# CLASSIFICATION OF THE LINEAR CANONICAL TRANSFORMATION AND ITS ASSOCIATED REAL SYMPLECTIC MATRIX


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## ABSTRACT

Based on the eigenvalues of the real symplectic $\mathbf{ABCD}$-matrix that characterizes the linear canonical integral transformation, a classification of this transformation and the associated $\mathbf{ABCD}$-system is proposed and some nuclei (i.e. elementary members) in each class are described. In the one-dimensional case, possible optical nuclei are the magnifier, the lens, and the fractional Fourier transformer; in the two-dimensional case, we have – in addition to the obvious concatenations of one-dimensional nuclei – the four combinations of a magnifier or a lens with a rotator or a shearing operator, where the rotator and the shearer are obviously inherently two-dimensional. Any $\mathbf{ABCD}$-system belongs to one of the classes described in this paper and is similar (in the sense of similarity of the respective symplectic matrices) to the corresponding nucleus.


## 1. INTRODUCTION

Many papers have been published about the linear canonical integral transformation [1, 2, 3] and about lossless first-order optical systems [4, 5] (also known as $\mathbf{ABCD}$-systems) associated to such canonical transformations. There seems, however, no complete overview of the different classes that can be distinguished for the canonical transformation, except for a paper by Pei and Ding [6], in which the one-dimensional case was treated. In the present paper we consider the two-dimensional case, i.e., lossless first-order optical systems with two transverse coordinates, $x$ and $y$, combined into a two-dimensional column vector $\mathbf{r} = (x, y)^t$, and with a $4 \times 4$ matrix $\mathbf{T}$ (also known as the ray transformation matrix or $\mathbf{ABCD}$-matrix in optics) through which such a system and the associated canonical transformation can be described. In particular, we propose a classification of first-order optical systems, based on the eigenvalues of the ray transformation matrix $\mathbf{T}$, where $\mathbf{T}$ is real and symplectic.

We recall that symplecticity of the $2D \times 2D$ matrix $\mathbf{T}$ can be expressed as

$$\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{D}^t & -\mathbf{B}^t \\ -\mathbf{C}^t & \mathbf{A}^t \end{bmatrix}, \qquad (1)$$

where all submatrices are $D \times D$; in the present paper we will restrict ourselves to the cases $D = 1$ and $D = 2$. We also recall that if $\lambda$ is an eigenvalue of a real symplectic matrix, then $\lambda^*$, $1/\lambda$, and $1/\lambda^*$ are eigenvalues, too [7].

For completeness, we recall that in the case of a non-singular submatrix $\mathbf{B}$ the canonical transformation $f_i(\mathbf{r}) \rightarrow f_o(\mathbf{r})$ can be represented in the form of Collins integral [1]:

$$f_o(\mathbf{r}_o) = \frac{\exp[i\pi \mathbf{r}_o^t \mathbf{D}\mathbf{B}^{-1}\mathbf{r}_o]}{\sqrt{\det i\mathbf{B}}} \int_{-\infty}^{\infty} f_i(\mathbf{r}_i) \times \exp[i\pi(\mathbf{r}_i^t \mathbf{B}^{-1}\mathbf{A}\mathbf{r}_i - 2\mathbf{r}_i^t \mathbf{B}^{-1}\mathbf{r}_o)]\, d\mathbf{r}_i. \qquad (2)$$

## 2. EIGENVALUES OF REAL SYMPLECTIC MATRICES

We recall that each matrix $\mathbf{T}$ is similar to a Jordan matrix $\mathbf{\Delta}$ [8]: $\mathbf{T} = \mathbf{Q}\mathbf{\Delta}\mathbf{Q}^{-1}$. Let us consider the possibilities for $D = 1$ and $D = 2$.

For $D = 1$ we have the following three cases:

**1.** A pair of real eigenvalues $s$ and $s^{-1}$ ($s \neq \pm 1$), with two linearly independent eigenvectors; the Jordan matrix reads

$$\mathbf{\Lambda}_m(s) = \begin{bmatrix} s & 0 \\ 0 & s^{-1} \end{bmatrix}. \qquad (3)$$

**2.** Two real eigenvalues $\lambda = 1$ or $\lambda = -1$, with only one eigenvector; the Jordan matrix reads

$$\mathbf{J}_+(\lambda) = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad \text{or} \quad \mathbf{J}_-(\lambda) = \begin{bmatrix} \lambda & 0 \\ -1 & \lambda \end{bmatrix}. \qquad (4)$$

For convenience, when $\lambda = 1$ we will simply write the short-hand forms $\mathbf{J}_+ = \mathbf{J}_+(1)$ and $\mathbf{J}_- = \mathbf{J}_-(1)$.

**3.** A pair of two unimodular, complex conjugated eigenvalues $\exp(i\theta)$ and $\exp(-i\theta)$, with two linearly independent eigenvectors; the Jordan matrix reads

$$\mathbf{\Lambda}_f(\theta) = \begin{bmatrix} \exp(i\theta) & 0 \\ 0 & \exp(-i\theta) \end{bmatrix}. \qquad (5)$$

For $D = 2$ we have of course concatenations of the one-dimensional Jordan matrices and in addition to these concatenations, the following four inherently two-dimensional cases arise:

**4.** A complex quartet of eigenvalues $s\exp(i\theta)$, $s\exp(-i\theta)$, $s^{-1}\exp(-i\theta)$, and $s^{-1}\exp(i\theta)$ ($s \neq \pm 1$), with four linearly independent eigenvectors; the Jordan matrix reads

$$\begin{bmatrix} s\exp(i\theta) & 0 & 0 & 0 \\ 0 & s\exp(-i\theta) & 0 & 0 \\ 0 & 0 & s^{-1}\exp(-i\theta) & 0 \\ 0 & 0 & 0 & s^{-1}\exp(i\theta) \end{bmatrix}. \quad (6)$$

**5.** Two identical pairs of unimodular, complex conjugated eigenvalues $\exp i\theta$ and $\exp(-i\theta)$, with only two linearly independent eigenvectors; the Jordan matrix reads

$$\begin{bmatrix} \mathbf{J}_+(\exp(i\theta)) & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_+(\exp(-i\theta)) \end{bmatrix}$$
$$\text{or} \quad \begin{bmatrix} \mathbf{J}_-(\exp(i\theta)) & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_-(\exp(-i\theta)) \end{bmatrix}. \quad (7)$$

**6.** Two identical pairs of real eigenvalues $s$ and $s^{-1}$ ($s \neq \pm 1$), with only two linearly independent eigenvectors; the Jordan matrix reads

$$\begin{bmatrix} \mathbf{J}_+(s) & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_+(s^{-1}) \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \mathbf{J}_-(s) & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_-(s^{-1}) \end{bmatrix}. \quad (8)$$

**7.** Four real eigenvalues $\lambda = 1$ or $\lambda = -1$, with only one eigenvector; the Jordan matrix reads

$$\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \lambda & 0 & 0 & 0 \\ -1 & \lambda & 0 & 0 \\ 0 & -1 & \lambda & 0 \\ 0 & 0 & -1 & \lambda \end{bmatrix}. \quad (9)$$

The different cases can be represented elegantly in an $a_1$–$a_2$ plane, see Fig. 1, where $a_1 = a_3$ and $a_2$ are the coefficients in the characteristic equation

$$\det(\mathbf{T} - \lambda\mathbf{I}) = \lambda^4 - a_3\lambda^3 + a_2\lambda^2 - a_1\lambda + 1 = 0. \quad (10)$$

Two lines (1 and 2) and one parabola (3) are relevant:

1. $a_2 = 2a_1 - 2$
$$\begin{cases} a_2 > 6 & \text{occupied by case 1-2} \\ a_2 < -2 & \text{occupied by case 1-2} \\ -2 \le a_2 \le 6 & \text{occupied by case 2-3} \end{cases}$$
2. $a_2 = -2a_1 - 2$
$$\begin{cases} a_2 > 6 & \text{occupied by case 1-2} \\ a_2 < -2 & \text{occupied by case 1-2} \\ -2 \le a_2 \le 6 & \text{occupied by case 2-3} \end{cases}$$
3. $a_2 = 2 + a_1^2/4$
$$\begin{cases} a_2 > 6 & \text{occupied by case 6} \\ 2 \le a_2 \le 6 & \text{occupied by case 5} \end{cases}$$

together with the three points

$(a_1, a_2) = (4, 6)$ occupied by cases 2-2 and 7
$(a_1, a_2) = (-4, 6)$ occupied by cases 2-2 and 7
$(a_1, a_2) = (0, -2)$ occupied by case 2-2.

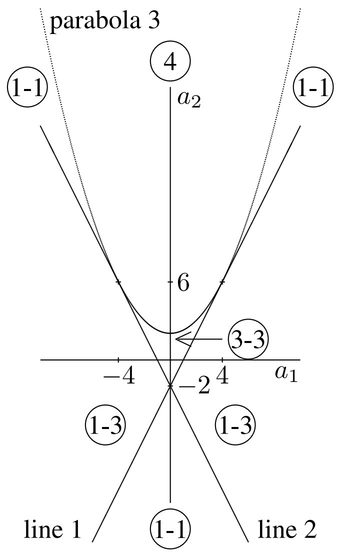


Figure 1: The different areas for the two-variable cases 1-1, 1-3, 3-3 (boundaries included), and 4. For the one-variable cases we have case 1-2: lines 1 and 2 ($a_2 > 6$ or $a_2 < -2$); case 2-3: lines 1 and 2 ($-2 \le a_2 \le 6$); case 5: parabola ($a_2 \le 6$); and case 6: parabola ($a_2 > 6$). For the isolated points we have case 2-2: $(a_1, a_2) = (\pm 4, 6)$ or $(a_1, a_2) = (0, -2)$; and case 7: $(a_1, a_2) = (\pm 4, 6)$. The cases 2-2, 2-3, 3-3, 5, and 7 correspond to unimodular eigenvalues, while the other cases have two (cases 1-2 and 1-3) or four (cases 1-1, 4, and 6) non-unimodular eigenvalues.

The problem with the common Jordan matrix associated to a real symplectic matrix is that it does not automatically preserve the properties of realness [see Eqs. (5), (6), and (7)] and symplecticity [see Eq. (9)]. Instead of the Jordan matrix, we therefore look for other matrices in these four cases: matrices with a minimum number of parameters, which do preserve these properties. Since we only deal with $D = 1$ and $D = 2$, we can easily formulate these matrices. For higher-dimensional real symplectic matrices, we refer to the paper by Lin et al. [9], in particular Theorem 41.

## 3. OPTICAL NUCLEI ASSOCIATED TO REAL SYMPLECTIC MATRICES

We will now present first-order optical nuclei for the three cases that may occur for $D = 1$.

### 3.1. Magnifier $\mathcal{M}(s)$

The magnifier $\mathcal{M}(s)$, with ray transformation matrix

$$\mathbf{T}_m(s) = \mathbf{\Lambda}_m(s) = \begin{bmatrix} s & 0 \\ 0 & s^{-1} \end{bmatrix}, \quad (11)$$

is an obvious nucleus for the case of a pair of two (generally different) real eigenvalues: $s$ and $s^{-1}$. In particular, we have the unity operator $\mathbf{T}_m(1) = \mathbf{I}$ when $s = 1$ and the coordinate reverter $\mathbf{T}_m(-1) = -\mathbf{I}$ when

$s = -1$. In general we will restrict ourselves to real eigenvalues $s$ that are positive; in the case of negative real eigenvalues we simply add an additional coordinate reverter $-\mathbf{I}$ to the nucleus.

### 3.2. Lens $\mathcal{L}(f)$ and free space $\mathcal{S}(z)$

The lens $\mathcal{L}(f)$ and a section of free space $\mathcal{S}(z)$ (operating on light with wavelength $\lambda_\circ$) with ray transformation matrices

$$\mathbf{T}_l(f) = \begin{bmatrix} 1 & 0 \\ -1/\lambda_\circ f & 1 \end{bmatrix} \text{ and } \mathbf{T}_s(z) = \begin{bmatrix} 1 & \lambda_\circ z \\ 0 & 1 \end{bmatrix}, \quad (12)$$

respectively, are obvious nuclei for the case of two identical eigenvalues $\lambda = 1$ with only one eigenvector. The case of two negative eigenvalues $\lambda = -1$ can easily be dealt with, again, by means of an additional coordinate reverter. The case of a double eigenvalue $\lambda = \pm 1$ has been extensively treated by Pei and Ding [6, Section IV].

### 3.3. Fractional Fourier transformer $\mathcal{F}(\theta; w)$

The fractional Fourier transformer $\mathcal{F}(\theta; w)$, with ray transformation matrix

$$\begin{aligned} \mathbf{T}_f(\theta; w) &= \begin{bmatrix} \cos\theta & w^2 \sin\theta \\ -w^{-2}\sin\theta & \cos\theta \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & iw^2 \\ iw^{-2} & 1 \end{bmatrix} \begin{bmatrix} \exp(i\theta) & 0 \\ 0 & \exp(-i\theta) \end{bmatrix} \\ \times \sqrt{2} & \begin{bmatrix} 1 & iw^2 \\ iw^{-2} & 1 \end{bmatrix}^{-1} \equiv \mathbf{Q}_f(w)\, \mathbf{\Lambda}_f(\theta)\, \mathbf{Q}_f^{-1}(w), \quad (13) \end{aligned}$$

is an obvious nucleus for the case of a pair of two (generally different) complex conjugated unimodular eigenvalues $\exp(i\theta)$ and $\exp(-i\theta)$. Note that the eigenvalue matrix $\mathbf{\Lambda}_f(\theta)$ [see Eq. (5)] and the eigenvector matrix $\mathbf{Q}_f(w)$ are complex, and that the eigenvalue matrix itself thus cannot act as the ray transformation matrix of a proper nucleus.

A class of separable nuclei for two-dimensional optical systems, $D = 2$, can be constructed by concatenation of one-dimensional nuclei. The four remaining two-dimensional nuclei, corresponding to the cases 4, 5, 6, and 7, are inherently two-dimensional: one nucleus (case 4) with still a sufficient number of four linearly independent eigenvectors, and the other three nuclei (case 5, 6, and 7) with less eigenvectors. We will be able to choose all four remaining nuclei such that they will perform operations between conjugate planes like a magnifier and a lens do [with $\mathbf{B} = \mathbf{0}$], and to realize them in practice we need combinations of a rotator or a shearer with an isotropic magnifier or an isotropic lens; note that the rotator and the shearer are inherently two-dimensional, whereas the isotropic magnifier and the isotropic lens are concatenations of their one-dimensional versions.

### 3.4. Rotator–magnifier combination $\mathcal{M}(s, s)\, \mathcal{R}(\theta)$

There is one obvious case in the realm of diagonalizable real symplectic matrices that can only occur for $D > 1$, viz., the one in which we have a full complex quartet of eigenvalues: $\lambda$, $\lambda^*$, $1/\lambda$, and $1/\lambda^*$, with $\lambda$ not unimodular and not real. A possible nucleus for this class, with eigenvalues $s\exp(\pm i\theta)$ and $s^{-1}\exp(\pm i\theta)$, is the system with ray transformation matrix

$$\mathbf{T}_{mr}(\theta, s) = \begin{bmatrix} s\mathbf{R}(\theta) & \mathbf{0} \\ \mathbf{0} & s^{-1}\mathbf{R}(\theta) \end{bmatrix}, \quad (14)$$

where the $2 \times 2$ matrix $\mathbf{R}(\theta)$ is the rotation matrix

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \quad (15)$$

Note again that the eigenvalue matrix [see Eq. (6)] and the eigenvector matrix are complex, and that the eigenvalue matrix thus cannot act as the ray transformation matrix of a proper nucleus.

The matrix $\mathbf{T}_{mr}(\theta, s)$ is clearly the ray transformation matrix of a combination of a rotator $\mathcal{R}(\theta)$ with ray transformation matrix

$$\mathbf{T}_r(\theta) = \begin{bmatrix} \mathbf{R}(\theta) & \mathbf{0} \\ \mathbf{0} & \mathbf{R}(\theta) \end{bmatrix} \quad (16)$$

and an isotropic magnifier $\mathcal{M}(s, s)$ [see Eq. (11]:

$$\mathbf{T}_{mr}(\theta, s) = \mathbf{T}_m(s, s)\, \mathbf{T}_r(\theta) = \mathbf{T}_r(\theta)\, \mathbf{T}_m(s, s). \quad (17)$$

Note that the matrices $\mathbf{T}_r(\theta)$ and $\mathbf{T}_m(s, s)$ in the latter expression commute and that the order in the cascade does not matter.

From the general input-output relationship

$$f_o(\mathbf{r}) = f_i(\mathbf{A}^{-1}\mathbf{r}) \exp(i\pi \mathbf{r}^t \mathbf{C}\mathbf{A}^{-1}\mathbf{r}) / \sqrt{|\det \mathbf{A}|}, \quad (18)$$

which holds for a linear canonical transformation $f_i(\mathbf{r}) \to f_o(\mathbf{r})$ in the special case that $\mathbf{B} = \mathbf{0}$, we immediately derive the input-output relation for the rotator–magnifier combination [with $\mathbf{A} = s\mathbf{R}(\theta)$ and $\mathbf{C} = \mathbf{0}$] as

$$s f_o(sx, sy) = f_i(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta) \quad (19)$$

or in polar coordinates (with $x = r\cos\varphi$ and $y = r\sin\varphi$):

$$s f_o(sr, \varphi) = f_i(r, \varphi + \theta). \quad (20)$$

### 3.5. Rotator–lens combination $\mathcal{L}(f, f)\, \mathcal{R}(\theta)$

From the non-diagonalizable matrices, let us first consider the case of a double pair of complex conjugated unimodular eigenvalues. A possible nucleus for this class, with double eigenvalues $\exp(\pm i\theta)$, is the system with ray transformation matrix

$$\mathbf{T}_{lr}(\theta; f) = \begin{bmatrix} \mathbf{R}(\theta) & \mathbf{0} \\ -(1/\lambda_\circ f)\, \mathbf{R}(\theta) & \mathbf{R}(\theta) \end{bmatrix}, \quad (21)$$

which is a (commuting) combination of a rotator $\mathcal{R}(\theta)$ [see Eq. (16)] and an isotropic lens $\mathcal{L}(f, f)$ [see Eq. (12)]:

$$\mathbf{T}_{lr}(\theta; f) = \mathbf{T}_l(f, f)\, \mathbf{T}_r(\theta) = \mathbf{T}_r(\theta)\, \mathbf{T}_l(f, f). \quad (22)$$

The input-output relation of this system [with $\mathbf{A} = \mathbf{R}(\theta)$ and $\mathbf{C} = -(1/\lambda_\circ f)\,\mathbf{R}(\theta)$] reads

$$f_o(x,y) = f_i(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta) \times \exp[-i\pi\,(1/\lambda_\circ f)\,(x^2+y^2)] \quad (23)$$

or in polar coordinates again:

$$f_o(r,\varphi) = f_i(r,\varphi+\theta)\,\exp[-i\pi\,(1/\lambda_\circ f)\,r^2]. \quad (24)$$

Like a lens and a section of free space are each others' Fourier transforms, we can also find an alternative nucleus $\mathcal{F}(\pi/2;w)\,\mathcal{L}(f,f)\,\mathcal{R}(\theta)\,\mathcal{F}(-\pi/2;w) = \mathcal{S}(w^4/\lambda_\circ f)\,\mathcal{R}(\theta)$ by embedding the rotator–lens combination in between a Fourier transformer and its inverse.

### 3.6. Shearer–magnifier combination $\mathcal{M}(s,s)\,\mathcal{Z}$

Let us now consider the case of a double pair of real eigenvalues; as mentioned before, we will restrict ourselves to the case of positive real eigenvalues. A possible nucleus for this class, with double eigenvalues $s$ and $s^{-1}$, is the system with ray transformation matrix

$$\mathbf{T}_{mz}(s) = \begin{bmatrix} s\mathbf{J}_+ & \mathbf{0} \\ \mathbf{0} & s^{-1}\mathbf{J}_- \end{bmatrix} = \begin{bmatrix} s\mathbf{I} & \mathbf{0} \\ \mathbf{0} & s^{-1}\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{J}_+ & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_- \end{bmatrix}, \quad (25)$$

which is a (commuting) combination of a shearer $\mathcal{Z}$ with ray transformation matrix

$$\mathbf{T}_z = \begin{bmatrix} \mathbf{J}_+ & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_- \end{bmatrix}, \quad (26)$$

and an isotropic magnifier $\mathcal{M}(s,s)$:

$$\mathbf{T}_{mz}(s) = \mathbf{T}_m(s,s)\,\mathbf{T}_z = \mathbf{T}_z\mathbf{T}_m(s,s). \quad (27)$$

The input-output relation of this system [with $\mathbf{A} = s\mathbf{J}_+$ and $\mathbf{C} = \mathbf{0}$] reads

$$s f_o(sx, sy) = f_i(x-y, y), \quad (28)$$

which represents – apart from a magnification with $s$ – a simple shearing of the $x$ coordinate. Note that a shearer can be realized as a separable magnifier embedded in between two rotators:

$$\mathcal{Z} = \mathcal{R}(-\operatorname{arccot}\phi)\,\mathcal{M}(\phi,\phi^{-1})\,\mathcal{R}(\operatorname{arccot}\phi + \operatorname{arccot} 2), \quad (29)$$

with $\phi$ the golden ratio: $\phi^2 - \phi - 1 = 0$.

### 3.7. Shearer–lens combination $\mathcal{L}(f,f)\,\mathcal{Z}$

We finally consider the case of a four-fold eigenvalue $\lambda = 1$; for convenience we restrict ourselves again to the case $\lambda = 1$, while the case $\lambda = -1$ would simply require an additional coordinate reverter $-\mathbf{I}$. A possible nucleus for this class is the system with ray transformation matrix

$$\mathbf{T}_{lz}(f) = \begin{bmatrix} \mathbf{J}_+ & \mathbf{0} \\ -(1/\lambda_\circ f)\,\mathbf{J}_+ & \mathbf{J}_- \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -(1/\lambda_\circ f)\,\mathbf{I} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{J}_+ & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_- \end{bmatrix}, \quad (30)$$

which is a (not commuting !) combination of a shearer $\mathcal{Z}$ with an isotropic lens $\mathcal{L}(f,f)$:

$$\mathbf{T}_{lz} = \mathbf{T}_l(f,f)\,\mathbf{T}_z. \quad (31)$$

The input-output relation of this system [with $\mathbf{A} = \mathbf{J}_+$ and $\mathbf{C} = -(1/\lambda_\circ f)\,\mathbf{J}_+$] reads

$$f_o(x,y) = f_i(x-y,y)\,\exp[-i\pi\,(1/\lambda_\circ f)\,(x^2+y^2)]. \quad (32)$$

Like a lens and a section of free space are each others' Fourier transforms, we can also find an alternative nucleus $\mathcal{F}(\pi/2;w)\,\mathcal{L}(f,f)\,\mathcal{Z}\,\mathcal{F}(-\pi/2;w) = \mathcal{S}(w^4/\lambda_\circ f)\,\mathcal{Z}'$ by embedding the shearer–lens combination in between a Fourier transformer and its inverse. The ray transformation matrices of the shearers $\mathcal{Z}'$ and $\mathcal{Z}$ differ only in the positions of the Jordan blocks $\mathbf{J}_+$ and $\mathbf{J}_-$ [see Eq. (26)], and we have $\mathbf{T}_{z'}^{-1} = \mathbf{T}_z^t$.

## 4. CONCLUSION

We have presented a classification of one- and two-dimensional real symplectic matrices, based on the distribution of the eigenvalues of the matrix and on whether or not it can be diagonalized. Subsequently, for each class we have formulated simple lossless first-order optical systems that can be considered as a nucleus for this class: the nucleus is described by as few parameters as possible and its ray transformation matrix satisfies the properties of realness and symplecticity. We have thus suggested: (1) the magnifier, (2) the lens, (and a section of free space), and (3) a fractional Fourier transformer as nuclei for the three classes that arise in the one-dimensional case. For the two-dimensional case we have suggested – in addition to the obvious concatenations of one-dimensional nuclei – the four inherently two-dimensional combinations: (4) an isotropic magnifier with a rotator, (5) an isotropic lens (or section of free space) with a rotator, (6) an isotropic magnifier with a shearing operator, and (7) an isotropic lens (or section of free space) with a shearing operator.

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