On the propagation of the twist of Gaussian light in first-order optical systems

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A measure for the twist of Gaussian light is expressed in terms of the second-order moments of the Wigner distribution function. The propagation law for these second-order moments between the input plane and the output plane of a first-order optical system is used to express the twist in one plane in terms of moments in the other plane. Although in general the twist in one plane is determined not only by the twist in the other plane, but also by other combinations of the moments, several special cases are considered for which a direct relationship between the twists can be formulated. In particular it is shown under what conditions zero twist is preserved in a first-order optical system.

1 INTRODUCTION

In recent years there has been some interest in the twist of Gaussian light [1, 2, 3, 4, 5, 6]. In this paper we will consider the propagation of this twist in first-order optical systems and we will use the Wigner distribution function as a mathematical tool to do so.

In Section 2 we will first represent Gaussian light by means of its cross-spectral density. From this cross-spectral density, we will derive the Wigner distribution function in Section 3. Moreover, we will introduce the moments of the Wigner distribution function and we will define a measure for the twist, based on these moments. In Section 4 we introduce a first-order optical system, and describe its input-output relationship in terms of a ray transformation matrix. The propagation of the moments of the Wigner distribution function through first-order optical systems is presented in Section 5, which section will lead to a general relationship between the twist in the output plane and the moments in the input plane, and vice versa. Although the general relationships do not show an easy interpretation, they will form a basis to consider some special cases in Section 6.

Before we start, we make some remarks about notation. We will throughout consider an optical signal in a plane z = constant; the signal thus depends on the transverse coordinates x and y, only, which coordinates are combined into a 2-dimensional column vector $\boldsymbol{r}$. The spatial-frequency variable will be denoted by the 2-dimensional column vector $\boldsymbol{q}$, whereas the temporal-frequency dependence of the optical signal will not be taken into account, since it is not relevant in the context of this paper. While vectors will be denoted by bold-face, lower-case characters, matrices will throughout be denoted by bold-face, upper-case characters. Finally, the superscript ^{t} will be used to denote transposition of vectors and matrices.

2 CROSS-SPECTRAL DENSITY OF GAUSSIAN LIGHT

The cross-spectral density [7, 8, 9] of the most general partially coherent Gaussian light can be written in the form [10]

$$\Gamma(\boldsymbol{r}_1,\boldsymbol{r}_2) = \frac{1}{\pi}\sqrt{\det \boldsymbol{G}_1} \times \exp\left(-\tfrac{1}{4}\begin{bmatrix} \boldsymbol{r}_1+\boldsymbol{r}_2 \\ \boldsymbol{r}_1-\boldsymbol{r}_2 \end{bmatrix}^t \begin{bmatrix} \boldsymbol{G}_1 & -i\boldsymbol{H} \\ -i\boldsymbol{H}^t & \boldsymbol{G}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{r}_1+\boldsymbol{r}_2 \\ \boldsymbol{r}_1-\boldsymbol{r}_2 \end{bmatrix}\right), \tag{1}$$

where we have chosen a representation that enables us to determine the Wigner distribution function of such light in an easy way. The exponent shows a quadratic form in which a 4-dimensional column vector $[(\boldsymbol{r}_1+\boldsymbol{r}_2)^t, (\boldsymbol{r}_1-\boldsymbol{r}_2)^t]^t$ arises, together with a 4×4 symmetric matrix. This matrix consists of four real 2×2 submatrices $\boldsymbol{G}_1$, $\boldsymbol{G}_2$, $\boldsymbol{H}$, and $\boldsymbol{H}^t$, where, moreover, the matrices $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ (as well as the matrix $\boldsymbol{G}_2-\boldsymbol{G}_1$) are positive definite symmetric. The special form of the matrix is a direct consequence of the fact that the cross-spectral density is a nonnegative definite Hermitian function [8, 9].

In a more common way, the cross-spectral density of Gaussian light can be expressed in the form

$$\begin{aligned}\Gamma(\boldsymbol{r}_1,\boldsymbol{r}_2) \;=\; & \frac{1}{\pi}\sqrt{\det \boldsymbol{G}_1} \\ & \times \exp\{-\tfrac{1}{4}(\boldsymbol{r}_1-\boldsymbol{r}_2)^t(\boldsymbol{G}_2-\boldsymbol{G}_1)(\boldsymbol{r}_1-\boldsymbol{r}_2)\} \\ & \times \exp\{-\tfrac{1}{2}\boldsymbol{r}_1^t[\boldsymbol{G}_1-i\tfrac{1}{2}(\boldsymbol{H}+\boldsymbol{H}^t)]\boldsymbol{r}_1\} \\ & \times \exp\{-\tfrac{1}{2}\boldsymbol{r}_2^t[\boldsymbol{G}_1+i\tfrac{1}{2}(\boldsymbol{H}+\boldsymbol{H}^t)]\boldsymbol{r}_2\} \\ & \times \exp\{-\tfrac{1}{2}\boldsymbol{r}_1^t i(\boldsymbol{H}-\boldsymbol{H}^t)\boldsymbol{r}_2\}. \end{aligned} \tag{2}$$

Note that the asymmetry of the matrix $\boldsymbol{H}$ is a measure for the twist [1, 2, 3, 4, 5, 6] of Gaussian light, and that general Gaussian light reduces to zero-twist Gaussian Schell-model light [11, 12], if the matrix $\boldsymbol{H}$ is symmetric, $\boldsymbol{H}-$

$\boldsymbol{H}^t = \boldsymbol{0}$. In that case the light can be considered as spatially stationary light with a Gaussian cross-spectral density $(1/\pi)\sqrt{\det \boldsymbol{G}_1}\exp\{-\frac{1}{4}(\boldsymbol{r}_1-\boldsymbol{r}_2)^t(\boldsymbol{G}_2-\boldsymbol{G}_1)(\boldsymbol{r}_1-\boldsymbol{r}_2)\}$, modulated by a Gaussian modulator with modulation function $\exp\{-\frac{1}{2}\boldsymbol{r}^t(\boldsymbol{G}_1-i\boldsymbol{H})\boldsymbol{r}\}$.

3 WIGNER DISTRIBUTION FUNCTION OF GAUSSIAN LIGHT

We now introduce the Wigner distribution function [13], which is defined as the spatial Fourier transform of the cross-spectral density $\Gamma(\boldsymbol{r}+\frac{1}{2}\boldsymbol{r}',\boldsymbol{r}-\frac{1}{2}\boldsymbol{r}')$ with respect to the difference coordinate $\boldsymbol{r}'$ [10, 14, 15]:

$$F(\boldsymbol{r},\boldsymbol{q}) = \int_{-\infty}^{+\infty}\Gamma(\boldsymbol{r}+\tfrac{1}{2}\boldsymbol{r}',\boldsymbol{r}-\tfrac{1}{2}\boldsymbol{r}')\exp\{-i\boldsymbol{q}^t\boldsymbol{r}'\}d\boldsymbol{r}'. \quad (3)$$

Note that, due to the Hermitian character $\Gamma(\boldsymbol{r}_1,\boldsymbol{r}_2) = \Gamma^*(\boldsymbol{r}_2,\boldsymbol{r}_1)$ of the cross-spectral density, the Wigner distribution function is real, $F(\boldsymbol{r},\boldsymbol{q}) = F^*(\boldsymbol{r},\boldsymbol{q})$.

Whereas the cross-spectral density $\Gamma(\boldsymbol{r}_1,\boldsymbol{r}_2)$ describes an optical signal in the space domain, the Wigner distribution function $F(\boldsymbol{r},\boldsymbol{q})$ describes the signal in a so-called phase space, i.e., a combined space-frequency domain. The Wigner distribution function thus resembles the ray concept in geometrical optics, where the position and direction of an optical ray are given simultaneously, too. In a way, $F(\boldsymbol{r},\boldsymbol{q})$ is the amplitude of a ray that passes through the position $\boldsymbol{r}$ and has a direction (or spatial frequency) $\boldsymbol{q}$.

Upon substituting from the cross-spectral density (1) into Eq. (3), it can easily be shown that the Wigner distribution function of Gaussian light takes the form [10, 16, 17]

$$F(\boldsymbol{r},\boldsymbol{q}) = 4\sqrt{\frac{\det \boldsymbol{G}_1}{\det \boldsymbol{G}_2}}$$

$$\exp\left(-\begin{bmatrix}\boldsymbol{r}\\ \boldsymbol{q}\end{bmatrix}^t\begin{bmatrix}\boldsymbol{G}_1+\boldsymbol{H}\boldsymbol{G}_2^{-1}\boldsymbol{H}^t & -\boldsymbol{H}\boldsymbol{G}_2^{-1}\\ -\boldsymbol{G}_2^{-1}\boldsymbol{H}^t & \boldsymbol{G}_2^{-1}\end{bmatrix}\begin{bmatrix}\boldsymbol{r}\\ \boldsymbol{q}\end{bmatrix}\right). \quad (4)$$

The 4×4 real symmetric matrix $\boldsymbol{M}$ of normalized second-order moments of the Wigner distribution function is defined by [10, 15, 17]

$$\boldsymbol{M} = \frac{\displaystyle\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\begin{bmatrix}\boldsymbol{r}\boldsymbol{r}^t & \boldsymbol{r}\boldsymbol{q}^t\\ \boldsymbol{q}\boldsymbol{r}^t & \boldsymbol{q}\boldsymbol{q}^t\end{bmatrix}F(\boldsymbol{r},\boldsymbol{q})d\boldsymbol{r}d\frac{\boldsymbol{q}}{2\pi}}{\displaystyle\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}F(\boldsymbol{r},\boldsymbol{q})d\boldsymbol{r}d\frac{\boldsymbol{q}}{2\pi}}. \quad (5)$$

It yields not only such quantities as the commonly known effective spatial width of a light beam, corresponding to the second-order moments of the spatial variable $\boldsymbol{r}$, and the effective directional width of the beam, corresponding to the second-order moments of the directional variable $\boldsymbol{q}$, but also mixed quantities, which follow from the products $\boldsymbol{r}\boldsymbol{q}^t$ and $\boldsymbol{q}\boldsymbol{r}^t$. It can be shown [10] that the symmetric moment matrix $\boldsymbol{M}$ is positive definite; an easy proof of this property will be presented in Section 5, following Eq. (16).

In the case of Gaussian light the moment matrix $\boldsymbol{M}$ takes the form

$$\begin{aligned}\boldsymbol{M} &= \begin{bmatrix}\boldsymbol{R} & \boldsymbol{P}\\ \boldsymbol{P}^t & \boldsymbol{Q}\end{bmatrix}\\ &= \frac{1}{2}\begin{bmatrix}\boldsymbol{G}_1+\boldsymbol{H}\boldsymbol{G}_2^{-1}\boldsymbol{H}^t & -\boldsymbol{H}\boldsymbol{G}_2^{-1}\\ -\boldsymbol{G}_2^{-1}\boldsymbol{H}^t & \boldsymbol{G}_2^{-1}\end{bmatrix}^{-1}\\ &= \frac{1}{2}\begin{bmatrix}\boldsymbol{G}_1^{-1} & \boldsymbol{G}_1^{-1}\boldsymbol{H}\\ \boldsymbol{H}^t\boldsymbol{G}_1^{-1} & \boldsymbol{G}_2+\boldsymbol{H}^t\boldsymbol{G}_1^{-1}\boldsymbol{H}\end{bmatrix}. \end{aligned} \quad (6)$$

Note that the positive definiteness of the matrix $\boldsymbol{M}$ leads to the positivity of the quadratic form

$$2\begin{bmatrix}\boldsymbol{q}\\ \boldsymbol{r}\end{bmatrix}^t\boldsymbol{M}\begin{bmatrix}\boldsymbol{q}\\ \boldsymbol{r}\end{bmatrix} = \boldsymbol{r}^t\boldsymbol{G}_2\boldsymbol{r} + (\boldsymbol{q}+\boldsymbol{H}\boldsymbol{r})^t\boldsymbol{G}_1^{-1}(\boldsymbol{q}+\boldsymbol{H}\boldsymbol{r}) \quad (7)$$

for any vectors $\boldsymbol{r}$ and $\boldsymbol{q}$, which immediately leads to the positive definiteness of the matrices $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$, as already mentioned.

From Eq. (6) we conclude that the matrices $\boldsymbol{G}_1$, $\boldsymbol{G}_2$, and $\boldsymbol{H}$ follow directly from the moment matrices $\boldsymbol{R}$, $\boldsymbol{Q}$, and $\boldsymbol{P}$; in particular we have the relationships

$$\boldsymbol{P} = \boldsymbol{R}\boldsymbol{H} \quad \text{and} \quad \boldsymbol{P}^t = \boldsymbol{H}^t\boldsymbol{R}. \quad (8)$$

As a consequence we can express the twistedness of the Gaussian light directly in terms of the moment matrices $\boldsymbol{R}$ and $\boldsymbol{P}$, and we therefore introduce as a measure $\boldsymbol{X}$ for the twist the expression

$$\begin{aligned}\boldsymbol{X} = \boldsymbol{P}\boldsymbol{R} - \boldsymbol{R}\boldsymbol{P}^t = \boldsymbol{R}(\boldsymbol{H}-\boldsymbol{H}^t)\boldsymbol{R}\\ = \tfrac{1}{4}\boldsymbol{G}_1^{-1}(\boldsymbol{H}-\boldsymbol{H}^t)\boldsymbol{G}_1^{-1} = \tfrac{1}{4}(\boldsymbol{H}-\boldsymbol{H}^t)\det \boldsymbol{G}_1^{-1}.\end{aligned} \quad (9)$$

4 DESCRIPTION OF A FIRST-ORDER OPTICAL SYSTEM

In this section we will consider the propagation of Gaussian light through a first-order optical system [18], described by the input-output relationship [10, 15]

$$F_o(\boldsymbol{r},\boldsymbol{q}) = F_i(\boldsymbol{A}\boldsymbol{r}+\boldsymbol{B}\boldsymbol{q},\boldsymbol{C}\boldsymbol{r}+\boldsymbol{D}\boldsymbol{q}), \quad (10)$$

where $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ are real 2×2 matrices. For such a first-order optical system a single input ray, entering the system at the position $\boldsymbol{r}_i$ with direction $\boldsymbol{q}_i$, yields a single output ray, leaving the system at the position $\boldsymbol{r}_o$ with direction $\boldsymbol{q}_o$. The input and output positions and directions are related by the matrix relationship [18]

$$\begin{bmatrix}\boldsymbol{r}_i\\ \boldsymbol{q}_i\end{bmatrix} = \begin{bmatrix}\boldsymbol{A} & \boldsymbol{B}\\ \boldsymbol{C} & \boldsymbol{D}\end{bmatrix}\begin{bmatrix}\boldsymbol{r}_o\\ \boldsymbol{q}_o\end{bmatrix}. \quad (11)$$

The latter relation is a well-known geometric-optical matrix description of a first-order optical system; the 4×4 $\boldsymbol{ABCD}$ matrix in this relationship is known as the ray transformation matrix [19].

The ray transformation matrix is symplectic [10, 15, 18, 19]. To express symplecticity in an easy way, we introduce the 4×4 matrix $\boldsymbol{J}$ according to

$$\boldsymbol{J} = i\begin{bmatrix}\boldsymbol{0} & -\boldsymbol{I}\\ \boldsymbol{I} & \boldsymbol{0}\end{bmatrix}, \quad (12)$$

where $\boldsymbol{I}$ denotes the 2×2 identity matrix. The matrix $\boldsymbol{J}$ has the properties $\boldsymbol{J}=\boldsymbol{J}^{-1}=\boldsymbol{J}^{\dagger}=-\boldsymbol{J}^{t}$, where $\boldsymbol{J}^{\dagger}$ is the adjoint of $\boldsymbol{J}$. Symplecticity of the ray transformation matrix can then be expressed by the relationship

$$\boldsymbol{T}^{-1}=\boldsymbol{J}\boldsymbol{T}^{t}\boldsymbol{J}, \tag{13}$$

where we have used the short-hand notation for the 4×4 ray transformation matrix

$$\boldsymbol{T}=\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}. \tag{14}$$

In terms of the submatrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$, symplecticity implies the relations

$$\boldsymbol{A}\boldsymbol{B}^{t}=\boldsymbol{B}\boldsymbol{A}^{t},\quad \boldsymbol{B}^{t}\boldsymbol{D}=\boldsymbol{D}^{t}\boldsymbol{B},\quad \boldsymbol{D}\boldsymbol{C}^{t}=\boldsymbol{C}\boldsymbol{D}^{t},\quad \boldsymbol{C}^{t}\boldsymbol{A}=\boldsymbol{A}^{t}\boldsymbol{C},$$
$$\text{and}\quad \boldsymbol{A}\boldsymbol{D}^{t}-\boldsymbol{B}\boldsymbol{C}^{t}=\boldsymbol{I}=\boldsymbol{A}^{t}\boldsymbol{D}-\boldsymbol{C}^{t}\boldsymbol{B}. \tag{15}$$

5 PROPAGATION OF THE MOMENTS AND THE TWIST

The propagation of the moment matrix $\boldsymbol{M}$ through a first-order optical system with ray transformation matrix $\boldsymbol{T}$ can be described by the relationship [10, 15, 17, 20, 21]

$$\boldsymbol{M}_i=\boldsymbol{T}\boldsymbol{M}_o\boldsymbol{T}^{t}. \tag{16}$$

This relationship can readily be derived by combining the input-output relationship (11) of the first-order optical system with the definition (5) of the moment matrices $\boldsymbol{M}_i$ and $\boldsymbol{M}_o$ of the input and the output signal, respectively.

We remark that we can easily prove the positive definiteness of the moment matrix $\boldsymbol{M}$ with the help of the input-output relationship (16). We therefore choose an $\boldsymbol{ABCD}$ matrix as follows:

- the matrix entries a_{11}, a_{12}, b_{11}, and b_{12} are chosen arbitrarily, and are combined into the 4-dimensional column vector $\boldsymbol{t}=[a_{11},a_{12},b_{11},b_{12}]^{t}$;
- the matrix entries a_{21} and b_{21} are chosen equal to zero;
- the matrix entries a_{22} and b_{22} are chosen such that $a_{12}b_{22}=a_{22}b_{12}$;
- $\boldsymbol{C}=\boldsymbol{0}$; and
- $\boldsymbol{D}=(\boldsymbol{A}^{t})^{-1}$.

It can easily be seen that such an $\boldsymbol{ABCD}$ matrix is symplectic and, as a consequence, can be interpreted as the ray transformation matrix of a physically realizable first-order optical system. We now consider the upper left entry of the matrix $\boldsymbol{M}_i$ in the left hand side of Eq. (16). This entry, being on the main diagonal of $\boldsymbol{M}_i$ and thus representing the square of an effective width, is positive. On the other hand, this entry equals $\boldsymbol{t}^{t}\boldsymbol{M}_o\boldsymbol{t}$, where the vector $\boldsymbol{t}$ can be chosen arbitrarily. We thus conclude that the quadratic form $\boldsymbol{t}^{t}\boldsymbol{M}_o\boldsymbol{t}$ is positive for any vector $\boldsymbol{t}$, with which we have proved that the real symmetric moment matrix $\boldsymbol{M}$ is positive definite.

In terms of the submatrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$, $\boldsymbol{P}$, $\boldsymbol{Q}$, and $\boldsymbol{R}$, Eq. (16) takes the form

$$\begin{bmatrix} \boldsymbol{R}_i & \boldsymbol{P}_i \\ \boldsymbol{P}_i^{t} & \boldsymbol{Q}_i \end{bmatrix}=\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}\begin{bmatrix} \boldsymbol{R}_o & \boldsymbol{P}_o \\ \boldsymbol{P}_o^{t} & \boldsymbol{Q}_o \end{bmatrix}\begin{bmatrix} \boldsymbol{A}^{t} & \boldsymbol{C}^{t} \\ \boldsymbol{B}^{t} & \boldsymbol{D}^{t} \end{bmatrix} \tag{17}$$

and hence

$$\begin{aligned} \boldsymbol{R}_i &= \boldsymbol{A}\boldsymbol{R}_o\boldsymbol{A}^{t}+\boldsymbol{A}\boldsymbol{P}_o\boldsymbol{B}^{t}+\boldsymbol{B}\boldsymbol{P}_o^{t}\boldsymbol{A}^{t}+\boldsymbol{B}\boldsymbol{Q}_o\boldsymbol{B}^{t} \\ \boldsymbol{P}_i &= \boldsymbol{A}\boldsymbol{R}_o\boldsymbol{C}^{t}+\boldsymbol{A}\boldsymbol{P}_o\boldsymbol{D}^{t}+\boldsymbol{B}\boldsymbol{P}_o^{t}\boldsymbol{C}^{t}+\boldsymbol{B}\boldsymbol{Q}_o\boldsymbol{D}^{t} \\ \boldsymbol{P}_i^{t} &= \boldsymbol{C}\boldsymbol{R}_o\boldsymbol{A}^{t}+\boldsymbol{C}\boldsymbol{P}_o\boldsymbol{B}^{t}+\boldsymbol{D}\boldsymbol{P}_o^{t}\boldsymbol{A}^{t}+\boldsymbol{D}\boldsymbol{Q}_o\boldsymbol{B}^{t} \\ \boldsymbol{Q}_i &= \boldsymbol{C}\boldsymbol{R}_o\boldsymbol{C}^{t}+\boldsymbol{C}\boldsymbol{P}_o\boldsymbol{D}^{t}+\boldsymbol{D}\boldsymbol{P}_o^{t}\boldsymbol{C}^{t}+\boldsymbol{D}\boldsymbol{Q}_o\boldsymbol{D}^{t}. \end{aligned} \tag{18}$$

While $\boldsymbol{X}_o=\boldsymbol{P}_o\boldsymbol{R}_o-\boldsymbol{R}_o\boldsymbol{P}_o^{t}=\frac{1}{4}\boldsymbol{G}_1^{-1}(\boldsymbol{H}-\boldsymbol{H}^{t})\boldsymbol{G}_1^{-1}=\frac{1}{4}(\boldsymbol{H}-\boldsymbol{H}^{t})\det\boldsymbol{G}_1^{-1}$ is a measure of the twist in the output plane [cf. Eq. (9)], for the twist $\boldsymbol{X}_i=\boldsymbol{P}_i\boldsymbol{R}_i-\boldsymbol{R}_i\boldsymbol{P}_i^{t}$ in the input plane we have (see the Appendix)

$$4\,\boldsymbol{X}_i=(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{H}^{t})\boldsymbol{G}_1^{-1}(\boldsymbol{H}-\boldsymbol{H}^{t})\boldsymbol{G}_1^{-1}(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{H}^{t})^{t}$$
$$-(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{H}^{t})\boldsymbol{G}_1^{-1}\boldsymbol{G}_2\boldsymbol{B}^{t}+\boldsymbol{B}\boldsymbol{G}_2\boldsymbol{G}_1^{-1}(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{H}^{t})^{t}. \tag{19}$$

The above equation expresses the input twist $\boldsymbol{X}_i$ in terms of the output-plane matrices $\boldsymbol{G}_1$, $\boldsymbol{G}_2$, and $\boldsymbol{H}$. We could as well express the output twist $\boldsymbol{X}_o$ in terms of input-plane matrices, which would lead to the equation

$$4\,\boldsymbol{X}_o=(\boldsymbol{D}-\boldsymbol{H}\boldsymbol{B})^{t}\boldsymbol{G}_1^{-1}(\boldsymbol{H}-\boldsymbol{H}^{t})\boldsymbol{G}_1^{-1}(\boldsymbol{D}-\boldsymbol{H}\boldsymbol{B})$$
$$+(\boldsymbol{D}-\boldsymbol{H}\boldsymbol{B})^{t}\boldsymbol{G}_1^{-1}\boldsymbol{G}_2\boldsymbol{B}-\boldsymbol{B}^{t}\boldsymbol{G}_2\boldsymbol{G}_1^{-1}(\boldsymbol{D}-\boldsymbol{H}\boldsymbol{B}), \tag{20}$$

where $\boldsymbol{G}_1$, $\boldsymbol{G}_2$, and $\boldsymbol{H}$ now represent input-plane matrices.

6 SPECIAL CASES

The relationships (19) and (20) show that in general the twist in one plane is determined not only by the twist in the other plane, i.e., by the combination $\boldsymbol{G}_1^{-1}(\boldsymbol{H}-\boldsymbol{H}^{t})\boldsymbol{G}_1^{-1}$, but by other combinations of the matrices $\boldsymbol{G}_1$, $\boldsymbol{G}_2$, and $\boldsymbol{H}$, as well. However, there are some special cases in which a more direct relationship between the twists in the input and the output plane can be formulated.

6.1 The case $\boldsymbol{B}=\boldsymbol{0}$

For the special first-order optical system for which $\boldsymbol{B}=\boldsymbol{0}$ [and also, due to the condition of symplecticity, $\boldsymbol{A}^{t}\boldsymbol{D}=\boldsymbol{I}$ and $\boldsymbol{C}^{t}\boldsymbol{A}=\boldsymbol{A}^{t}\boldsymbol{C}$, see Eq. (15)], we have a simple relationship between the twists $\boldsymbol{X}_i$ and $\boldsymbol{X}_o$, reading

$$\boldsymbol{X}_i=\boldsymbol{A}\boldsymbol{X}_o\boldsymbol{A}^{t}=\boldsymbol{X}_o\det\boldsymbol{A} \tag{21}$$

or, equivalently,

$$\boldsymbol{X}_o=\boldsymbol{D}^{t}\boldsymbol{X}_i\boldsymbol{D}=\boldsymbol{X}_i\det\boldsymbol{D}, \tag{22}$$

which relations follow immediately from the general expressions (19) and (20), respectively. Note that the case $\boldsymbol{B}=\boldsymbol{0}$ applies when we consider propagation between two conjugate planes [22].

6.2 The case $\det \boldsymbol{B} \neq 0$

If $\boldsymbol{B} \neq \boldsymbol{0}$, simple relationships of the form (21) and (22) no longer exist. However, there exists a certain signal for which, again, a special relationship can be formulated.

In the special case that the output signal (with matrix $\boldsymbol{H}_o$) is adapted to the system according to

$$\boldsymbol{A} + \boldsymbol{B}\boldsymbol{H}_o^t = \boldsymbol{0}, \tag{23}$$

Eq. (19) immediately shows that the input twist is zero, $\boldsymbol{X}_i = \boldsymbol{0}$. We remark that, due to the symplecticity condition $\boldsymbol{A}\boldsymbol{B}^t = \boldsymbol{B}\boldsymbol{A}^t$, the adaptation condition (23) implies $\boldsymbol{B}(\boldsymbol{H}_o - \boldsymbol{H}_o^t)\boldsymbol{B}^t = \boldsymbol{0}$ and hence $\boldsymbol{X}_o \det \boldsymbol{B} = \boldsymbol{0}$; note that $\boldsymbol{X}_o = \boldsymbol{0}$ if $\det \boldsymbol{B} \neq 0$. Similar results follow from Eq. (20) and the symplecticity condition $\boldsymbol{B}^t\boldsymbol{D} = \boldsymbol{D}^t\boldsymbol{B}$, for the case that the input signal (with matrix $\boldsymbol{H}_i$) is adapted to the system according to

$$\boldsymbol{D} - \boldsymbol{H}_i\boldsymbol{B} = \boldsymbol{0}, \tag{24}$$

in which case we have $\boldsymbol{X}_o = \boldsymbol{0}$ and $\boldsymbol{X}_i \det \boldsymbol{B} = \boldsymbol{0}$.

Let us assume that the output signal is adapted. If we substitute from Eq. (23) and from the relation $\boldsymbol{P}_o = \boldsymbol{R}_o\boldsymbol{H}_o$ [cf. Eq. (8)] into Eqs. (18), we get

$$\begin{aligned} \boldsymbol{R}_i &= \boldsymbol{B}\boldsymbol{Q}_o\boldsymbol{B}^t - \boldsymbol{A}\boldsymbol{R}_o\boldsymbol{A}^t \\ \boldsymbol{P}_i &= \boldsymbol{B}\boldsymbol{Q}_o\boldsymbol{D}^t + \boldsymbol{A}\boldsymbol{R}_o\boldsymbol{H}_o\boldsymbol{D}^t. \end{aligned} \tag{25}$$

Using Eq. (23) and the symplecticity condition $\boldsymbol{D}^t\boldsymbol{B} = \boldsymbol{B}^t\boldsymbol{D}$ again, we are immediately led to the relationship

$$\boldsymbol{P}_i\boldsymbol{B} = \boldsymbol{R}_i\boldsymbol{D}. \tag{26}$$

With $\boldsymbol{P}_i = \boldsymbol{R}_i\boldsymbol{H}_i$ [cf. Eq. (8)], Eq. (26) takes the form $\boldsymbol{R}_i(\boldsymbol{D} - \boldsymbol{H}_i\boldsymbol{B}) = \boldsymbol{0}$, and we conclude that the input signal is adapted as well, see Eq. (24).

We are thus led to the important conclusion that for every first-order optical system (with the mild condition $\det \boldsymbol{B} \neq 0$) there exists a signal which is adapted in both the input and the output plane and has zero twist in these planes. In general, however, a zero-twist signal in one plane does not correspond to a zero-twist signal in the other plane.

6.3 The case $\boldsymbol{G}_1 = \sigma^2\boldsymbol{G}_2$

In the special case that $\boldsymbol{G}_1 = \sigma\boldsymbol{G}$ and $\boldsymbol{G} = \sigma\boldsymbol{G}_2$ (with $0 < \sigma \leq 1$), the general relationships (19) and (20) reduce to

$$\boldsymbol{X}_i = (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{H}_o^t)\boldsymbol{X}_o(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{H}_o^t)^t + \boldsymbol{B}\boldsymbol{G}_o\boldsymbol{X}_o\boldsymbol{G}_o\boldsymbol{B}^t \tag{27}$$

and

$$\boldsymbol{X}_o = (\boldsymbol{D} - \boldsymbol{H}_i\boldsymbol{B})^t\boldsymbol{X}_i(\boldsymbol{D} - \boldsymbol{H}_i\boldsymbol{B}) + \boldsymbol{B}^t\boldsymbol{G}_i\boldsymbol{X}_i\boldsymbol{G}_i\boldsymbol{B}, \tag{28}$$

respectively. From the latter relations we conclude that if a signal for which $\boldsymbol{G}_1$ is proportional to $\boldsymbol{G}_2$, has zero twist in the input plane, it will also have zero twist in the output plane, and vice versa.

This conclusion is in accordance with the fact that a signal for which $\boldsymbol{G}_1 = \sigma\boldsymbol{G} = \sigma^2\boldsymbol{G}_2$ and $\boldsymbol{H} = \boldsymbol{H}^t$, leads to a moment matrix $\boldsymbol{M}$ that is proportional to a symplectic matrix with proportionality factor σ, and that symplecticity is preserved when such a moment matrix propagates through a (symplectic) first-order optical $\boldsymbol{ABCD}$ system [10, 17]. Moreover, it has been shown [10, 17] that the proportionality factor σ is a measure of the coherence of the signal, and that the propagation of such a signal is governed by the bilinear relationship

$$\boldsymbol{H}_i \pm i\boldsymbol{G}_i = [\boldsymbol{C} + \boldsymbol{D}(\boldsymbol{H}_o \pm i\boldsymbol{G}_o)][\boldsymbol{A} + \boldsymbol{B}(\boldsymbol{H}_o \pm i\boldsymbol{G}_o)]^{-1} \tag{29}$$

and by the fact that the factor σ remains constant. In the context of this paper it is important to conclude that the (zero) twist of symplectic signals is invariant under propagation through any first-order optical system.

7 CONCLUSION

In this paper we have introduced a measure for the twist of Gaussian light, expressed in terms of the second-order moments of the Wigner distribution function. Using the propagation law for these second-order moments between the input plane and the output plane of a first-order optical $\boldsymbol{ABCD}$ system, we were able to express the twist in one plane in terms of moments in the other plane. These general expressions [see Eqs. (19) and (20)] show that the twist in one plane is determined not only by the twist in the other plane, but also by other combinations of the moments.

Starting from the general relationships (19) and (20) we were able to conclude the following.

- For the special first-order optical $\boldsymbol{ABCD}$ system for which $\boldsymbol{B} = \boldsymbol{0}$ – i.e., if we consider propagation of light between conjugate planes – the twist in one plane is determined by the twist in the other, and not by other combinations of the moments, see Eqs. (21) and (22). Moreover, the relationship between the twists is linear, and a zero-twist signal in one plane thus corresponds to a zero-twist signal in the other plane.

- For a general first-order optical $\boldsymbol{ABCD}$ system, zero twist is not preserved. For any first-order optical system (with the mild condition $\det \boldsymbol{B} \neq \boldsymbol{0}$) there does exist, however, a zero-twist signal that is adapted to the system in the sense that it corresponds to a zero-twist signal in the other plane. The adaptation only affects the matrix $\boldsymbol{H}$; the matrices $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ are not affected, see Eqs. (23) and (24).

- Zero twist is preserved if we put some additional requirements to the matrices $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$; in particular we require that the moment matrix is symplectic. In that case zero twist is preserved (as is symplecticity of the moment matrix, in general) in any first-order optical system.

  Although symplecticity might seem rather special, we should realize that symplectic signals form a large subclass of Gaussian light [10]. Symplecticity applies, for instance, in (i) the completely coherent case, (ii) the (partially coherent) one-dimensional case, and (iii) the (partially coherent) rotationally symmetric case.

APPENDIX. DERIVATION OF EQ. (19)

We substitute from Eq. (18) into $\boldsymbol{X}_i = \boldsymbol{P}_i\boldsymbol{R}_i - \boldsymbol{R}_i\boldsymbol{P}_i^t$ [cf. Eq. (9)] and get

$$\begin{aligned}
\boldsymbol{X}_i &= \boldsymbol{A}\boldsymbol{R}_o(\boldsymbol{C}^t\boldsymbol{A}-\boldsymbol{A}^t\boldsymbol{C})\boldsymbol{R}_o\boldsymbol{A}^t \\
&+ \boldsymbol{A}\boldsymbol{R}_o(\boldsymbol{C}^t\boldsymbol{A}-\boldsymbol{A}^t\boldsymbol{C})\boldsymbol{P}_o\boldsymbol{B}^t \\
&+ \boldsymbol{A}\boldsymbol{R}_o(\boldsymbol{C}^t\boldsymbol{B}-\boldsymbol{A}^t\boldsymbol{D})\boldsymbol{P}_o^t\boldsymbol{A}^t \\
&+ \boldsymbol{A}\boldsymbol{R}_o(\boldsymbol{C}^t\boldsymbol{B}-\boldsymbol{A}^t\boldsymbol{D})\boldsymbol{Q}_o\boldsymbol{B}^t \\
&+ \boldsymbol{A}\boldsymbol{P}_o(\boldsymbol{D}^t\boldsymbol{A}-\boldsymbol{B}^t\boldsymbol{C})\boldsymbol{R}_o\boldsymbol{A}^t \\
&+ \boldsymbol{A}\boldsymbol{P}_o(\boldsymbol{D}^t\boldsymbol{A}-\boldsymbol{B}^t\boldsymbol{C})\boldsymbol{P}_o\boldsymbol{B}^t \\
&+ \boldsymbol{A}\boldsymbol{P}_o(\boldsymbol{D}^t\boldsymbol{B}-\boldsymbol{B}^t\boldsymbol{D})\boldsymbol{P}_o^t\boldsymbol{A}^t \\
&+ \boldsymbol{A}\boldsymbol{P}_o(\boldsymbol{D}^t\boldsymbol{B}-\boldsymbol{B}^t\boldsymbol{D})\boldsymbol{Q}_o\boldsymbol{B}^t \\
&+ \boldsymbol{B}\boldsymbol{P}_o^t(\boldsymbol{C}^t\boldsymbol{A}-\boldsymbol{A}^t\boldsymbol{C})\boldsymbol{R}_o\boldsymbol{A}^t \\
&+ \boldsymbol{B}\boldsymbol{P}_o^t(\boldsymbol{C}^t\boldsymbol{A}-\boldsymbol{A}^t\boldsymbol{C})\boldsymbol{P}_o\boldsymbol{B}^t \\
&+ \boldsymbol{B}\boldsymbol{P}_o^t(\boldsymbol{C}^t\boldsymbol{B}-\boldsymbol{A}^t\boldsymbol{D})\boldsymbol{P}_o^t\boldsymbol{A}^t \\
&+ \boldsymbol{B}\boldsymbol{P}_o^t(\boldsymbol{C}^t\boldsymbol{B}-\boldsymbol{A}^t\boldsymbol{D})\boldsymbol{Q}_o\boldsymbol{B}^t \\
&+ \boldsymbol{B}\boldsymbol{Q}_o(\boldsymbol{D}^t\boldsymbol{A}-\boldsymbol{B}^t\boldsymbol{C})\boldsymbol{R}_o\boldsymbol{A}^t \\
&+ \boldsymbol{B}\boldsymbol{Q}_o(\boldsymbol{D}^t\boldsymbol{A}-\boldsymbol{B}^t\boldsymbol{C})\boldsymbol{P}_o\boldsymbol{B}^t \\
&+ \boldsymbol{B}\boldsymbol{Q}_o(\boldsymbol{D}^t\boldsymbol{B}-\boldsymbol{B}^t\boldsymbol{D})\boldsymbol{P}_o^t\boldsymbol{A}^t \\
&+ \boldsymbol{B}\boldsymbol{Q}_o(\boldsymbol{D}^t\boldsymbol{B}-\boldsymbol{B}^t\boldsymbol{D})\boldsymbol{Q}_o\boldsymbol{B}^t .
\end{aligned}$$

After substituting from the symplecticity conditions (15), the latter expression reduces to

$$\begin{aligned}
\boldsymbol{X}_i &= (\boldsymbol{A}\boldsymbol{P}_o\boldsymbol{R}_o\boldsymbol{A}^t - \boldsymbol{A}\boldsymbol{R}_o\boldsymbol{P}_o^t\boldsymbol{A}^t) \\
&+ (\boldsymbol{A}\boldsymbol{P}_o\boldsymbol{P}_o\boldsymbol{B}^t - \boldsymbol{B}\boldsymbol{P}_o^t\boldsymbol{P}_o^t\boldsymbol{A}^t) \\
&+ (\boldsymbol{B}\boldsymbol{Q}_o\boldsymbol{R}_o\boldsymbol{A}^t - \boldsymbol{A}\boldsymbol{R}_o\boldsymbol{Q}_o\boldsymbol{B}^t) \\
&+ (\boldsymbol{B}\boldsymbol{Q}_o\boldsymbol{P}_o\boldsymbol{B}^t - \boldsymbol{B}\boldsymbol{P}_o^t\boldsymbol{Q}_o\boldsymbol{B}^t).
\end{aligned}$$

We now substitute from the relations $\boldsymbol{R}_o = \frac{1}{2}\boldsymbol{G}_1^{-1}$, $\boldsymbol{Q}_o = \frac{1}{2}(\boldsymbol{G}_2 + \boldsymbol{H}^t\boldsymbol{G}_1^{-1}\boldsymbol{H})$, and $\boldsymbol{P}_o = \frac{1}{2}\boldsymbol{G}_1^{-1}\boldsymbol{H}$ [see Eq. (6)], after which the latter expression takes the form

$$\begin{aligned}
4\,\boldsymbol{X}_i &= \boldsymbol{A}\boldsymbol{G}_1^{-1}(\boldsymbol{H}-\boldsymbol{H}^t)\boldsymbol{G}_1^{-1}\boldsymbol{A}^t \\
&+ (\boldsymbol{A}\boldsymbol{G}_1^{-1}\boldsymbol{H}\boldsymbol{G}_1^{-1}\boldsymbol{H}\boldsymbol{B}^t - \boldsymbol{B}\boldsymbol{H}^t\boldsymbol{G}_1^{-1}\boldsymbol{H}^t\boldsymbol{G}_1^{-1}\boldsymbol{A}^t) \\
&+ (\boldsymbol{B}\boldsymbol{G}_2\boldsymbol{G}_1^{-1}\boldsymbol{A}^t - \boldsymbol{A}\boldsymbol{G}_1^{-1}\boldsymbol{G}_2\boldsymbol{B}^t) \\
&+ (\boldsymbol{B}\boldsymbol{H}^t\boldsymbol{G}_1^{-1}\boldsymbol{H}\boldsymbol{G}_1^{-1}\boldsymbol{A}^t - \boldsymbol{A}\boldsymbol{G}_1^{-1}\boldsymbol{H}^t\boldsymbol{G}_1^{-1}\boldsymbol{H}\boldsymbol{B}^t) \\
&+ (\boldsymbol{B}\boldsymbol{G}_2\boldsymbol{G}_1^{-1}\boldsymbol{H}\boldsymbol{B}^t - \boldsymbol{B}\boldsymbol{H}^t\boldsymbol{G}_1^{-1}\boldsymbol{B}^t) \\
&+ \boldsymbol{B}\boldsymbol{H}^t\boldsymbol{G}_1^{-1}(\boldsymbol{H}-\boldsymbol{H}^t)\boldsymbol{G}_1^{-1}\boldsymbol{H}\boldsymbol{B}^t .
\end{aligned}$$

We finally combine terms to get Eq. (19)

$$\begin{aligned}
4\,\boldsymbol{X}_i &= (\boldsymbol{A}+\boldsymbol{B}\boldsymbol{H}^t)\boldsymbol{G}_1^{-1}(\boldsymbol{H}-\boldsymbol{H}^t)\boldsymbol{G}_1^{-1}(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{H}^t)^t \\
&- (\boldsymbol{A}+\boldsymbol{B}\boldsymbol{H}^t)\boldsymbol{G}_1^{-1}\boldsymbol{G}_2\boldsymbol{B}^t + \boldsymbol{B}\boldsymbol{G}_2\boldsymbol{G}_1^{-1}(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{H}^t)^t .
\end{aligned}$$

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