, and $\boldsymbol{J}^t$ are the inverse, the adjoint, and the transpose of $\boldsymbol{J}$, respectively; moreover, $\det\boldsymbol{J}=1$. Symplecticity of the ray transformation matrix can then be expressed by the relationship

$$\boldsymbol{T}^{-1}=\boldsymbol{J}\boldsymbol{T}^t\boldsymbol{J},\tag{5.21}$$

where we have used the short-hand notation for the 4×4 ray transformation matrix

$$\boldsymbol{T}=\left[\begin{array}{cc}\boldsymbol{A} & \boldsymbol{B}\\ \boldsymbol{C} & \boldsymbol{D}\end{array}\right]. \tag{5.22}$$

In terms of the submatrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$, symplecticity implies the relations

$$\boldsymbol{A}\boldsymbol{B}^t=\boldsymbol{B}\boldsymbol{A}^t,\quad \boldsymbol{B}^t\boldsymbol{D}=\boldsymbol{D}^t\boldsymbol{B},\quad \boldsymbol{D}\boldsymbol{C}^t=\boldsymbol{C}\boldsymbol{D}^t,\quad \boldsymbol{C}^t\boldsymbol{A}=\boldsymbol{A}^t\boldsymbol{C},$$

$$\text{and}\quad \boldsymbol{A}\boldsymbol{D}^t-\boldsymbol{B}\boldsymbol{C}^t=\boldsymbol{I}=\boldsymbol{A}^t\boldsymbol{D}-\boldsymbol{C}^t\boldsymbol{B}. \tag{5.23}$$

We remark that spherical waves are intimately related to first-order optical systems, because both the Wigner distribution function of a spherical way and the ray-spread function of a first-order system consist of Dirac functions. If a spherical wave with curvature matrix $\boldsymbol{H}_i$, thus having a Wigner distribution function of the form [see Eq. (3.26)]

$$f_i(\boldsymbol{r},\boldsymbol{q})=\delta\left(\frac{\boldsymbol{q}-\boldsymbol{H}_i\boldsymbol{r}}{2\pi}\right), \tag{5.24}$$

forms the input signal of a first-order $\boldsymbol{ABCD}$-system, then the Wigner distribution function of the output signal reads

$$f_o(\boldsymbol{r},\boldsymbol{q})=\frac{1}{\det(\boldsymbol{D}-\boldsymbol{H}_i\boldsymbol{B})}\times\delta\left(\frac{\boldsymbol{q}+(\boldsymbol{D}-\boldsymbol{H}_i\boldsymbol{B})^{-1}(\boldsymbol{C}-\boldsymbol{H}_i\boldsymbol{A})\boldsymbol{r}}{2\pi}\right). \tag{5.25}$$

We conclude that the output signal is again a spherical wave with curvature matrix $\boldsymbol{H}_o=-(\boldsymbol{D}-\boldsymbol{H}_i\boldsymbol{B})^{-1}(\boldsymbol{C}-\boldsymbol{H}_i\boldsymbol{A})$, hence

$$\boldsymbol{H}_i=(\boldsymbol{C}+\boldsymbol{D}\boldsymbol{H}_o)(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{H}_o)^{-1}. \tag{5.26}$$

The latter bilinear relationship is known as the $\boldsymbol{ABCD}$-law. In Section 6 we will show that the $\boldsymbol{ABCD}$-law can be applied to describe the propagation of symplectic Gaussian light through first-order systems, as well.

5.2 Transport equations for the Wigner distribution function

In the previous section we studied, in example 5.1.2, the propagation of the Wigner distribution function through free space by considering a section of free space as an optical system with an input plane and an output plane. It is possible, however, to find the propagation of the Wigner distribution function through free space directly from the differential equation that the signal must satisfy. We therefore let the longitudinal variable z enter into the formulas and remark that the propagation of coherent light in free space (at least in the Fresnel approximation) is governed by the differential equation (see [13], p. 358)

$$-i\frac{\partial\varphi}{\partial z}=\left(k+\frac{1}{2k}\frac{\partial^2}{\partial\boldsymbol{r}^2}\right)\varphi, \tag{5.27}$$

with $\frac{\partial^2}{\partial\boldsymbol{r}^2}$ representing the scalar operator $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$; partially coherent light must satisfy the differential equation

$$-i\frac{\partial\Gamma}{\partial z}=\left[\left(k+\frac{1}{2k}\frac{\partial^2}{\partial\boldsymbol{r}_1^2}\right)-\left(k+\frac{1}{2k}\frac{\partial^2}{\partial\boldsymbol{r}_2^2}\right)\right]\Gamma. \tag{5.28}$$

The propagation of the Wigner distribution function is now described by a *transport equation* [48, 49, 50, 51, 52, 53], which in this case takes the form

$$\frac{\boldsymbol{q}^t}{k}\frac{\partial F}{\partial\boldsymbol{r}}+\frac{\partial F}{\partial z}=0, \tag{5.29}$$

with $\frac{\partial}{\partial\boldsymbol{r}}$ representing the vectorial operator $\left[\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right]^t$. The transport equation (5.29) has the solution

$$F(\boldsymbol{r},\boldsymbol{q};z)=F\left(\boldsymbol{r}-\frac{\boldsymbol{q}}{k}z,\boldsymbol{q};0\right), \tag{5.30}$$

which is equivalent to the result in section 5.1.2, with $\boldsymbol{B}=-(z/k)\boldsymbol{I}$ [see Eq. (5.14)].

The differential equation (5.28) is a special case of the more general equation

$$-i\frac{\partial\Gamma}{\partial z}=\left[L\left(\boldsymbol{r}_1,-i\frac{\partial}{\partial\boldsymbol{r}_1};z\right)-L^*\left(\boldsymbol{r}_2,-i\frac{\partial}{\partial\boldsymbol{r}_2};z\right)\right]\Gamma, \tag{5.31}$$

where L is some explicit function of the space variables $\boldsymbol{r}$ and z and of the partial derivatives of Γ contained in the operator $\partial/\partial\boldsymbol{r}$. The transport equation that corresponds to this differential equation reads as [4, 50, 52, 53]

$$\frac{\partial F}{\partial z}=2\,\mathrm{Im}\left[L\left(\boldsymbol{r}+\tfrac{1}{2}i\frac{\partial}{\partial\boldsymbol{q}},\boldsymbol{q}-\tfrac{1}{2}i\frac{\partial}{\partial\boldsymbol{r}};z\right)\right]F, \tag{5.32}$$

in which Im denotes the imaginary part. In the *Liouville approximation* (or *geometric-optical approximation*) the transport equation (5.32) reduces to

$$\begin{aligned}-\frac{\partial F}{\partial z}&=2\,(\mathrm{Im}L)\,F\\&+\left(\frac{\partial\mathrm{Re}L}{\partial\boldsymbol{r}}\right)^t\frac{\partial F}{\partial\boldsymbol{q}}-\left(\frac{\partial\mathrm{Re}L}{\partial\boldsymbol{q}}\right)^t\frac{\partial F}{\partial\boldsymbol{r}},\end{aligned} \tag{5.33}$$

in which Re denotes the real part. Relation (5.33) is a first-order partial differential equation, which can be solved by the method of characteristics [54]: along a path described in a parameter notation by $\boldsymbol{r}=\boldsymbol{r}(s)$, $z=z(s)$, and $\boldsymbol{q}=\boldsymbol{q}(s)$, and defined by the differential equations

$$\frac{d\boldsymbol{r}}{ds}=-\frac{\partial\mathrm{Re}L}{\partial\boldsymbol{q}},\quad\frac{dz}{ds}=1,\quad\frac{d\boldsymbol{q}}{ds}=\frac{\partial\mathrm{Re}L}{\partial\boldsymbol{r}}, \tag{5.34}$$

the *partial* differential equation (5.33) reduces to the *ordinary* differential equation

$$-\frac{dF}{ds}=2\,(\mathrm{Im}L)\,F. \tag{5.35}$$

In the special case that $L(\boldsymbol{r}, \boldsymbol{q}; z)$ is a *real* function of $\boldsymbol{r}$, $\boldsymbol{q}$, and z, Eq. (5.35) implies that, along the path defined by relations (5.34), the Wigner distribution function has a *constant* value (see also, for instance, [55]).

Let us consider some examples of transport equations.

5.2.1 Free space in the Fresnel approximation. In *free space in the Fresnel approximation*, the signal is governed by equation (5.28), and the function L reads as

$$L(\boldsymbol{r}, \boldsymbol{q}; z) = k - \frac{\boldsymbol{q}^t\boldsymbol{q}}{2k}. \tag{5.36}$$

The corresponding transport equation (5.29) and its solution (5.30) have already been mentioned in the introductory paragraph of section 5.2.

5.2.2 Free space. In *free space* (but not necessarily in the Fresnel approximation), a coherent signal must satisfy the Helmholtz equation, whereas the propagation of partially coherent light is governed by the differential equation

$$-i\frac{\partial \Gamma}{\partial z} = \left[\sqrt{k^2 + \frac{\partial^2}{\partial \boldsymbol{r}_1^2}} - \sqrt{k^2 + \frac{\partial^2}{\partial \boldsymbol{r}_2^2}}\right]\Gamma. \tag{5.37}$$

In this case, the function L reads as

$$L(\boldsymbol{r}, \boldsymbol{q}; z) = \sqrt{k^2 - \boldsymbol{q}^t\boldsymbol{q}}. \tag{5.38}$$

We can again derive a transport equation for the Wigner distribution function; the exact transport equation is rather complicated, but in the Liouville approximation it takes the simple form

$$\frac{\boldsymbol{q}^t}{k}\frac{\partial F}{\partial \boldsymbol{r}} + \frac{\sqrt{k^2 - \boldsymbol{q}^t\boldsymbol{q}}}{k}\frac{\partial F}{\partial z} = 0. \tag{5.39}$$

This transport equation can again be solved explicitly, and the solution reads as

$$F(\boldsymbol{r}, \boldsymbol{q}; z) = F\left(\boldsymbol{r} - \frac{\boldsymbol{q}}{\sqrt{k^2 - \boldsymbol{q}^t\boldsymbol{q}}}z, \boldsymbol{q}; 0\right). \tag{5.40}$$

The difference from the previous solution (5.30), in which we considered the Fresnel approximation, is that the sine $\boldsymbol{q}/k$ has been replaced by the tangent $\boldsymbol{q}/\sqrt{k^2 - \boldsymbol{q}^t\boldsymbol{q}}$. Note that in the Fresnel approximation, the relations (5.37), (5.39), and (5.40) reduce to the relations (5.28), (5.29), and (5.30), respectively. When we integrate the transport equation (5.39) over the frequency variable $\boldsymbol{q}$ and use the definitions (3.9), (3.10) and (3.11), we get the relation

$$\left(\frac{\partial}{\partial \boldsymbol{r}}\right)^t \boldsymbol{j}_r + \frac{\partial j_z}{\partial z} = 0, \tag{5.41}$$

which shows that the geometrical vector flux $\boldsymbol{j}$ has zero divergence [25].

5.2.3 Weakly inhomogeneous medium. In a *weakly inhomogeneous medium*, the differential equation that the signal must satisfy again has the form of Eq. (5.37), but now with $k = k(\boldsymbol{r}, z)$. The transport equation in the Liouville approximation now takes the form

$$\frac{\boldsymbol{q}^t}{k}\frac{\partial F}{\partial \boldsymbol{r}} + \frac{\sqrt{k^2 - \boldsymbol{q}^t\boldsymbol{q}}}{k}\frac{\partial F}{\partial z} + \left(\frac{\partial k}{\partial \boldsymbol{r}}\right)^t\frac{\partial F}{\partial \boldsymbol{q}} = 0, \tag{5.42}$$

which, in general, cannot be solved explicitly. With the method of characteristics we conclude that along a path defined by

$$\frac{d\boldsymbol{r}}{ds} = \frac{\boldsymbol{q}}{k}, \quad \frac{dz}{ds} = \frac{\sqrt{k^2 - \boldsymbol{q}^t\boldsymbol{q}}}{k}, \quad \frac{d\boldsymbol{q}}{ds} = \frac{\partial k}{\partial \boldsymbol{r}}, \tag{5.43}$$

the Wigner distribution function has a constant value. When we eliminate the frequency variable $\boldsymbol{q}$ from Eqs. (5.43), we are immediately led to

$$\frac{d}{ds}\left(k\frac{d\boldsymbol{r}}{ds}\right) = \frac{\partial k}{\partial \boldsymbol{r}}, \quad \frac{d}{ds}\left(k\frac{dz}{ds}\right) = \frac{\partial k}{\partial z}, \tag{5.44}$$

which are the equations for an optical ray in geometrical optics [56]. We are thus led to the general conclusion that in the Liouville approximation the Wigner distribution function has a constant value along the geometric-optical ray paths. Note that in a homogeneous medium, i.e., $\partial k/\partial \boldsymbol{r} = \boldsymbol{0}$ and $\partial k/\partial z = 0$, the transport equation (5.42) reduces to Eq. (5.39) and that the ray paths become straight lines.

5.2.4 Rotationally symmetric fiber. Our final example is a *rotationally symmetric inhomogeneous medium*, a fiber, for instance, that extends along the z axis. The transport equation takes the general form (5.42), but now with $k = k(\sqrt{\boldsymbol{r}^t\boldsymbol{r}}) = k(\rho)$. When we apply the coordinate transformation

$$x = \rho\cos\theta, \quad y = \rho\sin\theta, \quad h = vx - uy, \quad k^2 = u^2 + v^2 + w^2, \tag{5.45}$$

with the corresponding relationships for the differentials $d\rho = \cos\theta dx + \sin\theta dy$, $\rho d\theta = -\sin\theta dx + \cos\theta dy$, $dh = vdx - udy - ydu + xdv$, and $wdw = k(dk/d\rho)\cos\theta dx + k(dk/d\rho)\sin\theta dy - udu - vdv$, we arrive at the transport equation

$$\sqrt{k^2 - w^2 - \frac{h^2}{\rho^2}}\frac{\partial F}{\partial \rho} + \frac{h}{\rho^2}\frac{\partial F}{\partial \theta} + w\frac{\partial F}{\partial z} = 0. \tag{5.46}$$

We remark that the derivatives of the Wigner distribution function with respect to the *ray invariants* h and w do not enter the transport equation. From the definition of the characteristics

$$w\frac{dh}{dz} = w\frac{dw}{dz} = 0, \quad w\frac{d\rho}{dz} = \sqrt{k^2 - w^2 - \frac{h^2}{\rho^2}}, \quad w\frac{d\theta}{dz} = \frac{h}{\rho^2}, \tag{5.47}$$

we conclude that

$$\frac{dh}{dz} = \frac{dw}{dz} = 0, \tag{5.48}$$

and h and w are indeed invariant along a ray.

5.3 Geometric-optical systems

Let us start by studying a *modulator* described – in the case of partially coherent light – by the input-output relationship $\Gamma_o(\boldsymbol{r}_1, \boldsymbol{r}_2) = m(\boldsymbol{r}_1)\Gamma_i(\boldsymbol{r}_1, \boldsymbol{r}_2)m^*(\boldsymbol{r}_2)$. The input and output Wigner distribution functions are related by the relationship

$$F_o(\boldsymbol{r}, \boldsymbol{q}_o) = \int F_i(\boldsymbol{r}, \boldsymbol{q}_i) d\frac{\boldsymbol{q}_i}{2\pi}$$
$$\times \int m(\boldsymbol{r} + \tfrac{1}{2}\boldsymbol{r}')m^*(\boldsymbol{r} - \tfrac{1}{2}\boldsymbol{r}')e^{-i(\boldsymbol{q}_o - \boldsymbol{q}_i)^t\boldsymbol{r}'} d\boldsymbol{r}'. \quad (5.49)$$

This input-output relationship can be written in two distinct forms. On the one hand we can represent it in a *differential* format reading as follows:

$$F_o(\boldsymbol{r}, \boldsymbol{q}) = m\left(\boldsymbol{r} + \tfrac{1}{2}i\frac{\partial}{\partial \boldsymbol{q}}\right) m^*\left(\boldsymbol{r} - \tfrac{1}{2}i\frac{\partial}{\partial \boldsymbol{q}}\right) F_i(\boldsymbol{r}, \boldsymbol{q}). \quad (5.50)$$

On the other hand, we can represent it in an *integral* format that reads as follows:

$$F_o(\boldsymbol{r}, \boldsymbol{q}) = \int f_m(\boldsymbol{r}, \boldsymbol{q} - \boldsymbol{q}_i) F_i(\boldsymbol{r}, \boldsymbol{q}_i) d\frac{\boldsymbol{q}_i}{2\pi}, \quad (5.51)$$

where $f_m(\boldsymbol{r}, \boldsymbol{q})$ is the Wigner distribution function of the modulation function $m(\boldsymbol{r})$. Which of these two forms is superior depends on the problem.

We now confine ourselves to the case of a *pure phase* modulation function $m(\boldsymbol{r}) = \exp[i\gamma(\boldsymbol{r})]$. We then get

$$m(\boldsymbol{r} + \tfrac{1}{2}\boldsymbol{r}')m^*(\boldsymbol{r} - \tfrac{1}{2}\boldsymbol{r}') = e^{i\left\{\gamma(\boldsymbol{r} + \frac{1}{2}\boldsymbol{r}') - \gamma(\boldsymbol{r} - \frac{1}{2}\boldsymbol{r}')\right\}}$$
$$= \exp\left[i\left\{\left(\frac{\partial\gamma}{\partial\boldsymbol{r}}\right)^t \boldsymbol{r}' + \text{higher-order terms}\right\}\right]. \quad (5.52)$$

If we consider only the first-order derivative in relation (5.52), we arrive at the following expressions:

$$m\left(\boldsymbol{r} + \tfrac{1}{2}i\frac{\partial}{\partial \boldsymbol{q}}\right) m^*\left(\boldsymbol{r} - \tfrac{1}{2}i\frac{\partial}{\partial \boldsymbol{q}}\right) \simeq \exp\left[-\left(\frac{\partial\gamma}{\partial\boldsymbol{r}}\right)^t \frac{\partial}{\partial\boldsymbol{q}}\right], \quad (5.53)$$

$$f_m(\boldsymbol{r}, \boldsymbol{q}) \simeq 2\pi\delta\left(\boldsymbol{q} - \frac{\partial\gamma}{\partial\boldsymbol{r}}\right), \quad (5.54)$$

and the input-output relationship of the pure phase modulator becomes

$$F_o(\boldsymbol{r}, \boldsymbol{q}) \simeq F_i\left(\boldsymbol{r}, \boldsymbol{q} - \frac{\partial\gamma}{\partial\boldsymbol{r}}\right), \quad (5.55)$$

which is a mere coordinate transformation. We conclude that a single input ray yields a single output ray.

The ideas described above have been applied to the design of optical coordinate transformers [57, 58] and to the theory of aberrations [59]. Now, if the first-order approximation is not sufficiently accurate, i.e., if we have to take into account higher-order derivatives in relation (5.52), the Wigner distribution function allows us to overcome this problem. Indeed, we still have the exact input-output relationships (5.50) and (5.51), and we can take into account as many derivatives in relation (5.52) as necessary. We thus end up with a more general differential form [60] than expression (5.53) or a more general integral form [61] than expression (5.54). The latter case, for instance, will yield an Airy function [62] instead of a Dirac function, when we take not only the first but also the third derivative into account.

From expression (5.55) we concluded that a single input ray yields a single output ray. This may also happen in more general – not just modulation-type – systems; we call such systems *geometric-optical systems*. These systems have the simple input-output relationship

$$F_o(\boldsymbol{r}, \boldsymbol{q}) \simeq F_i\left(\boldsymbol{g}_x(\boldsymbol{r}, \boldsymbol{q}), \boldsymbol{g}_u(\boldsymbol{r}, \boldsymbol{q})\right), \quad (5.56)$$

where the $\simeq$ sign becomes an $=$ sign in the case of linear functions $\boldsymbol{g}_r$ and $\boldsymbol{g}_q$, i.e., in the case of Luneburg's first-order optical systems, which we have considered in Section 5.1.5. There appears to be a close relationship to the description of such geometric-optical systems by means of the Hamilton characteristics [42].

Instead of the black-box approach of a geometric-optical system, which leads to the input-output relationship (5.56), we can also consider the system as a continuous medium and formulate transport equations, as we did in Section 5.2. For geometric-optical systems, this transport equation takes the form of a first-order partial differential equation [63], which can be solved by the method of characteristics. In Section 5.2 we reached the general conclusion that these characteristics represent the geometric-optical ray paths and that along these ray paths the Wigner distribution function has a constant value.

The use of the transport equation is not restricted to *deterministic* media; Bremmer [51] has applied it to *stochastic* media. Neither is the transport equation restricted to the *scalar* treatment of wave fields; Bugnolo and Bremmer [64] have applied it to study the propagation of *vectorial* wave fields. In the vectorial case, the concept of the Wigner distribution function leads to a Hermitian matrix rather than to a scalar function and permits the description of *nonisotropic* media as well.

6 MORE ON SECOND-ORDER MOMENTS

In this final section we study some miscellaneous topics related to second-order moments of the Wigner distribution function and to their propagation through first-order optical systems [5, 6, 7].

6.1 Second-order moments of the Wigner distribution function

The propagation of the matrix $\boldsymbol{M}$ of second-order moments of the Wigner distribution function through a first-order system with ray transformation matrix $\boldsymbol{T}$, can be described by the input-output relationship [47, 65]

$$\boldsymbol{M}_i = \boldsymbol{T}\boldsymbol{M}_o\boldsymbol{T}^t. \quad (6.1)$$

This relationship can readily be derived by combining the input-output relationship (5.19) of the first-order system with the definition (3.6) of the moment matrices of the input and the output signal. Since the ray transformation matrix $\boldsymbol{T}$ is symplectic, we can immediately formulate a property for that particular kind of partially coherent light for which the moment matrix $\boldsymbol{M}$ is symplectic as well:

- Symplecticity is preserved in a first-order system: if $\boldsymbol{M}_i$ is proportional to a symplectic matrix, then $\boldsymbol{M}_o$ is proportional to a symplectic matrix as well, with the same proportionality factor.

If we multiply Eq. (6.1) from the right by $\boldsymbol{J}$, and use the symplecticity property (5.21) and the properties of $\boldsymbol{J}$, the input-output relationship (6.1) can be written as [6]

$$\boldsymbol{M}_i\boldsymbol{J} = \boldsymbol{T}(\boldsymbol{M}_o\boldsymbol{J})\boldsymbol{T}^{-1}. \tag{6.2}$$

From the latter relationship we conclude that the matrices $\boldsymbol{M}_o\boldsymbol{J}$ and $\boldsymbol{M}_i\boldsymbol{J}$ are related to each other by a *similarity transformation*. As a consequence of this similarity transformation, and writing the matrix $\boldsymbol{MJ}$ in terms of its *eigenvalues* and *eigenvectors* according to

$$\boldsymbol{MJ} = \boldsymbol{S}\Lambda\boldsymbol{S}^{-1}, \tag{6.3}$$

we can formulate the relationships

$$\Lambda_i = \Lambda_o \tag{6.4}$$

and

$$\boldsymbol{S}_i = \boldsymbol{T}\boldsymbol{S}_o. \tag{6.5}$$

We are thus led to the important property [6]:

- The eigenvalues of the matrix $\boldsymbol{MJ}$ (and any combination of these eigenvalues) remain invariant under propagation through a first-order system, while the matrix of eigenvectors $\boldsymbol{S}$ transforms in the same way as the ray vector $[\boldsymbol{r}^t, \boldsymbol{q}^t]^t$ does.

A similar property holds for the matrix $\boldsymbol{JM}$, but in the present paper we will concentrate on the matrix $\boldsymbol{MJ}$. In Section 6.3 we will use the invariance property to derive invariant expressions in terms of the second-order moments of the Wigner distribution function.

We state the following properties for the matrices $\boldsymbol{M}$ and $\boldsymbol{MJ}$; the proofs of these properties can be found in [6].

- If λ is an eigenvalue of $\boldsymbol{MJ}$, then $-\lambda$ is an eigenvalue, too. This property is a consequence of the fact that the matrix $\boldsymbol{M}$ is a *symmetric* matrix. We remark that the fact that both $+\lambda$ and $-\lambda$ are eigenvalues of the matrix $\boldsymbol{MJ}$, implies that the characteristic polynomial $\det(\boldsymbol{MJ} - \lambda\boldsymbol{I})$, with the help of which we determine the eigenvalues, is a polynomial of λ^2.

- The eigenvalues of $\boldsymbol{MJ}$ are real. This property is a consequence of the fact that $\boldsymbol{M}$ is a *real*, *positive definite* symmetric matrix (and thus a *positive definite Hermitian* matrix, $\boldsymbol{M} = \boldsymbol{M}^\dagger$).

- If $\boldsymbol{M}$ is proportional to a symplectic matrix, then it can be expressed in the form

$$\boldsymbol{M} = m\begin{bmatrix} \boldsymbol{G}^{-1} & \boldsymbol{G}^{-1}\boldsymbol{H} \\ \boldsymbol{H}\boldsymbol{G}^{-1} & \boldsymbol{G}+\boldsymbol{H}\boldsymbol{G}^{-1}\boldsymbol{H} \end{bmatrix}, \tag{6.6}$$

  with m a positive scalar, $\boldsymbol{G}$ and $\boldsymbol{H}$ real symmetric 2×2 matrices, and $\boldsymbol{G}$ positive-definite.

- If $\boldsymbol{M}$ is proportional to a symplectic matrix with a (positive) proportionality factor m ($m^4 = \det \boldsymbol{M}$), then the two positive eigenvalues of $\boldsymbol{MJ}$ are equal to $+m$ and the two negative eigenvalues are equal to $-m$.

Let us consider the case of a moment matrix $\boldsymbol{M}$ that is proportional to a symplectic matrix with proportionality factor m in more detail. From the equality of the positive and of the negative eigenvalues of $\boldsymbol{MJ}$, we conclude that these eigenvalues read as $\lambda = \pm m$, where each value of λ is 2-fold. The two eigenvectors that correspond to the positive eigenvalue $+m$ can be combined into the 4×2 matrix

$$\begin{bmatrix} \boldsymbol{R}^+ \\ \boldsymbol{Q}^+ \end{bmatrix}; \tag{6.7}$$

a similar matrix (with superscripts $-$) can be associated with the two negative eigenvalues $-m$.

The eigenvector equation for the matrix $\boldsymbol{MJ}$,

$$\boldsymbol{MJS} = \boldsymbol{S}\Lambda, \tag{6.8}$$

now takes the form

$$i\begin{bmatrix} \boldsymbol{G}^{-1} & \boldsymbol{G}^{-1}\boldsymbol{H} \\ \boldsymbol{H}\boldsymbol{G}^{-1} & \boldsymbol{G}+\boldsymbol{H}\boldsymbol{G}^{-1}\boldsymbol{H} \end{bmatrix}\begin{bmatrix} \boldsymbol{0} & -\boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{0} \end{bmatrix}\begin{bmatrix} \boldsymbol{R}^\pm \\ \boldsymbol{Q}^\pm \end{bmatrix} = \pm\begin{bmatrix} \boldsymbol{R}^\pm \\ \boldsymbol{Q}^\pm \end{bmatrix}, \tag{6.9}$$

and the first block row of this relationship reads

$$i(-\boldsymbol{G}^{-1}\boldsymbol{Q}^\pm + \boldsymbol{G}^{-1}\boldsymbol{H}\boldsymbol{R}^\pm) = \pm\boldsymbol{R}^\pm, \tag{6.10}$$

which yields the relation

$$\boldsymbol{Q}^\pm = (\boldsymbol{H} \pm i\boldsymbol{G})\boldsymbol{R}^\pm. \tag{6.11}$$

We remark that, since the eigenvectors satisfy the input-output relationship (5.19), and thus

$$\begin{bmatrix} \boldsymbol{R}_i^\pm \\ \boldsymbol{Q}_i^\pm \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}\begin{bmatrix} \boldsymbol{R}_o^\pm \\ \boldsymbol{Q}_o^\pm \end{bmatrix}, \tag{6.12}$$

the *bilinear relationship*

$$\boldsymbol{H}_i \pm i\boldsymbol{G}_i = [\boldsymbol{C} + \boldsymbol{D}(\boldsymbol{H}_o \pm i\boldsymbol{G}_o)][\boldsymbol{A} + \boldsymbol{B}(\boldsymbol{H}_o \pm i\boldsymbol{G}_o)]^{-1} \tag{6.13}$$

holds. This bilinear relationship, together with the invariance of $\det \boldsymbol{M}$, completely describes the propagation of a symplectic matrix $\boldsymbol{M}$ through a first-order system. Note that the bilinear relationship (6.13) is identical to the $\boldsymbol{ABCD}$-law (5.26) for spherical waves: for spherical waves we have $\boldsymbol{H}_i = [\boldsymbol{C} + \boldsymbol{D}\boldsymbol{H}_o][\boldsymbol{A} + \boldsymbol{B}\boldsymbol{H}_o]^{-1}$, and we have only replaced the (real) curvature matrix $\boldsymbol{H}$ by the (generally complex) matrix $\boldsymbol{H} \pm i\boldsymbol{G}$. We are thus led to the important result:

- If the matrix $\boldsymbol{M}$ of second-order moments is symplectic (up to a positive constant) as described in Eq. (6.6), its propagation through a first-order system is completely described by the invariance of this positive constant and the $\boldsymbol{ABCD}$-law (6.13).

6.2 Second-order moments of the Gaussian signal

Let us consider Gaussian light again, described by the Wigner distribution function [see Eq. (3.28)]

$$F(\boldsymbol{r},\boldsymbol{q}) = 4\sqrt{\frac{\det \boldsymbol{G}_1}{\det \boldsymbol{G}_2}}$$

$$\times \exp\left(-\begin{bmatrix} \boldsymbol{r} \\ \boldsymbol{q} \end{bmatrix}^t \begin{bmatrix} \boldsymbol{G}_1 + \boldsymbol{H}\boldsymbol{G}_2^{-1}\boldsymbol{H}^t & -\boldsymbol{H}\boldsymbol{G}_2^{-1} \\ -\boldsymbol{G}_2^{-1}\boldsymbol{H}^t & \boldsymbol{G}_2^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{r} \\ \boldsymbol{q} \end{bmatrix}\right). \tag{6.14}$$

The matrix of second-order moments $\boldsymbol{M}$ for such light reads as

$$\begin{aligned} \boldsymbol{M} &= \begin{bmatrix} \boldsymbol{R} & \boldsymbol{P} \\ \boldsymbol{P}^t & \boldsymbol{Q} \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} \boldsymbol{G}_1 + \boldsymbol{H}\boldsymbol{G}_2^{-1}\boldsymbol{H}^t & -\boldsymbol{H}\boldsymbol{G}_2^{-1} \\ -\boldsymbol{G}_2^{-1}\boldsymbol{H}^t & \boldsymbol{G}_2^{-1} \end{bmatrix}^{-1} \\ &= \frac{1}{2}\begin{bmatrix} \boldsymbol{G}_1^{-1} & \boldsymbol{G}_1^{-1}\boldsymbol{H} \\ \boldsymbol{H}^t\boldsymbol{G}_1^{-1} & \boldsymbol{G}_2 + \boldsymbol{H}^t\boldsymbol{G}_1^{-1}\boldsymbol{H} \end{bmatrix}; \end{aligned} \tag{6.15}$$

hence, the matrices $\boldsymbol{G}_1$, $\boldsymbol{G}_2$, and $\boldsymbol{H}$ follow directly from the submatrices $\boldsymbol{P}$, $\boldsymbol{Q}$, and $\boldsymbol{R}$ of the moment matrix $\boldsymbol{M}$:

$$\boldsymbol{G}_1 = \tfrac{1}{2}\boldsymbol{R}^{-1} = \boldsymbol{G}_1^t, \quad \boldsymbol{G}_2 = 2\left(\boldsymbol{Q} - \boldsymbol{P}^t\boldsymbol{R}^{-1}\boldsymbol{P}\right) = \boldsymbol{G}_2^t,$$

$$\boldsymbol{H} = \boldsymbol{R}^{-1}\boldsymbol{P}, \quad \text{and} \quad \boldsymbol{H}^t = \boldsymbol{P}^t\boldsymbol{R}^{-1}. \tag{6.16}$$

For symplectic Gaussian light (with $\boldsymbol{H} = \boldsymbol{H}^t$, $\boldsymbol{G}_1 = \sigma\boldsymbol{G}$, and $\boldsymbol{G}_2 = \boldsymbol{G}/\sigma$) we have

$$\boldsymbol{M} = \frac{1}{2\sigma}\begin{bmatrix} \boldsymbol{G}^{-1} & \boldsymbol{G}^{-1}\boldsymbol{H} \\ \boldsymbol{H}\boldsymbol{G}^{-1} & \boldsymbol{G} + \boldsymbol{H}\boldsymbol{G}^{-1}\boldsymbol{H} \end{bmatrix}, \tag{6.17}$$

and the matrices $\boldsymbol{G}$ and $\boldsymbol{H}$ follow the $\boldsymbol{ABCD}$-law (6.13).

The matrix decomposition described in Eq. (6.15) is, in fact, a general decomposition which holds for any symmetric matrix $\boldsymbol{M}$. From the positive definiteness of the matrix $\boldsymbol{M}$ we conclude that both $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ are positive definite. This can be seen when we consider the quadratic form

$$2\begin{bmatrix} \boldsymbol{q} \\ \boldsymbol{r} \end{bmatrix}^t \boldsymbol{M} \begin{bmatrix} \boldsymbol{q} \\ \boldsymbol{r} \end{bmatrix} = 2(\boldsymbol{q}^t\boldsymbol{R}\boldsymbol{q} + \boldsymbol{q}^t\boldsymbol{P}\boldsymbol{r} + \boldsymbol{r}^t\boldsymbol{P}^t\boldsymbol{q} + \boldsymbol{r}^t\boldsymbol{Q}\boldsymbol{r})$$

$$= \boldsymbol{r}^t\boldsymbol{G}_2\boldsymbol{r} + (\boldsymbol{q} + \boldsymbol{H}\boldsymbol{r})^t\boldsymbol{G}_1^{-1}(\boldsymbol{q} + \boldsymbol{H}\boldsymbol{r}), \tag{6.18}$$

which must be positive for any vectors $\boldsymbol{q}$ and $\boldsymbol{r}$.

We remark that the twistedness of Gaussian light – related to the skewness of the matrix $\boldsymbol{H}$ – can directly be expressed in terms of the moment matrices $\boldsymbol{R}$ and $\boldsymbol{P}$. We therefore introduce as a measure of the twist the expression [8]

$$\boldsymbol{X} = \boldsymbol{P}\boldsymbol{R} - \boldsymbol{R}\boldsymbol{P}^t = \boldsymbol{R}(\boldsymbol{H} - \boldsymbol{H}^t)\boldsymbol{R}$$

$$= \tfrac{1}{4}\boldsymbol{G}_1^{-1}(\boldsymbol{H} - \boldsymbol{H}^t)\boldsymbol{G}_1^{-1} = \tfrac{1}{4}(\boldsymbol{H} - \boldsymbol{H}^t)\det \boldsymbol{G}_1^{-1}. \tag{6.19}$$

We are now interested in the propagation of the twist parameter through first-order optical systems. It can be shown [8] that, while $\boldsymbol{X}_o = \boldsymbol{P}_o\boldsymbol{R}_o - \boldsymbol{R}_o\boldsymbol{P}_o^t = \frac{1}{4}\boldsymbol{G}_1^{-1}(\boldsymbol{H} - \boldsymbol{H}^t)\boldsymbol{G}_1^{-1} = \frac{1}{4}(\boldsymbol{H} - \boldsymbol{H}^t)\det \boldsymbol{G}_1^{-1}$ is a measure of the twist in the output plane [cf. Eq. (6.19)], for the twist $\boldsymbol{X}_i = \boldsymbol{P}_i\boldsymbol{R}_i - \boldsymbol{R}_i\boldsymbol{P}_i^t$ in the input plane we have

$$4\,\boldsymbol{X}_i = (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{H}^t)\boldsymbol{G}_1^{-1}(\boldsymbol{H} - \boldsymbol{H}^t)\boldsymbol{G}_1^{-1}(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{H}^t)^t$$

$$-(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{H}^t)\boldsymbol{G}_1^{-1}\boldsymbol{G}_2\boldsymbol{B}^t + \boldsymbol{B}\boldsymbol{G}_2\boldsymbol{G}_1^{-1}(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{H}^t)^t. \tag{6.20}$$

The above equation expresses the input twist $\boldsymbol{X}_i$ in terms of the output-plane matrices $\boldsymbol{G}_1$, $\boldsymbol{G}_2$, and $\boldsymbol{H}$. We could as well express the output twist $\boldsymbol{X}_o$ in terms of input-plane matrices, which would lead to the equation

$$4\,\boldsymbol{X}_o = (\boldsymbol{D} - \boldsymbol{H}\boldsymbol{B})^t\boldsymbol{G}_1^{-1}(\boldsymbol{H} - \boldsymbol{H}^t)\boldsymbol{G}_1^{-1}(\boldsymbol{D} - \boldsymbol{H}\boldsymbol{B})$$

$$+(\boldsymbol{D} - \boldsymbol{H}\boldsymbol{B})^t\boldsymbol{G}_1^{-1}\boldsymbol{G}_2\boldsymbol{B} - \boldsymbol{B}^t\boldsymbol{G}_2\boldsymbol{G}_1^{-1}(\boldsymbol{D} - \boldsymbol{H}\boldsymbol{B}), \tag{6.21}$$

where $\boldsymbol{G}_1$, $\boldsymbol{G}_2$, and $\boldsymbol{H}$ now represent input-plane matrices. The general expressions (6.20) and (6.21) show that the twist in one plane is determined not only by the twist in the other plane, but also by other combinations of the moments.

Starting from the relationships (6.20) and (6.21) we can conclude the following [8].

- For the special first-order optical $\boldsymbol{ABCD}$ system for which $\boldsymbol{B} = \boldsymbol{0}$ – i.e., if we consider propagation of light between conjugate planes – the twist in one plane is determined by the twist in the other, and not by other combinations of the moments:

$$\boldsymbol{X}_i = \boldsymbol{A}\boldsymbol{X}_o\boldsymbol{A}^t = \boldsymbol{X}_o \det \boldsymbol{A}, \tag{6.22}$$

$$\boldsymbol{X}_o = \boldsymbol{D}^t\boldsymbol{X}_i\boldsymbol{D} = \boldsymbol{X}_o \det \boldsymbol{D}. \tag{6.23}$$

  Moreover, the relationship between the twists is linear, and a zero-twist signal in one plane thus corresponds to a zero-twist signal in the other plane.
- For a general first-order optical $\boldsymbol{ABCD}$ system, zero twist is not preserved. For any first-order optical system (with the mild condition $\det \boldsymbol{B} \neq \boldsymbol{0}$) there does exist, however, a zero-twist signal that is adapted to the system in the sense that it corresponds to a zero-twist signal in the other plane: $\boldsymbol{X}_i = \boldsymbol{X}_o = \boldsymbol{0}$. The adaptation only affects the matrix $\boldsymbol{H}$:

$$\boldsymbol{A} + \boldsymbol{B}\boldsymbol{H}_o^t = \boldsymbol{0}, \tag{6.24}$$

$$\boldsymbol{D} - \boldsymbol{H}_i\boldsymbol{B} = \boldsymbol{0}; \tag{6.25}$$

  the matrices $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ are not affected:
- Zero twist is preserved if we put some additional requirements to the matrices $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$; in particular we require that the moment matrix is symplectic. In that case zero twist is preserved (as is symplecticity of the moment matrix, in general) in any first-order optical system.

6.3 Invariants for the second-order moments

From the invariance of the eigenvalues of $\boldsymbol{MJ}$ we can derive invariants for the second-order moments of the Wigner distribution function. Let us consider this in more detail.

6.3.1 1-dimensional case. In the *one-dimensional case*, the 2×2 matrix of second-order moments $\boldsymbol{M}$ takes the form

$$\boldsymbol{M} = \begin{bmatrix} m_{xx} & m_{xu} \\ m_{xu} & m_{uu} \end{bmatrix}. \tag{6.26}$$

The eigenvalues λ of the matrix $\boldsymbol{MJ}$ follow from the characteristic equation

$$0 = \det(\boldsymbol{MJ} - \lambda \boldsymbol{I}) = \lambda^2 - (m_{xx}m_{uu} - m_{xu}^2) = \lambda^2 - \det \boldsymbol{M}. \tag{6.27}$$

The eigenvalues read $\lambda^{\pm} = \pm\sqrt{\det \boldsymbol{M}}$, and we conclude that $\det \boldsymbol{M} = m_{xx}m_{uu} - m_{xu}^2$ is an invariant.

6.3.2 2-dimensional case. In the *two-dimensional case*, the 4×4 matrix $\boldsymbol{M}$ takes the form

$$\boldsymbol{M} = \begin{bmatrix} m_{xx} & m_{xy} & m_{xu} & m_{xv} \\ m_{xy} & m_{yy} & m_{yu} & m_{yv} \\ m_{xu} & m_{yu} & m_{uu} & m_{uv} \\ m_{xv} & m_{yv} & m_{uv} & m_{vv} \end{bmatrix}. \tag{6.28}$$

The eigenvalues λ of $\boldsymbol{MJ}$ follow again from the characteristic equation which now reads

$$\begin{aligned} 0 &= \det(\boldsymbol{MJ} - \lambda \boldsymbol{I}) \\ &= \lambda^4 - [(m_{xx}m_{uu} - m_{xu}^2) + (m_{yy}m_{vv} - m_{yv}^2) \\ &\quad +2(m_{xy}m_{uv} - m_{xv}m_{yu})]\lambda^2 + \det \boldsymbol{M}. \end{aligned} \tag{6.29}$$

We remark that the constant that arises with the term λ^2 is the sum of four of the 2×2 minors of $\boldsymbol{M}$:

$$\begin{aligned} &\begin{vmatrix} m_{xx} & m_{xu} \\ m_{xu} & m_{uu} \end{vmatrix} + \begin{vmatrix} m_{yy} & m_{yv} \\ m_{yv} & m_{vv} \end{vmatrix} \\ + &\begin{vmatrix} m_{xy} & m_{xv} \\ m_{yu} & m_{uv} \end{vmatrix} + \begin{vmatrix} m_{xy} & m_{yu} \\ m_{xv} & m_{uv} \end{vmatrix}. \end{aligned} \tag{6.30}$$

Since the eigenvalues are invariant under propagation through a first-order system, the same holds for any combination of the eigenvalues, in particular for the constants that arise in the characteristic polynomial. Therefore, not only the determinant of $\boldsymbol{M}$ is invariant under propagation – as it was in the one-dimensional case – but also the sum (6.30) of the four minors of $\boldsymbol{M}$ mentioned above. See also [66].

7 CONCLUSION

We have introduced the (real-valued) Wigner distribution function of partially coherent light as the Fourier transform of its (Hermitian) mutual power spectrum with respect to the difference coordinate. We have shown that the Wigner distribution function leads to a physically attractive description of partially coherent light, and that its properties and its propagation through linear systems can directly be interpreted in terms of such well-known concepts as radiometry, ray optics, matrix optics, geometrical optics, etc. Special attention has been paid to the second-order moments of partially coherent light beams.

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