Fractional Cyclic Transforms in Optics: Theory and Applications

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ABSTRACT

In this paper we review fractional linear integral transforms, which have been actively used during the last decade in optical information processing. The general algorithm for the fractionalization of the linear cyclic integral transforms is discussed and the main properties of fractional transforms are considered. It is shown that there is an infinite number of continuous fractional transforms related with a given cyclic integral transform.

The optical fractional Fourier transform used for different applications such as adaptive filter design, phase retrieval, encryption, watermarking, etc., is discussed in detail. Other fractional cyclic transforms that can be implemented in optics, such as fractional Hankel, Sine, Cosine, Hartley, and Hilbert transforms, are investigated.

1. INTRODUCTION

The development of optical engineering has attracted much attention to the investigation of new methods of analysis, transformation, compression, and reconstruction of information by using optical systems [1, 2]. Nowadays, data security has become a great problem. Indeed, rapid technological development in computers, printers, scanners, and copiers, made it possible to produce authentic looking images including money bills, logos, signatures, etc. In the area of encoding and watermarking, optical techniques are very promising [3], thanks to their natural parallel processing capabilities and the existence of several parameters in which information can be hidden (phase, wavelength, spatial frequency, and polarization).

On the other hand, the approaches for optical signal/image processing become unified with the ones used in electrical engineering. Thus, the traditional area of optical processing – Fourier optics – is enlarged by the introduction of bilinear distributions such as the Wigner distribution, the ambiguity function, spectrograms (squared moduli of the short-time Fourier transform), scalograms (squared moduli of the wavelet transform), and other distributions that represent a signal in a position-frequency or a position-scale plane [4].

Although the application of the ambiguity function (AF) and the Wigner distribution (WD) for the description of coherent and partially coherent optical fields was proposed almost 20 years ago [5]-[8], it became even more attractive after the introduction in optics of the fractional Fourier transform (FT) [9]-[12]. The fractional FT is a generalization of the ordinary FT [13]-[16]. A simple optical set up performing a fractional FT, is very similar to the one used for the ordinary optical FT [17]. The fractional FT depends on a parameter that is associated with the rotation angle in phase space. Thus, the fractional FT produces a rotation of the AF and the WD. These functions can be reconstructed from the knowledge of only squared moduli of the fractional FT related to the intensity distributions [18].

Recent achievements of the applications of the fractional FT for signal analysis, filtering, encoding, watermarking, phase retrieval, etc. [9]-[50], increase the interest to the general fractionalization procedure [50]-[62]. The fractional generalization of the Hilbert, Hankel, Hartley, Cosine, and Sine transforms, and different types of the fractional FT, were proposed for signal and image processing. The advantage of various types of fractional transforms is that they can be optically implemented, enhancing the usefulness in optical signal processing and imaging techniques, in general.

In this paper we briefly establish a general algorithm for the fractionalization of linear cyclic integral transforms and consider their main characteristics and fundamental properties. From the examples of the fractional Fourier, Hankel, Sine, Cosine, Hartley, and Hilbert transforms (all of which can be obtained in paraxial optics), we discuss the applications of fractional transforms for new filter design, edge enhancement, watermarking, encoding, analysis of fractal objects, etc. Finally, we make some conclusions about future development of fractional optics.

2. FRACTIONALIZATION IN OPTICS

The word "fraction" is nowadays very popular in different fields of knowledge. We only mention the fractional derivatives in mathematics, the fractal dimension in geometry, fractal noise, and fractional transformations in signal processing. In general, it means that some parameter has a noninteger value.

Regarding to linear paraxial – also called first-order [63] – optics, we can always divide any optical set up S into several parts, called subsets s_k, which are somehow fractions of the whole set up. Moreover, every first-order optical system is described by a 2×2 matrix M whose determinant equals 1, and the composition of two of them is characterized by a matrix $M_3 = M_1 \times M_2$, which expresses the additivity of first-order systems. We can demand that certain rules of the dividing procedure hold; for example, that the fractional subsets were defined by the same matrix [64]. It is often possible to separate the original set up into equal subsets characterized by a one-parametric matrix. This is the case for Fresnel diffraction and for the scaling operation.

The original set up S can also be considered as a $1/N$ part of the more global one G. If after propagation through a sequence of N identical set ups S the output and the input optical wavefronts will be the same, then the global set up G performs the identity transformation, and the set up S produces a cyclic transformation [41, 62]. This paper is devoted to the construction and investigation of fractional one-parametric subsets (transforms) s of a one-parametric cyclic set up (transform) S.

3. FRACTIONALIZATION OF CYCLIC TRANSFORMS

There is a long list of linear transforms, actively used in optics and signal/image processing, which belong to the class of cyclic transforms. Thus, if R is an operator of a linear integral transform,

$$R\left[f(x)\right](u) = \int_{-\infty}^{\infty} K(x,u) f(x) dx, \qquad (1)$$

a linear integral transform is a cyclic one, if its N-time acting produces the identity transform

$$R^N = I. \qquad (2)$$

For example, the Fourier and Hilbert transforms are cyclic with a period $N = 4$. The Hankel and Hartley transforms have a period $N = 2$. The cyclic canonical transforms of period N with kernel

$$K(x,u) = \frac{1}{\sqrt{iB}} \exp\left[\frac{i\pi\left(Ax^2 + Du^2 - 2ux\right)}{B}\right], \qquad (3)$$

where $A + D = 2\cos(2\pi m/N)$ and m and N are integers, were studied in [41].

All these transforms have some common properties. Thus, the eigenvalues of the cyclic transforms can be represented as $A = \exp(i2\pi L/N)$, where L is an integer. Indeed, let $\Psi(x)$ be an eigenfunction of R with eigenvalue $A = a\exp(i\varphi)$, where a and φ are real and $a > 0$. From equation (2) one gets that $A^N = 1$, and hence $a = 1$ and $\varphi = 2\pi L/N$.

In order to construct a fractional transform for a given cyclic transform, we first formulate the desirable properties of the fractional R-transform R^α, where α is the parameter of the fractionalization:

- continuity of R^α for any real value α;
- additivity of R^α with respect to the parameter α: $R^{\alpha+\beta} = R^\alpha R^\beta$;
- reproducibility of the ordinary transform for integer values of α, in particular $R^1 = R$ and $R^0 = R^N = I$.

Let us analyze the structure of the kernel $K(\alpha, x, u)$ of a fractional R-transform with period N. Due to its periodicity with respect to the parameter α, one can represent $K(\alpha, x, u)$ in the form

$$K(\alpha, x, u) = \sum_{n=-\infty}^{\infty} k_n(x, u) \exp\left(i2\pi\alpha n/N\right), \quad (4)$$

where the coefficients $\{k_n\}$ have to satisfy the system of N equations [62]

$$K(l, x, u) = \sum_{n=-\infty}^{\infty} k_n(x, u) \exp\left(i2\pi l n/N\right) \quad (5)$$

with $l = 0, \ldots, N-1$. Moreover, the coefficients have to be orthonormal to each other,

$$\int_{-\infty}^{\infty} k_n(x, u) k_m(u, y) du = \delta_{n,m} k_n(x, y), \quad (6)$$

where $\delta_{n,m}$ denotes the Kronecker delta. Indeed, from the additivity property

$$\int_{-\infty}^{\infty} K(\alpha, x, u) K(\beta, u, y) du = K(\alpha + \beta, x, y), \quad (7)$$

we get the condition

$$\begin{aligned} &\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \exp\left[i2\pi(\alpha n + \beta m)/N\right] \\ &\quad \times \int_{-\infty}^{\infty} k_n(x, u) k_m(u, y) du \\ &= \sum_{l=-\infty}^{\infty} k_l(x, y) \exp\left[i2\pi(\alpha + \beta) l/N\right], \end{aligned} \quad (8)$$

which can only be satisfied if the orthogonality condition (6) holds; otherwise, the coefficients k_n in the left-hand and the right-hand side could not be equal, except for the obvious case $k_n(x, u) = k_m(x, u) = 0$.

Note that all coefficients k_{n+mN} for fixed n and an arbitrary integer m, have the same exponent factor in the system of equations (5). Therefore we can rewrite Eq. (5) as

$$K(l, x, u) = \sum_{n=0}^{N-1} \exp\left(i2\pi l n/N\right) \sum_{m=-\infty}^{\infty} k_{n+mN}(x, u). \quad (9)$$

If we introduce the new variables $C_n(x, u)$, which are the partial sums of the coefficients in the Fourier expansions (4) and (5),

$$C_n(x, u) = \sum_{m=-\infty}^{\infty} k_{n+mN}(x, u), \quad (10)$$

Eq. (9) reduces to a system of N linear equations with N variables. This system has a unique solution [62]

$$C_n(x, u) = \frac{1}{N} \sum_{l=0}^{N-1} \exp\left(-i2\pi l n/N\right) K(l, x, u). \quad (11)$$

It is easy to see that the variables C_n satisfy a condition similar to (6):

$$\int_{-\infty}^{\infty} C_n(x, u) C_m(u, y) du = \delta_{n,m} C_n(x, y). \quad (12)$$

Note that some partial sums for certain transforms may be equal to zero. As we will see further on, this is the case for the Hilbert transform, for instance.

So, if we find the coefficients k_n that satisfy the condition (6) and whose partial sums are given by (11), we can construct the fractional transform. In general, there are a number of sets $\{k_n\}$ that generate fractional transforms of a given R-transform.

4. N PERIODIC CYCLIC FRACTIONAL TRANSFORM KERNELS WITH N HARMONICS

As we have seen above, the kernel of a fractional cyclic transform can be represented as a superposition of harmonics with complex amplitudes k_n. In this section we suppose that the number of harmonics is limited by N, where N is a period of the cyclic transform. Then every sum $C_n(x, u)$ ($n \in [0, N-1]$) contains only one element $k_{n+\varphi_n}(x, u) = C_n(x, u)$ from the decomposition (4), where $\varphi_n = mN$ and m is an arbitrary integer. Therefore, in the general case, the kernel of the fractional R-transform with N harmonics can be written as

$$\begin{aligned} K(\alpha, x, u) &= \sum_{n=0}^{N-1} k_{n+\varphi_n}(x, u) \exp\left[i2\pi\alpha(n + \varphi_n)/N\right] \\ &= \frac{1}{N} \sum_{l=0}^{N-1} K(l, x, u) \sum_{n=0}^{N-1} \exp\left(-i2\pi l n/N\right) \\ &\qquad \times \exp\left[i2\pi\alpha(n + \varphi_n)/N\right]. \end{aligned} \quad (13)$$

This equation provides a formula for recovering the continuous periodic function $K(\alpha, x, u)$ from its N samples $K(l, x, u)$, under the assumption that the spectrum of $K(\alpha, x, u)$ contains only N harmonics at the frequencies $\{\varphi_0, 1 + \varphi_1, \ldots, n + \varphi_n, \ldots, N - 1 + \varphi_{N-1}\}$.

If we put $\varphi_n = 0$ ($n = 0, 1, \ldots, N-1$), we obtain the fractional transform with the kernel

$$\begin{aligned} K(\alpha, x, u) &= \frac{1}{N} \sum_{l=0}^{N-1} \exp\left[i\pi(N-1)(\alpha - l)/N\right] \\ &\qquad \times \frac{\sin\left[\pi(\alpha - l)\right]}{\sin\left[\pi(\alpha - l)/N\right]} K(l, x, u) \end{aligned} \quad (14)$$

proposed by Shih in [58]. In particular, this formula is used as the definition of a kind of fractional FT (for the continuous and the discrete case) [58, 59].

With N an odd integer and choosing N nonzero coefficients in the decomposition (4) with indices $j = -\frac{1}{2}(N-1), \ldots, 0, \ldots, \frac{1}{2}(N-1)$ [corresponding to the indices $n+mN$ for $m=0$ and $n = 0, 1, \ldots, \frac{1}{2}(N-1)$, and $m=-1$ and $n = \frac{1}{2}(N-1)+1, \ldots, N-1$], we obtain the kernel

$$K(\alpha, x, u) = \frac{1}{N}\sum_{l=0}^{N-1} \frac{\sin\left[\pi(\alpha - l)\right]}{\sin\left[\pi(\alpha - l)/N\right]} K(l, x, u). \quad (15)$$

This equation corresponds to the recovering procedure of a band-limited periodic function from its values on equidistant sampling points [65]. In particular, if $K(l, x, u)$ is real for integer $l = 0, 1, \ldots, N-1$, then the kernel of the fractional transform determined by Eq. (15) is real, too. It also means that the Fourier spectrum of $K(\alpha, x, u)$ with respect to the parameter α is symmetric: $|k_j| = |k_{-j}|$.

As an example, let us consider the general expression (13) for the kernel of the fractional R-transform with period 4 (which is the case for the Fourier and Hilbert transforms):

$$K(\alpha, x, u) = \frac{1}{4}\sum_{l=0}^{3} K(l, x, u) S(l) \quad (16)$$

with $S(l) = \sum_{n=0}^{3} \exp\left(-\frac{1}{2} i\pi l n\right) \exp\left[\frac{1}{2} i\pi\alpha(n + \varphi_n)\right]$.

Note that for the Hilbert transform, the number of harmonics reduces to two, because $C_0(x, u) = C_2(x, u) = 0$, which follows from $K(0, x, u) = -K(2, x, u)$ and $K(1, x, u) = -K(3, x, u)$, From Eq. (13) we then conclude that the fractional Hilbert transform kernel can be written as

$$\begin{aligned} K(\alpha, x, u) &= \exp\left[i\pi\alpha(1 + m_1 + m_3)\right] \\ &\times \left\{K(0, x, u)\cos\left[\pi\alpha\left(\tfrac{1}{2} + m_3 - m_1\right)\right]\right. \\ &\left. - K(1, x, u)\sin\left[\pi\alpha\left(\tfrac{1}{2} + m_3 - m_1\right)\right]\right\}, \end{aligned} \quad (17)$$

where m_1 and m_3 are integers. In particular, for the case $m_1 = m_3 = 0$ ($k_n = 0$ if $n \neq 1, 3$), one gets

$$\begin{aligned} K(\alpha, x, u) &= \exp\left(i\pi\alpha\right) \\ &\times \left[K(0, x, u)\cos\left(\tfrac{1}{2}\pi\alpha\right) - K(1, x, u)\sin\left(\tfrac{1}{2}\pi\alpha\right)\right], \end{aligned} \quad (18)$$

while for the case $m_1 = 0$ and $m_3 = -1$ ($k_n = 0$ if $n \neq -1, 1$), the common form for the fractional Hilbert transform [51] with a real kernel is obtained:

$$\begin{aligned} K(\alpha, x, u) &= K(0, x, u)\cos\left(\tfrac{1}{2}\pi\alpha\right) \\ &+ K(1, x, u)\sin\left(\tfrac{1}{2}\pi\alpha\right). \end{aligned} \quad (19)$$

Therefore, even for the same number of harmonics, there are several ways for the fractionalization of cyclic transforms.

5. FRACTIONAL TRANSFORM KERNELS AND EIGENFUNCTIONS OF CYCLIC TRANSFORMS

In this section we consider the fractional transform kernel representation through the set of eigenfunctions of the cyclic transform. This is one of the methods to construct fractional kernels with a number of harmonics $M > N$, where N is the period of the cyclic transform.

Suppose that there is a complete set of orthonormal eigenfunctions $\{\Phi_n\}$ of the operator R with eigenvalues $\{A_n = \exp\left(i2\pi L_n/N\right)\}$, $n = 0, 1, \ldots$ (see Section 3):

$$\int_{-\infty}^{\infty} \Phi_n(x)\Phi_m^*(x)dx = \delta_{n,m}. \quad (20)$$

This is the case, for instance, for the Fourier and Hartley transforms, where $\Phi_n(x)$ are the Hermite-Gauss modes [9, 10]

$$\Phi_n(x) = \left(\sqrt{\pi}2^n n!\right)^{-\frac{1}{2}} \exp\left(-\tfrac{1}{2}x^2\right) H_n(x) \quad (21)$$

and $H_n(u)$ are the Hermite polynomials; for the Hankel transform of different orders, where $\Phi_n(x)$ are the normalized Laguerre-Gauss functions [54, 55]; and for many other (but not all) cyclic transforms. The Hilbert operator, for example, does not satisfy this assumption. Indeed, because of the orthogonal property for the Hilbert transform,

$$\int_{-\infty}^{\infty} f(x) R\left[f(u)\right](x)dx = 0, \quad (22)$$

its eigenfunctions $\Psi(x)$ are self-orthogonal:

$$\int_{-\infty}^{\infty} \Psi^2(x)dx = 0. \quad (23)$$

If the set of orthonormal eigenfunctions exists, we can represent a kernel of the R-transform of the integer power p as

$$K(p, x, u) = \sum_{n=0}^{\infty} \Phi_n(x) A_n^p \Phi_n^*(u)$$

$$= \sum_{n=0}^{\infty} \Phi_n(x) \exp\left(i2\pi p L_n/N\right) \Phi_n^*(u). \tag{24}$$

One of the possible series of kernels for the fractional R-transform can then be written in the form

$$K(\alpha, x, u) = \sum_{n=0}^{\infty} \Phi_n(x) \exp\left[i2\pi(L_n/N + l_n)\alpha\right] \Phi_n^*(u), \tag{25}$$

where l_n is an integer. This kernel satisfies the additivity condition due to the orthonormality of the eigenfunctions $\Phi_n(x)$.

In particular, the fractional Fourier and Hankel transforms, based on this definition of fractionalization for $L_n = -n$ and $l_n = 0$, describe the propagation of optical beams through a medium with a quadratic refractive index [9, 10, 54, 55]. We will call these transforms the optical fractional transforms. Their kernels contain an infinite number of harmonics. Thus the optical fractional FT has the following kernel

$$\begin{aligned} K_F(\alpha, x, u) &= \sum_{n=0}^{\infty} \Phi_n(x) \exp\left(-\tfrac{1}{2} i\pi n\alpha\right) \Phi_n^*(u) \\ &= \frac{\exp(\frac{1}{4} i\pi\alpha)}{\sqrt{i2\pi \sin(\frac{1}{2}\pi\alpha)}} \\ &\times \exp\left[i \frac{(x^2+u^2)\cos(\frac{1}{2}\pi\alpha) - 2ux}{2\sin(\frac{1}{2}\pi\alpha)}\right]. \end{aligned} \tag{26}$$

Let us rewrite Eq. (25) in the form

$$K(\alpha, x, u) = \sum_{n=-\infty}^{\infty} z_n(x, u) \exp\left(i2\pi n\alpha/N\right). \tag{27}$$

Here $z_n(x, u)$ is a sum of the elements $\Phi_j(x)\Phi_j^*(u)$ over j, where $\Phi_j(x)$ is the eigenfunction of the R-transform with eigenvalue $\exp\left(i2\pi n/N\right)$. Thus for the case of the optical fractional FT,

$$\begin{aligned} K_F(\alpha, x, u) &= \sum_{n=0}^{\infty} \Phi_n(x) \exp\left(-\tfrac{1}{2} i\pi n\alpha\right) \Phi_n^*(u) \\ &= \sum_{n=-\infty}^{0} z_n(x, u) \exp\left(\tfrac{1}{2} i\pi n\alpha\right), \end{aligned} \tag{28}$$

the coefficients $z_n(x, u)$ vanish for positive n and $z_n(x, u) = \Phi_n(x)\Phi_n^*(u)$ for $n \leq 0$. As we will see below, the fractional Hartley transform [62] can be represented in the form

$$\begin{cases} K(\alpha, x, u) = \sum_{n=0}^{\infty} \exp\left(-i\pi n\alpha\right) z_{-n}(x, u) \\ z_{-n}(x, u) = \Phi_{2n}(x)\Phi_{2n}(u) + \Phi_{2n+1}(x)\Phi_{2n+1}(u). \end{cases} \tag{29}$$

It is easy to see from (27) that we can generate another kernel series with M harmonics,

$$K(\alpha, x, u) = \sum_{n=0}^{M-1} \exp\left(i2\pi n\alpha/M\right) \sum_{m=-\infty}^{\infty} z_{n+mM}(x, u), \tag{30}$$

which satisfy the requirements for the fractional transforms. Here the sums of the elements $z_j(x, u)$

$$k_n(M) = \sum_{m=-\infty}^{\infty} z_{n+mM}(x, u) \tag{31}$$

are used as the coefficients $k_n(x, u)$ in (4). Note that the relationship (6) holds for the coefficients $k_n(M)$ and $k_m(M)$, because they are constructed from the disjoint series of orthonormal elements. The number of harmonics M has to be defined by $M = Nl$ with l an integer, in order to satisfy the conditions (9).

One can prove that the kernel (30) for $\alpha = 1$ reduces to (24). In particular, if $\{\Phi_n\}$ is the Hermite-Gauss mode set and $z_{-n}(x, u) = \Phi_n(x)\Phi_n^*(u)$ for $n = 0, 1, \ldots$ and $z_{-n}(x, u) = 0$ for negative n, then Eq. (30) corresponds to the series of the M-harmonic fractional FTs proposed in [60],

$$\begin{aligned} K(\alpha, x, u) &= \sum_{n=0}^{M-1} \exp\left[-i2\pi n\alpha(1-M)/M\right] \\ &\times \sum_{m=0}^{\infty} \Phi_{n+mM}(x)\Phi_{n+mM}^*(u) \\ &= \frac{1}{M} \sum_{n=0}^{M-1} \exp\left[i\pi(M-1)(\alpha l - n)/M\right] \\ &\times \frac{\sin\left[\pi(\alpha l - n)\right]}{\sin\left[\pi(\alpha l - n)/M\right]} K_F(n/l, x, u), \end{aligned} \tag{32}$$

where $K_F(n/l, x, u)$ is the kernel of the optical fractional FT. Application of such types of fractional FT for image encryption was reported in [42]. If $M = N$ ($l = 1$), we obtain that the kernel of the Shih fractional transform defined by (14) can also be represented as

$$\begin{aligned} K(\alpha, x, u) &= \sum_{n=0}^{N-1} \exp\left[-i2\pi n\alpha(1-N)/N\right] \\ &\times \sum_{m=0}^{\infty} \Phi_{n+mN}(x)\Phi_{n+mN}^*(u). \end{aligned} \tag{33}$$

Finally we can conclude that if a complete orthonormal set of eigenfunctions for a given cyclic transform exists, then an infinite number of fractional transform kernels with an arbitrary number of harmonics can be constructed using the procedure (25). Some examples of fractional FTs whose kernels contain different numbers of harmonics were considered in [62].

6. PROPERTIES OF FRACTIONAL CYCLIC TRANSFORMS

Although there is a variety of schemes for the construction of fractional transforms, all of them have some common properties.

If the coefficients $k_n(x, u)$ in the decomposition (4) are real, then the following relationship holds for the fractional transforms of a real function $f(x)$:

$$\{R^\alpha [f(x)] (u)\}^* = R^{-\alpha} [f(x)] (u). \tag{34}$$

This is the case for the optical fractional FT, the related fractional Hartley transform, and the optical fractional Hankel transform.

Eigenfunctions of fractional transforms. By analogy with the analysis of the fractional FT eigenfunctions, made in [46, 47], the eigenfunction $\Psi_{1/M}(x)$ for the fractional transform R^α for $\alpha = 1/M$ with eigenvalue $A = \exp(i2\pi L/M)$, can be constructed from the arbitrary generator function by the following procedure:

$$\Psi_{1/M}(x) = \frac{1}{M} \sum_{n=0}^{M-1} \exp\left(-i2\pi nL/M\right) R^{n/M} [g(u)] (x). \tag{35}$$

In the limiting case $M \to \infty$, one gets the eigenfunction for any value α with eigenvalue $\exp(i2\pi\alpha L)$:

$$\Psi_\alpha^L(x) = \frac{1}{N} \int_0^N \exp\left(-i2\pi\alpha L\right) R^\alpha [g(u)] (x) d\alpha. \tag{36}$$

In particular for fractional transforms generated by Eq. (25) (as it was shown by the example for the fractional FT [47]), the functions $\Psi_\alpha^L(x)$ correspond to the elements of the orthogonal set $\{a_L \Phi_L\}$, where the constant factors depend on the generator function.

For the Shih definition of the fractional transform (14) or (33), there are only N different functions

$$\Psi_\alpha^L(x) = a_L \sum_{m=0}^{\infty} \Phi_{L+mN}(x), \tag{37}$$

which are self reproducible under the fractional transformation for any α:

$$R^\alpha \left[\Psi_\alpha^L(u)\right] (x) = \exp\left(i2\pi\alpha L\right) \Psi_\alpha^L(x), \tag{38}$$

where a_L is a constant. The functions with different indices L are orthogonal to each other.

Complex and real fractional transform kernels. We have seen in the previous section that if there exists a complete orthonormal set of eigenfunctions $\{\Phi_n\}$ for the R-transform, then any coefficient in the harmonic decomposition of the fractional kernel $k_n(x, u)$ (4) can be expressed as a linear composition of the elements $\Phi_j(x)\Phi_j^*(u)$. For the kernel of the fractional transform to be real, the Fourier spectrum of the fractional kernel with respect to the parameter α has to be symmetric; this means that $|k_{-n}(x, u)| = |k_n(x, u)|$. Since the coefficients $k_n(x, u)$ with different indices n contain disjoint series of the orthogonal elements, their amplitudes cannot be equal. In the case that there exists a complete orthonormal set of eigenfunctions $\{\Phi_n\}$ for the R-transform, the fractional kernel of the R^α-transform cannot be real, even if the R-transform kernel is real.

As we have seen above the fractional Hilbert kernel can be real, because there is no complete orthonormal set of eigenfunctions for the Hilbert transform.

7. OPTICAL FRACTIONAL FOURIER TRANSFORM

Although the FT can be divided into different fractions, the particular way of fractionalization corresponding to Eq. (26) certainly has advantages for application in optical signal processing. First, because this fractional FT can be obtained experimentally by using a simple optical set up, and secondly, because it produces a rotation of two fundamental joint distributions: the Wigner distribution (WD) and the ambiguity function (AF).

The fractional FT in this form was introduced more than 60 years ago in mathematical literature [13]; after that, it was reinvented for applications in quantum mechanics [14, 15], optics [9, 10, 12], and signal processing [16]. Thus, after the main properties of the fractional FT were established, the perspectives for its implementations in filter design, signal analysis, phase retrieval, watermarking, etc., became clear. Moreover, using refractive optics for an analog realization of the fractional FT opens a way for fractional Fourier optical signal processing. In this section we will point out the basic properties of the fractional FT and its applications in optics.

In Sections 7, 8, and 9, we will use a slightly different meaning of the parameter α, compared to the previous sections. The parameter α is now in the range from 0 to 2π instead of 0 to 4, and the ordinary FT arises for $\alpha = \frac{1}{2}\pi$ instead of $\alpha = 1$.

We define the fractional FT of an optical signal $f(x)$ as

$$\mathcal{R}^{\alpha}\left[f(x)\right](u) = F_{\alpha}(u) = \int_{-\infty}^{\infty} K(\alpha, x, u) f(x) dx, \tag{39}$$

where the kernel $K(\alpha, x, u)$ is given by

$$K(\alpha, x, u) = \frac{\exp(\frac{1}{2}i\alpha)}{\sqrt{i \sin\alpha}} \times \exp\left[i\pi \frac{(x^2+u^2)\cos\alpha - 2ux}{\sin\alpha}\right]. \tag{40}$$

The different meaning of the parameter α becomes apparent when we compare Eq. (40) with Eq. (26). The fractional FT can be considered as a generalization of the ordinary FT for the parameter α, which is now interpreted as the rotation angle in the phase plane. The phase plane is constructed by using Cartesian coordinates, where the x-axis and the y-axis correspond to the position and spatial-frequency variables, respectively. A signal is uniquely defined in the position domain, $f(x)$ $(\alpha = 0)$, or as its FT in frequency domain, $F_{\pi/2}(y)$. It follows from the additivity property that if one produces the fractional FT $F_{\alpha}(u)$ of a signal $f(x)$, then its FT $F_{\alpha+\pi/2}(v)$ is the fractional FT for the parameter α of $F_{\pi/2}(y)$. Thus the fractional FT is a signal representation along an axis u rotated at an angle α in the phase plane:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{41}$$

It was shown in [9, 10, 12] that the fractional FT describes the propagation of optical fields through a quadratic refractive index medium under the paraxial approximation of the scalar diffraction theory. Throughout this paper we will consider monochromatic waves under the paraxial approximation. Thus, if we consider an optical fiber of length L with a quadratic refractive index profile $n^2 = n_0^2(1 - g\mathbf{r}^2)$, where $\mathbf{r}$ is a transversal vector, then the complex field amplitude observed at the output plane is the fractional FT for the angle $\alpha = gL$ of the complex field amplitude at the input plane [9]-[12].

Two optical set ups with discrete refractive index elements (thin lenses) performing a fractional FT were proposed by Lohmann [17]. The first one contains a thin lens with focal length $f = \csc\alpha$, situated at equal distances $z = \tan(\frac{1}{2}\alpha)$ from the input and output planes. The second set up has two lenses of focal length $f = \cot(\frac{1}{2}\alpha)$, which are positioned just after the input plane and before the output plane; the distance between the two lenses equals $z = \sin\alpha$. In both cases the output complex field amplitude is related to the input one through the fractional FT for the angle α.

Another optical scheme for the fractional FT was described in [36]-[38]. In that scheme, the complex amplitudes at two spherical surfaces of given curvature and spacing are related by a fractional FT, where the angle is proportional to the Gouy phase shift between the two surfaces. This relationship can be helpful for the analysis of quasi-confocal resonators and transfer between a spherical emitter and receiver.

In the sequel, optical systems performing a fractional FT will be called fractional FT systems. Note that using cylindrical refractive index media allows to perform a separable, two-dimensional fractional FT for different angles in the two dimensions [39, 40]. For the sake of simplicity we consider the one dimensional case in this paper, only.

Table 1: Fractional Fourier Transform Properties

Linearity:

$$\mathcal{R}^{\alpha}\left[af(x) + bg(x)\right](u) = a\mathcal{R}^{\alpha}\left[f(x)\right](u) + b\mathcal{R}^{\alpha}\left[g(x)\right](u)$$

Parseval's equality:

$$\int_{-\infty}^{\infty} f(x) g^*(x) dx = \int_{-\infty}^{\infty} F_{\alpha}(u) G_{\alpha}^*(u) du$$

Shift theorem:

$$\mathcal{R}^{\alpha}\left[f(x-\tau)\right](u) = F_{\alpha}(u - \tau\cos\alpha) \times \exp\left(i\pi\sin\alpha\left(\tau^2\cos\alpha - 2u\tau\right)\right)$$

Modulation theorem:

$$\mathcal{R}^{\alpha}\left[f(x)\exp(i2\pi\nu x)\right](u) = F_{\alpha}(u - \nu\sin\alpha) \times \exp\left[-i\pi\cos\alpha\left(\nu^2\sin\alpha - 2u\nu\right)\right]$$

Scaling theorem:

$$\mathcal{R}^{\alpha}\left[f(cx)\right](u) = \sqrt{\cos\beta/\cos\alpha}\, \exp\left[\tfrac{1}{2}i(\alpha-\beta)\right] \times \exp\left[i\pi u^2 \cot\alpha\left(1 - \frac{\cos^2\beta}{\cos^2\alpha}\right)\right] F_{\beta}\left(u\frac{\sin\beta}{c\sin\alpha}\right),$$

where $\tan\beta = c^2 \tan\alpha$

The main theorems known for the ordinary FT

Table 2: Fractional Fourier Transform of Some Common Functions

$f(x)$	$F_\alpha(u)$
$\delta(x-\tau)$	$\exp(\frac{1}{2}i\alpha)/\sqrt{i\sin\alpha}$ $\times\ \exp\left[i\pi\frac{(\tau^2+u^2)\cos\alpha-2u\tau}{\sin\alpha}\right]$
$\exp(i2\pi x\nu)$	$\exp(\frac{1}{2}i\alpha)/\sqrt{\cos\alpha}$ $\times\ \exp\left[-i\pi(\nu^2+u^2)\tan\alpha\right]$ $\times\ \exp(i2\pi u\nu\sec\alpha)$
$\exp\left(ic\pi x^2\right)$	$\exp(\frac{1}{2}i\alpha)/\sqrt{\cos\alpha+c\sin\alpha}$ $\times\ \exp\left(i\pi u^2\frac{c-\tan\alpha}{1+c\tan\alpha}\right)$
$H_n(\sqrt{2\pi}x)$ $\times\exp(-\pi x^2)$	$H_n(\sqrt{2\pi}u)\exp(-\pi u^2)\exp(-in\alpha)$, H_n is the n-th Hermite polynomial
$\exp\left(-c\pi x^2\right)$	$\exp(\frac{1}{2}i\alpha)/\sqrt{\cos\alpha+ic\sin\alpha}$ $\times\exp\left[\pi u^2\frac{i(c^2-1)\cot\alpha-c\csc^2\alpha}{c^2+\cot^2\alpha}\right]$

were generalized to the case of the fractional FT [12, 16, 23] and can be found in Table 1. Table 2 contains some selected fractional FT pairs.

As we have mentioned above, the important property of the fractional FT is that it produces a rotation of the Wigner distribution and the ambiguity function [16, 17]. The Wigner distribution of $f(x)$ is defined as [4]

$$W_f(x,y)=\int_{-\infty}^{\infty}f\left(x+\tfrac{1}{2}\tau\right)f^*\left(x-\tfrac{1}{2}\tau\right)\times\ \exp\left(-i2\pi y\tau\right)d\tau. \tag{42}$$

The WD is always real-valued, but not necessarily positive, preserves position and frequency shifts, and satisfies the marginal properties, i.e., the frequency and position integrals of the WD correspond to the signal's instantaneous power $|f(x)|^2$ and its spectral energy density $|F_{\pi/2}(y)|^2$, respectively. The WD can be considered as the signal energy distribution over the phase plane, although the uncertainty principle prohibits the interpretation as a point position-frequency energy density. The ambiguity function is closely related to the WD through the double FT

$$A_f(\tau,\theta)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W_f(x,y)\times\ \exp\left[-i2\pi(x\tau-y\theta)\right]dxdy.$$

The application of the WD and the AF for the description of coherent and partially coherent optical fields was proposed more then 20 years ago [5]-[7] and became very attractive after the introduction of the optical fractional FT, due to the rather simple technique of reconstructing these functions from the fractional FT squared moduli. Note that for partially coherent fields the WD is related to the two-point correlation function, also called the mutual intensity, $\Gamma(x_1,x_2)=\langle f(x_1)f^*(x_2)\rangle$, where the brackets indicate an ensemble average over the set of realizations of the complex field amplitudes $f(x)$, as

$$W_f(x,y)=\int_{-\infty}^{\infty}\Gamma\left(x+\tfrac{1}{2}\tau,x-\tfrac{1}{2}\tau\right)\exp\left(-i2\pi y\tau\right)d\tau. \tag{43}$$

Since the fractional FT produces a rotation in the phase plane for an angle α,

$$\begin{array}{lcl} f(x) & \longleftrightarrow & W_f(x,y) \\ \downarrow \text{fractional FT} & & \downarrow \text{rotation of WD} \\ \mathcal{R}^\alpha\left[f\right] & \longleftrightarrow & W_{F_\alpha}(x,y) \\ & & =W_f(x\cos\alpha-y\sin\alpha, \\ & & \quad x\sin\alpha+y\cos\alpha), \end{array} \tag{44}$$

we obtain that the fractional power spectra $|F_\alpha(x)|^2$ are the projections of the Wigner distribution in the direction α in phase space

$$|F_\alpha(x)|^2=\int_{-\infty}^{\infty}W_f(x\cos\alpha-y\sin\alpha,x\sin\alpha+y\cos\alpha)dy. \tag{45}$$

The squared modulus of the fractional FT $|F_\alpha(x)|^2$, also called the fractional power spectrum, plays an important role in fractional optics. First, it is related to the intensity distribution in the corresponding fractional FT optical system and therefore can be measured in the experiments. Secondly, it corresponds to the projection of the Wigner distribution in a direction corresponding to the angle α in phase space [19, 50].

The set of fractional power spectra for the angles $\alpha\in[0,\pi)$ is called the Radon-Wigner transform (RWT), because it defines the Radon transform [4] of the WD. The WD can be obtained from the RWT by applying an inverse Radon transform. On the basis of this property, a phase space tomography method for the experimental determination of the amplitude and phase structure of a quasimonochromatic wave field in a transversal plane was proposed [18]. The implementation of cylindrical lenses permits to reconstruct the two-dimensional complex field amplitude $f(\mathbf{x})$ in the case of completely coherent wave fields, or the four-dimensional mutual intensity $\Gamma(\mathbf{x_1}, \mathbf{x_2})$ in the case of partially coherent waves.

The RWT can be considered as a quadratic distribution of the signal, which has very advantageous properties. It is positive and invertible, and it ideally combines the concepts of the instantaneous power and the spectral energy density. The association of the RWT with intensity distributions allows its direct measurement in optics. Moreover, the knowledge of the fractional power spectra permits to construct any Cohen's class bilinear phase-space distribution [27] and to estimate the main signal parameters such as instantaneous frequency, bandwidth, etc. This means that for a proper analysis of an optical signal, only the intensity distributions in the fractional FT system have to be measured.

In signal processing, the RWT was primarily developed for detection and classification of multicomponent linear FM in noise [24, 25]. Since the fractional FT of a chirp-type signal $\exp\left(-ic\pi x^2\right)$ is a δ function for the angle $\alpha = \operatorname{arccot} c$ (see Table 2), it can be detected as a local maximum on the RWT map. Analogously, in order to remove chirp-type noise, a notch filter that minimizes the signal information loss, can be placed at the proper point in the corresponding fractional FT domain. Thus, instead of performing the filtering in the space or the spatial-frequency domain, it can be done in the appropriate fractional domain, for example in the domain that corresponds to the best signal/noise position-frequency separation [50].

It was shown that the analysis of the global fractional FT moments [26, 27]

$$\int_{-\infty}^{\infty} |F_\alpha(x)|^2 x^n dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n \times W_f(x\cos\alpha - y\sin\alpha, x\sin\alpha + y\cos\alpha)dxdy \quad (46)$$

is helpful, for example, in search for an appropriate fractional domain to perform filtering operations [28]. Thus, the minimum of the second-order central fractional FT moment corresponds to the fractional domain where a signal is best concentrated along the α-radial line in the phase plane. Note that all second-order FT moments can be obtained from the knowledge of only three fractional FT power spectra [26]. Smoothing interferograms in the optimal fractional domain leads to a weighted Wigner distribution with significantly reduced interference terms of multicomponent signals, while the auto terms remain almost the same as in the WD. In general, we can conclude that the analysis of the fractional FT moments provides a low-cost construction of a signal-adaptive filter.

Instead of global moments, one can consider local fractional FT moments, which are related, for instance, to the instantaneous spatial frequency (or the group delay, if we consider the Fourier conjugated domain) at the different fractional FT domains. The instantaneous spatial frequency at the fractional domain is defined as [26]

$$U_{F_\beta}(x) = \frac{\int_{-\infty}^{\infty} yW_f(x\cos\beta - y\sin\beta, x\sin\beta + y\cos\beta)dy}{\int_{-\infty}^{\infty} W_f(x\cos\beta - y\sin\beta, x\sin\beta + y\cos\beta)dy} = \tfrac{1}{2}\left|F_\beta(x)\right|^{-2}\int_{-\infty}^{\infty}\left.\frac{\partial\left|F_\alpha(z)\right|^2}{\partial\alpha}\right|_{\alpha=\beta}\operatorname{sgn}(z-x)dz. \quad (47)$$

It is related to the phase $\varphi_\beta(x) = \arg F_\beta(x)$ of the fractional FT through

$$2\pi U_{F_\beta}(x) = d\varphi_\beta(x)/dx. \quad (48)$$

This implies that the derivative of the fractional power spectra with respect to the angle α defines the instantaneous spatial frequency at the fractional domain, and that it can be used for solving the phase retrieval problem of coherent fields from only two very close fractional power spectra [26, 27, 29].

The fractional power spectra are also used for the analysis of fractal structures [30]-[33]. On the basis of the scaling theorem for the fractional FT (see Table 1), it was shown in [30]-[33] that the fractional power spectra of a homogeneous function $f(\lambda x) = \lambda^H f(x)$ (with a scaling factor λ) at the angles α and β such that $\cot\alpha = \lambda^2\cot\beta$, are affine:

$$\left|F_\alpha(u)\right|^2 = \lambda^{-2(H+1)}\frac{\sin\beta}{\sin\alpha}\left|F_\beta\left(\frac{\sin\beta}{\sin\alpha}\lambda^{-1}u\right)\right|^2. \quad (49)$$

In particular, for $\cot\alpha = \lambda$ and $\beta = \frac{1}{2}\pi - \alpha$ we obtain

$$\left|F_\alpha(u)\right|^2 = \lambda^{-2H-1}\left|F_{\pi/2-\alpha}(u)\right|^2. \quad (50)$$

Thus the fractional power spectra of a homogeneous function for appropriately symmetric angles are the same, except for a normalization factor. The Hurst exponent H can then be found from the ratio of the two fractional power spectra. According to Eqs. (49) and (50), the evolution of the intensity distributions of a fractal optical field in the fractional FT system provides information about its hierarchical structure and can be used to determine main characteristics such as the Hurst exponent, the scaling factor, and the fractal dimension. Note that the discussed method enables us to investigate deterministic as well as random fractal optical fields [32].

The fractional FT is also used for signal/image copyright protection [34, 35]. Instead of embedding a watermark in the spatial or the spatial-frequency domain, it is introduced in the fractional Fourier domain. The robustness of this watermarking to important image processing attacks such as translation, rotation, cropping, and filtering, was demonstrated in [34, 35]. Since for such kind of watermarking the chirp signal $s(x, y) = \cos(a_x x^2 + a_y y^2)$ is usually used as a signature, for its detection the fractional power spectra in the corresponding fractional α_x, α_y-domain are analyzed. Thus, for the α_x, α_y-domain with $\alpha_{x,y}$ such that $\cot \alpha_{x,y} = \pm a_{x,y}$, a concentration of the watermark will be achieved, while the image will still be spread. Only the owner of the image, who knows all the fractional domains where the signature is concentrated, will be able to remove it. In general, watermarking in the fractional Fourier domains offers more flexible protection algorithms than standard watermarking in the spatial or the spatial-frequency domain.

The application of the fractional FT for image encryption and for the creation of neural networks was reported in [41, 42] and [43, 44], respectively, while optical mode analysis through the fractional FT was considered in [45].

In this section we have given a short overview of the main properties of the optical fractional FT and of its applications for signal/image processing. Nevertheless, this research area is rapidly developing, and we believe that novel optical devices based on the fractional FT will appear in the near future.

8. OPTICAL FRACTIONAL HANKEL TRANSFORM

The two-dimensional fractional FT of a rotationally symmetric function leads to the fractional Hankel transform, analogous to the fact that its two-dimensional FT produces the Hankel transform. Although there are several ways for the fractionalization of the Hankel transform, we will consider here the one that is closely related to the optical fractional FT and that describes the propagation of rotationally symmetric beams through a spherical quadratic refractive index medium [54, 55].

The fractional Hankel transform of a function $f(r)$ is defined as

$$\mathcal{R}^{\alpha} \left[f(r) \right] (u) = H_{\alpha}(u) = \int_0^{\infty} K(\alpha, r, u) f(r) r dr, \tag{51}$$

where the kernel $K(\alpha, r, u)$ is given by

$$K(\alpha, r, u) = \frac{\exp(i\alpha)}{i \sin \alpha}$$

$$\times \; \exp\left[i\pi (r^2 + u^2) \cot \alpha \right] J_0 \left(2\pi r u / \sin \alpha \right) \tag{52}$$

with J_0 the first-type, zero-order Bessel function. One can represent the fractional Hankel kernel in the form (25), where $L_n = -n$, $l_n = 0$, and $\Phi_n(x)$ are the Laguerre-Gauss functions, which are the eigenfunctions of the fractional Hankel transform.

The fractional Hankel transform inherits the main properties of the fractional FT [54, 56] and can be performed by the fractional FT set ups described in the previous section, if the input optical field is rotationally symmetric: $f(\mathbf{r}) = f(r)$. Although the fractional Hankel transform can substitute the fractional FT in many optical signal processing tasks where rotationally symmetric beams are used, the particular algorithms have not been explored yet.

9. FRACTIONAL SINE, COSINE, AND HARTLEY TRANSFORMS

Attempts to introduce the fractional Sine, Cosine, and Hartley transforms were made in [50, 61], where the authors supposed that the kernels of these transforms are the real part of the kernel of the optical fractional FT, the imaginary part of this kernel, or the sum of these two parts, respectively. Nevertheless, they have mentioned that the transforms defined in such a manner, are not angle additive, and therefore, in our view, cannot be interpreted as fractional transforms. Here we discuss the fractional Sine, Cosine, and Hartley transforms (ST, CT, HT) [57, 62] that are angle additive and that are related to the optical fractional FT in the same way as the ordinary ST, CT, and HT are connected to the FT.

Whereas the ordinary FT is defined as

$$R_F^{\pi/2} \left[f(x) \right] (u) = \int_{-\infty}^{\infty} f(x) \exp(-i 2\pi u x) dx,$$

the ST, CT, and HT of a function $f(x)$ are defined as follows:

$$\begin{aligned} R_S[f(x)](u) &= 2\int_0^\infty f(x)\sin(2\pi ux)dx, \\ R_C[f(x)](u) &= 2\int_0^\infty f(x)\cos(2\pi ux)dx, \qquad (53) \\ R_H[f(x)](u) &= \int_{-\infty}^\infty f(x)\,\mathrm{cas}(2\pi ux)dx, \end{aligned}$$

where $\mathrm{cas}(x) = \cos x + \sin x = \sqrt{2}\cos(x - \frac{1}{4}\pi)$. The forward and inverse Sine, Cosine, and Hartley transforms are identical in forms, which implies that the transformations are cyclic with period $N = 2$. Moreover, if they are applied to real functions $f(x)$, the transforms are real, too, which might be important for optical image processing.

The HT R_H is closely related to the FT $R_F^{\pi/2}$ and the inverse FT $R_F^{-\pi/2} = R_F^{3\pi/2}$. Thus, an alternative method of defining the HT is

$$R_H[f(x)](u) = 2^{-1/2}\exp(\tfrac{1}{4}i\pi) \times \left\{R_F^{\pi/2}[f(x)](u) - iR_F^{3\pi/2}[f(x)](u)\right\}. \qquad (54)$$

The Hartley and Fourier transforms have the same set of orthogonal eigenfunctions $\Psi_n(x) = (2^{n-1/2}n!)^{-1/2}\exp(-i\pi x^2)H_n(\sqrt{2\pi}x)$, viz., Hermite-Gauss functions, but with different eigenvalues. Thus

$$R_H[\Psi_n(x)](u) = \begin{cases} \Psi_n(u) & \text{for } n = 4m \\ & \text{or } n = 4m+1, \\ -\Psi_n(u) & \text{for } n = 4m+2 \\ & \text{or } n = 4m+3, \end{cases} \qquad (55)$$

where m is a nonnegative integer. The eigenfunctions for the ST and CT are the odd- and even-order Hermite-Gauss functions, respectively. Moreover, it is easy to see that the functions $\sqrt{2}\Psi_{2n+1}(x)$ are orthonormal on the half range,

$$2\int_0^\infty \Psi_{2n+1}(x)\Psi_{2m+1}(x)dx = \delta_{n,m}, \qquad (56)$$

and that the same holds for the functions $\sqrt{2}\Psi_{2n}(x)$. Since there exists a complete orthonormal set of eigenfunctions $\{\Phi_n\}$ for the ST, CT, and HT, the kernels of the corresponding fractional transforms $H(\alpha, x, u)$ cannot be real (see Section 6), in spite of the fact that the kernels of the ordinary ST, CT, and HT are real.

As we have seen in Section 5, one can easily construct a kernel for the fractional transform, if the set of orthonormal eigenfunctions for the integer transform is known. In order to preserve relationships between the fractional ST, CT, and HT, similar to the ones that hold between the ordinary transforms, we choose the following kernels for the fractional ST, CT, and HT:

$$\begin{aligned} K_S(\alpha,x,u) &= 2\sum_{n=0}^\infty \exp\left(-\tfrac{1}{2}i\alpha n\right)\Psi_{2n+1}(x)\Psi_{2n+1}(u), \\ K_C(\alpha,x,u) &= 2\sum_{n=0}^\infty \exp\left(-\tfrac{1}{2}i\alpha n\right)\Psi_{2n}(x)\Psi_{2n}(u), \\ K_H(\alpha,x,u) &= \sum_{n=0}^\infty \exp\left(-\tfrac{1}{2}i\alpha n\right)\left[\Psi_{2n}(x)\Psi_{2n}(u)\right. \\ &\quad \left. + \Psi_{2n+1}(x)\Psi_{2n+1}(u)\right]. \end{aligned} \qquad (57)$$

These fractional kernels can be rewritten in closed forms, which are similar to the fractional FT kernel

$$\begin{aligned} K_S(\alpha,x,u) &= 2\ k_\alpha(x,u)\sin\left(2\pi ux/\sin\alpha\right), \\ K_C(\alpha,x,u) &= 2\ k_\alpha(x,u)\cos\left(2\pi ux/\sin\alpha\right), \qquad (58) \\ K_H(\alpha,x,u) &= k_\alpha(x,u)\,\mathrm{cas}\left(2\pi ux/\sin\alpha\right), \end{aligned}$$

where

$$k_\alpha(x,u) = \frac{\exp(\frac{1}{2}i\alpha)}{\sqrt{i\sin\alpha}}\exp\left[i\pi\left(x^2+u^2\right)\cot\alpha\right]. \qquad (59)$$

Comparing Eqs. (40) and (58), we obtain the connection between the optical fractional HT operator R_H^α and the optical fractional FT operator R_F^α as

$$R_H^\alpha = \exp\left(\tfrac{1}{2}i\alpha\right)\left[\cos(\tfrac{1}{2}\alpha)R_F^\alpha - i\sin(\tfrac{1}{2}\alpha)R_F^{\alpha+\pi}\right]. \qquad (60)$$

This is the generalization of relationship (54) for an arbitrary parameter α. As the fractional FT optical set up is well known [50], this equation shows the way to an optical realization of the fractional HT. One of the possible schemes is given in [62].

On the other hand, it is easy to see that the fractional ST and CT describe the propagation of monochromatic waves through a refractive medium with a half quadratic profile, $n^2 = n_0^2(1 - gx^2)$ for $x > 0$ and $n^2 = \infty$ for $x < 0$. The fractional ST and CT satisfy the different boundary conditions for $u = 0$, $F_S^\alpha(u) = 0$ and $\partial F_C^\alpha(u)\partial u = 0$, respectively.

As the fractional optical ST, CT, and HT have been introduced only recently, their particular applications for optical signal analysis have not been well explored yet. Nevertheless, it seems that due to the similarity between the fractional FT and the fractional ST, CT, and HT, and the existence of optical devices that can perform these transforms, their application in optical signal/image processing is very promising.

10. FRACTIONAL HILBERT TRANSFORM

It has been shown in Section 4 that the kernel of the fractional Hilbert transform has only two harmonics (in spite of the fact that it is periodic with $N = 4$), which significantly reduces the number of possible fractionalization procedures. Here we will consider the fractional Hilbert transform introduced in [51, 52]. Its kernel is described by Eq. (19). The optical set up performing this transform was proposed in [51, 52]. As the fractional Hilbert transform is a weighted mixture of the optical field $f(u)$ itself and its Hilbert transform $H(u)$,

$$R^{\alpha}\left[f(x)\right](u) = f(u)\cos\left(\tfrac{1}{2}\pi\alpha\right) + H(u)\sin\left(\tfrac{1}{2}\pi\alpha\right), \tag{61}$$

an optical scheme similar to the one performing the ordinary Hilbert transform is used. We recall that the ordinary Hilbert transform can be optically implemented as a spatial-filtering process, whereby half the Fourier spectrum is π phase-shifted. The fractional degree α defines the weighting parameters and can be varied easily by a rotation of the polarizers used in the optical set up.

Like the ordinary Hilbert transform, the fractional Hilbert transform is used for edge enhancement [51]-[53]. It was shown that increasing the fractional order changes the nature of the edge enhancement. Thus, for $\alpha \approx \frac{1}{2}, 1, 1\frac{1}{2}$, the right-hand edges, both edges, and the left-hand edges of the input object are emphasized, respectively. In general we can conclude that the fractional Hilbert transform produces an output image that is selectively edge enhanced. This property of the fractional Hilbert transform makes it a perspective tool for image processing and pattern recognition.

11. CONCLUSIONS

In this paper we have considered a general method for the fractionalization of different types of cyclic transforms. The usefulness of a specific fractional transform is related with its optical feasibility as well as with its application in signal/image processing. We have shown that the analysis of the harmonic contents for various types of fractional transforms offers a procedure for their experimental realization.

The applications of the optical fractional FT for adaptive filter design, phase retrieval, complex signal/image characterization, watermarking, etc. were considered. Although the theoretical and numerical simulation works demonstrate an important impact of the optical implementation of the fractional FT, the experimental realization of the corresponding devices is still to be expected. We believe that the creation of fractional Fourier optics increases the importance of analog optical information processing significantly.

Another direction for further research is the investigation of the properties of the different types of the fractional FT and their possible implementations in optics and signal processing. From our point of view, these transforms can be suitable for signal/image encryption and watermarking.

A next research area is the exploration of other fractional cyclic transforms, including fractional Sine, Cosine, Hartley, Hankel, and Hilbert transforms. Thus, the fractional Hilbert transform seems to be very promising for selective edge enhancements. Meanwhile, the fractional Sine, Cosine, Hartley, and Hankel transforms, due to their similarity to the optical fractional FT, may act as a substitute for it in many tasks.

In general, the further development of linear fractional optics makes it competitive with nonlinear optics in applications for signal/image processing. The design of new devices based on fractional optics, will lead to unified approaches of signal/image processing used in optics and electrical engineering, which will significantly enrich the fields of optoelectronics, optical security technology, and optical computing.

ACKNOWLEDGMENTS

This work has been financially supported partially by the Rectorate of the Complutense University of Madrid (UCM), under Multidisciplinary Project PR486/97-7477/97. One of us (T. A.) would like to thank a grant from UCM.

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