Radon-Wigner transform for optical field analysis

Tatiana Alieva(1,2) and Martin J. Bastiaans(1)
(1) Technische Universiteit Eindhoven
Faculteit Elektrotechniek, EH 5.34
Postbus 513, 5600 MB Eindhoven, Netherlands
E-mail: M.J.Bastiaans@ele.tue.nl
(2) Katholieke Universiteit Leuven
Department of Electrical Engineering
Kardinaal Mercierlaan 94
B-3001 Leuven, Belgium

The Radon-Wigner transform, associated with the intensity distribution in the fractional Fourier transform system, is used for the analysis of complex structures of coherent as well as partially coherent optical fields. The application of the Radon-Wigner transform to the analysis of fractal fields is presented.

1 INTRODUCTION

The investigation of the local properties of an optical field requires its representation in phase space. Recently optical wavelet analysis has been used for the investigation of inhomogeneous fields [1, 2]. Mostly, one prefers to consider intensity distributions, which can easily be measured in optics. Unfortunately, scalograms (squared moduli of the wavelet transform) are not invertible in general and so they do not contain all information about a field.

A positive distribution of an optical field amplitude which is invertible, is its Radon-Wigner transform (RWT). The RWT of an optical field is the squared modulus of its fractional Fourier transform (FT) and therefore corresponds to the intensity distributions at different planes of a first-order system that provides an optical realization of the fractional FT. The RWT is used, in particular, for phase-space tomography [3], viz., reconstruction of the Wigner distribution function of an optical field [4, 5] from its projections corresponding to the RWT. In this paper we consider the application of the RWT in optical signal analysis.

2 FRACTIONAL FOURIER AND RADON-WIGNER TRANSFORMS

It has been shown in [6] that the RWT is closely related to the fractional Fourier transform [7, 8]. The fractional FT at an angle α of a complex field amplitude $f(x)$ is defined as

$$R^{\alpha}\left[f(x)\right](u) = F(u,\alpha) = \int_{-\infty}^{\infty} f(x) K_{\alpha}(x,u)\,dx, \qquad (1)$$

with the kernel

$$K_{\alpha}(x,u) = \frac{\exp(i\alpha/2)}{\sqrt{i\sin\alpha}} \exp\left(i\pi \frac{(x^2+u^2)\cos\alpha - 2xu}{\sin\alpha}\right). \qquad (2)$$

It describes, except for a phase shift $\alpha/2$, the evolution of a complex field amplitude during propagation through a quadratic refractive index medium in the paraxial approximation of scalar diffraction theory.

The Radon-Wigner transform of $f(x)$ is defined as the squared modulus of its fractional FT, $\left|R^{\alpha}\left[f(x)\right](u)\right|^2$ with $\alpha \in [0,\pi]$, which we will call the α-power spectrum. The RWT can be realized in a fractional FT system, which usually contains a thin lens as the quadratic refractive index element [9, 10, 11]. This method requires measurements of only intensity distributions at different planes corresponding to various angles α. At the angle $\alpha = 2\pi$, the RWT of a complex field amplitude corresponds to its intensity distribution, and at $\alpha = \pi/2$ to its Fourier (power) spectrum.

The RWT is alternatively defined as the projection of the Wigner distribution

$$W_f(x,k) = \int \left\langle f(x+y/2)\, f^*(x-y/2)\right\rangle \exp(-2\pi i k y)\,dy \qquad (3)$$

upon the direction at an angle α in the phase space [6]

$$\begin{aligned} &\left|R^{\alpha}\left[f(x)\right](u)\right|^2 = |F(u,\alpha)|^2 = \\ &= \int W_f(u\cos\alpha - k\sin\alpha,\, u\sin\alpha + k\cos\alpha)\,dk. \end{aligned} \qquad (4)$$

Note that the formalism of the RWT can be applied to coherent as well as partially coherent fields. The brackets in (3) indicate an ensemble average over the set of realizations of the amplitude $f(x)$ in the case of partial coherence. The inverse Radon transform permits to reconstruct the Wigner distribution from the set of projections $|F(u,\alpha)|^2$ for $\alpha \in [0,\pi]$, and then to find the field amplitude (except for a constant phase factor) for coherent fields,

$$f(x) = \frac{1}{f^*(0)} \int W_f(x/2, k) \exp(2\pi i k x) dk, \qquad (5)$$

or to find the two-point correlation function $G(x,y) = \langle f(x+y/2) f^*(x-y/2) \rangle$ for partially coherent ones,

$$G(x,y) = \int W_f(x,k) \exp(2\pi i k y) dk. \qquad (6)$$

This is the basic idea of phase-space tomography [3].

In this paper we investigate how the RWT, which can easily be measured in optics, can be used for the analysis of optical fields.

3 RADON-WIGNER TRANSFORM PROPERTIES

In order to extract the features of a field from its RWT map, we examine some properties of this transform. First of all, we generalize some well-known relations for Fourier spectra dealing with optical signal symmetry (see Table 1) to the case of RW spectra.

Table 1

optical signal	energy density	power spectrum
complex and even	even	even
complex and odd	even	even
real		even
imaginary		even

3.1 Radon-Wigner spectra of arbitrary optical signals

It is easy to see from (1) that for an arbitrary (real or complex) optical signal $f(x)$ we have

$$R^\alpha \left[f(x) \right](u) = R^{\alpha+\pi} \left[f(x) \right](-u), \qquad (7)$$

and correspondingly for the RW spectra:

$$\left| R^\alpha \left[f(x) \right](u) \right|^2 = \left| R^{\alpha+\pi} \left[f(x) \right](-u) \right|^2. \qquad (8)$$

An optical signal is thus completely defined by the RWTs ranging in the region $[0,\pi]$. The RW spectra for complex conjugate fields $f(x)$ and $f^*(x)$ enjoy the following properties:

$$\begin{aligned} \left| R^\alpha \left[f(x) \right](u) \right|^2 &= \left| R^{-\alpha} \left[f^*(x) \right](u) \right|^2 \\ &= \left| R^{\pi-\alpha} \left[f^*(x) \right](-u) \right|^2. \end{aligned} \qquad (9)$$

3.2 Radon-Wigner spectra of real optical signals

In several cases, optical systems are used for the analysis of objects that can be represented as real gratings. This means that the amplitude at the input plane of the fractional FT system is real.

For real optical signals, we have $R^{-\alpha}\left[f(x) \right](u) = \left(R^\alpha \left[f(x) \right](u) \right)^*$, and thus

$$\left| R^\alpha \left[f(x) \right](u) \right|^2 = \left| R^{-\alpha} \left[f(x) \right](u) \right|^2. \qquad (10)$$

Taking into account (8), we derive that the RW spectra for angles α and $\pi - \alpha$ are parity invertible:

$$\left| R^\alpha \left[f(x) \right](u) \right|^2 = \left| R^{\pi-\alpha} \left[f(x) \right](-u) \right|^2. \qquad (11)$$

This means that all information about a real optical signal is contained in the angle region $\left[0,\pi/2\right]$. Due to the similarity between the Fresnel transform and the fractional FT for angles in the region $\left[0,\pi/2\right]$, we can conclude that a real, one-dimensional optical signal can be completely determined from the diffraction patterns measured in free space corresponding to the region of Fresnel and Fraunhofer diffraction.

The power spectrum of an optical signal $R^\alpha\left[f(x) \right](u)$ is equal to $\left| R^{\pi/2-\alpha} \left[f(x) \right](-u) \right|^2$, because

$$\left| R^{\pi/2+\alpha} \left[f(x) \right](u) \right|^2 = \left| R^{\pi/2-\alpha} \left[f(x) \right](-u) \right|^2, \qquad (12)$$

which follows from (11). Note that the power spectrum of a real signal is always even: $\left| R^{\pi/2} \left[f(x) \right](u) \right|^2 = \left| R^{\pi/2} \left[f(x) \right](-u) \right|^2$. The angle $\alpha = \pi/4$ plays a central role in this symmetry: the power spectrum of $R^{\pi/4}\left[f(x) \right](u)$ is equal to $\left| R^{\pi/4} \left[f(x) \right](-u) \right|^2$.

The same properties hold for imaginary signals.

3.3 Radon-Wigner spectra of even and odd complex optical signals

From (7) it follows that the fractional FT of an even or odd optical signal satisfies the relationships

$$\begin{aligned} & R^\alpha \left[f(x) \right](u) = \pm R^\alpha \left[f(-x) \right](u) = \\ & = \pm R^{\alpha+\pi} \left[f(x) \right](u) = \pm R^\alpha \left[f(x) \right](-u), \end{aligned} \qquad (13)$$

where the $+$ sign stands for even, and the $-$ sign for odd signals. Hence, the RW spectra of even and odd optical signals are even:

$$\left| R^\alpha \left[f(x) \right](u) \right|^2 = \left| R^\alpha \left[f(x) \right](-u) \right|^2. \qquad (14)$$

Thus, that a signal is even or odd can clearly be seen from the energy density and the power spectrum. Odd signals have a RW spectrum which is 0 at zero frequency for all $\alpha \in [0,\pi]$.

3.4 Radon-Wigner transform of affine-transformed optical signals

One of the reason why scalograms (squared moduli of the wavelet transform) are widely used for the analysis of inhomogeneous optical signals is that they are covariant to affine signal transformations. Let us show that the RWT is affine variant under affine transformation of an optical signal.

It is easy to see that a shift in the position and a modulation of a signal result in a shift in the α-RW spectrum, which depends on α:

$$\begin{array}{lcl} f(x) & \rightarrow & f(x-x_0)\exp(-ik_0x) \\ \downarrow\alpha & & \downarrow\alpha \\ \left|F_f(u,\alpha)\right|^2 & \longrightarrow & \left|F_f(u-x_0\cos\alpha+k_0\sin\alpha,\alpha)\right|^2 \end{array} \quad (15)$$

The α-RW spectrum is covariant to position shift and invariant to modulation for $\alpha = 0$ and vice versa for $\alpha = \pi/2$. In general, there always exists an angle $\alpha_0 = \arctan(x_0/k_0)$ such that the corresponding α-RW spectrum is invariant to the combined action of a position shift x_0 and a modulation k_0.

After some algebraic manipulations one derives the following connection for the α-RW spectra of a signal $f(x)$ and the α-RW spectra of the affine transformed and modulated signal $a^{-1/2}f(\frac{x-x_0}{a})\exp(-ik_0x)$

$$\begin{array}{lcl} f(x) & \rightarrow & a^{-1/2}f(\frac{x-x_0}{a})\exp(-ik_0x) \\ \downarrow\alpha & & \downarrow\alpha \\ \left|F_f(u,\alpha)\right|^2 & \longrightarrow & p^{-1}\left|F_f\left(\frac{u-u_0}{p},\beta\right)\right|^2 \end{array} \quad (16)$$

where

$$\begin{cases} p = a^{-1}\sin\alpha/\sin\beta \\ u_0 = x_0\cos\alpha - k_0\sin\alpha \\ \tan\beta = a^{-2}\tan\alpha. \end{cases} \quad (17)$$

This means that affine transformations of an optical signal result in affine transformations of the α-RW spectra. In particular, $p = 1$ if $\beta = \pi/2-\alpha$ and $\tan\alpha = a$, in which case

$$\begin{array}{l} \left|R^{\alpha}\left[(\tan\alpha)^{-1/2}f(\frac{x-x_0}{\tan\alpha})\exp(-ik_0x)\right](u)\right|^2 = \\ = \left|R^{\pi/2-\alpha}\left[f(x)\right](u-x_0\cos\alpha+k_0\sin\alpha)\right|^2, \end{array} \quad (18)$$

and for $k_0 = x_0/a$ we have

$$\begin{array}{l} \left|R^{\alpha}\left[(\tan\alpha)^{-1/2}f(\frac{x-x_0}{\tan\alpha})\exp(-ix_0x/\tan\alpha)\right](u)\right|^2 = \\ = \left|R^{\pi/2-\alpha}\left[f(x)\right](u)\right|^2. \end{array} \quad (19)$$

4 PERIODICITY OF RADON-WIGNER TRANSFORM

In this section we consider optical fields whose RWTs are periodic in the angle or in the space coordinate.

4.1 Angular periodicity

A Wigner distribution that is invariant under rotation through an angle α in phase space, describes a self-imaging phenomenon in the fractional FT system. Thus the Radon-Wigner map of the eigenfields $f_\alpha(u)$ of the fractional FT system is periodic in the angle with period α:

$$|F(u,n\alpha+\beta)|^2 = |F(u,\beta)|^2. \quad (20)$$

In particular, for $\beta = 0$ we have $|F(u,n\alpha)|^2 = |f_\alpha(u)|^2$. The angle region of the Radon-Wigner map which completely characterizes such an optical field, reduces to $[0,\alpha]$.

4.2 Space periodicity

The Wigner distribution of the periodic wavefronts $f(x) = \sum f_n\exp(i2\pi xn)$, with period $d = 1$, is invariant under translation: $W_f(x+l,k) = W_f(x,k)$ $(l = 0, 1, ...)$. The RWT of a periodic optical signal at angle $\alpha \neq \pm\pi/2$ is periodic too, with decreasing period when $\alpha \rightarrow \pi/2$:

$$|F(u,\alpha)|^2 = \frac{\left|\sum f_n\exp\left(\frac{i2\pi xn}{\cos\alpha}\right)\exp\left(-i\pi n^2\tan\alpha\right)\right|^2}{|\cos\alpha|}. \quad (21)$$

In particular, we have for $\tan\alpha = 2m+e$:

$$|F(u,\alpha)|^2 = |f(u/\cos\alpha - e/2)|^2/|\cos\alpha|, \quad (22)$$

where m is an integer and $e = 0, 1$. So the RWT of a periodic field with period 1 at angles α such that $\tan\alpha$ is an integer, is a scaled replica of the squared modulus of the input intensity distribution.

Moreover, the RWTs for angles α and β such that $\tan\beta - \tan\alpha = 2m$ are similar to each other

$$|F(u,\alpha)|^2 = \frac{|\cos\beta|}{|\cos\alpha|}\left|F\left(u\frac{\cos\beta}{\cos\alpha},\beta\right)\right|^2. \quad (23)$$

5 RADON-WIGNER SPECTRA OF FRACTAL FIELDS

Properties of fields diffracted by fractal structures have attracted much attention of the optical community [12, 13, 14, 15, 16, 17, 18, 19, 20]. Usually such investigations are restricted to the consideration of Fraunhofer or Fresnel diffraction. Let us shortly discuss how the analysis of the Radon-Wigner map, corresponding to the intensity distribution of the diffraction patterns in the fractional FT system, can be used for the extraction of characteristics of fractal objects.

The typical property of fractal objects is that they keep their own shapes under a change of scale $f(\lambda x+x_0) = \lambda^H f(x+x_0)$. For real structures, $f(x)$ is usually self-affine for a limited sequence $\lambda = \lambda_0^k$, where $k = 1, ..., N$.

Using Eqs. (16) and (17), we derive that the RWT of a fractal field at the angles α and β $(\alpha,\beta \neq \pi m/2)$ connected by

$$\tan\beta = \lambda^2\tan\alpha \quad (24)$$

are similar:

$$\begin{array}{l} |F(u+x_0\cos\alpha,\alpha)|^2 = \left(\frac{\cot\beta}{\cos\alpha}\right)^H\left(\frac{\cos\beta}{\cos\alpha}\right) \\ \times\left|F\left(\sqrt{\left|\frac{\sin 2\beta}{\sin 2\alpha}\right|}u+x_0\cos\beta,\beta\right)\right|^2. \end{array} \quad (25)$$

Moreover the RWTs of the fractal fields at angles α and $\pi/2-\alpha$ are the same, except for a normalization factor and a dilation of the coordinate:

$$\begin{array}{ll} |F(u+x_0\cos\alpha,\pi/2-\alpha)|^2 & = (\cot\alpha)^{2H+1} \\ & \times|F(u+x_0\sin\alpha,\alpha)|^2. \end{array} \quad (26)$$

The ratio of the intensity distributions for such angles defines the scaling exponent H, which is closely related to the fractal dimension.

The above results have been demonstrated for the Mandelbrot-Weierstrass cosine fractal [18] and for homogeneous algebraic functions $|x|^H$ [20]. It has been shown numerically [20] for the example of the triadic Cantor set that the RWT map reveals the hierarchical structure of a fractal objects.

So, the scaling properties of fractal objects are reflected in the Radon-Wigner map: the RWTs for a certain sequence of angles are similar. The analysis of the Radon-Wigner map gives information about the hierarchical structure of a fractal field and allows to determine its main characteristics.

6 CONCLUSION

It has been shown how the Radon-Wigner spectra, corresponding to the intensity distributions of diffraction patterns in the fractional FT system, reveal the different types of optical field symmetries. The application of the RWT to the analysis of fractal optical fields has been discussed.

We believe that the application of fractional FT systems providing the RWT of optical fields, will allow to extend the methods used for optical fields analysis like Fourier or wavelet transforms.

REFERENCES

[1] E. Freysz, B. Pouligny, F. Argoul and A. Arneodo, "Optical wavelet transform of fractal aggregates," Phys. Rev. Lett. **64** (1990) 745-748.

[2] L. Onural, "Diffraction from a wavelet point of view," Opt. Lett. **18** (1993) 846-848.

[3] M.G. Raymer, M. Beck and D.F. McAlister, "Complex wavefield reconstruction using phase-space tomography," Phys. Rev. Lett. **72**, (1994) 1137-1140.

[4] M.J. Bastiaans, "The Wigner distribution function and its application to first-order optics," J. Opt. Soc. Am. **69** (1979) 1710-1716.

[5] M.J. Bastiaans, "The Wigner distribution function applied to optical signals and systems," Opt. Commun. **25** (1978) 26-30.

[6] A.W. Lohmann and B.H. Soffer, "Relationships between the Radon-Wigner and fractional Fourier transforms," J. Opt. Soc. Am. A **11** (1994) 1798-1801.

[7] V. Namias, "The fractional order Fourier transform and its application to quantum mechanics," J. Inst. Math. Appl. **25** (1980) 241-265.

[8] L.B. Almeida, "The fractional Fourier transform and time-frequency representations," IEEE Trans. Signal Process. **42** (1994) 3084-3091.

[9] D. Mendlovic and H.M. Ozaktas, "Fractional Fourier transforms and their optical implementation," J. Opt. Soc. Am. A **10** (1993) 1875-1881.

[10] A.W. Lohmann, "Image rotation, Wigner rotation, and the fractional Fourier transform," J. Opt. Soc. Am. A **10** (1993) 2181-2186.

[11] T. Alieva, V. Lopez, F. Agullo-Lopez and L. Almeida, "The fractional Fourier transform in the optical propagation problems," J. Mod. Opt. **41** (1994) 1037-1044.

[12] C. Allain and M. Cloitre, "Optical diffraction on fractals," Phys. Rev. B **33** (1986) 3566-3569.

[13] J. Uozumi, Y. Sakurada and T. Asakura, "Fraunhofer diffraction from apertures bounded by regular fractals," J. Mod. Opt. **12** (1995) 2309-2322.

[14] Y. Sakurada, J. Uozumi and T. Asakura, "Fresnel diffraction by one-dimensional regular fractals," Pure Appl. Opt. **1** (1992) 29-40.

[15] D.L. Jaggard and T. Spielman, "Triadic Cantor target diffraction," Micr. Opt. Tech. Lett. **5** (1992) 460-466.

[16] A. Lakhtakia, "Fractal self-Fourier functions," Optik No 1 (1993) 51-52.

[17] Y. Kim and D.L. Jaggard, "Diffraction by fractally serrated apertures," J. Opt. Soc. Am. A **8** (1991) 20-26.

[18] T. Alieva, "Fractional Fourier transform as a tool for investigation of fractal objects," J. Opt. Soc. Am. A **13** (1996) 1189-1871.

[19] T. Alieva and F. Agullo-Lopez, "Optical wave propagation of fractal fields," Opt. Commun. **125** (1996) 267-274.

[20] T. Alieva and A. Barbé, "Fractional Fourier analysis of objects with scaling symmetry," in *Fractals in Engineering*, J.L. Véhel, E. Lutton, and C. Tricot, eds., Springer Verlag, London (1997) 252-264.