GABOR'S SIGNAL EXPANSION AND THE GABOR TRANSFORM BASED ON A NON-ORTHOGONAL SAMPLING GEOMETRY

Martin J. Bastiaans and Arno J. van Leest

Technische Universiteit Eindhoven, Faculteit Elektrotechniek
Postbus 513, 5600 MB Eindhoven, Netherlands
phone: +31 40 2473319 – fax: +31 40 2448375 – e-mail: M.J.Bastiaans@ele.tue.nl

ABSTRACT

Gabor's signal expansion and the Gabor transform are formulated on a non-orthogonal time-frequency lattice instead of on the traditional rectangular lattice. The reason for doing so is that a non-orthogonal sampling geometry might be better adapted to the form of the window functions (in the time-frequency domain) than an orthogonal one: the set of shifted and modulated versions of the usual Gaussian synthesis window, for instance, corresponding to circular contour lines in the time-frequency domain, can be arranged more tightly in a hexagonal geometry than in a rectangular one. Oversampling in the Gabor scheme, which is required to have mathematically more attractive properties for the analysis window, then leads to better results in combination with less oversampling.

The new procedure presented in this paper is based on considering the non-orthogonal lattice as a sub-lattice of a denser orthogonal lattice that is oversampled by a rational factor. In doing so, Gabor's signal expansion on a non-orthogonal lattice can be related to the expansion on an orthogonal lattice (restricting ourselves, of course, to only those sampling points that are part of the non-orthogonal sub-lattice), and all the techniques that have been derived for rectangular sampling can be used, albeit in a slightly modified form.

1. INTRODUCTION

In 1946 [1], Gabor suggested the representation of a time signal in a combined time-frequency domain; in particular he proposed to represent the signal as a superposition of shifted and modulated versions of a so-called elementary signal or synthesis window $g(t)$. Moreover, as a synthesis window $g(t)$ he chose a Gaussian signal, because such a signal has a good localization, both in the time domain and in the frequency domain. The other choice that Gabor made, was that his signal expansion was formulated on a rectangular lattice in the time-frequency domain, $(mT, k\Omega)$, and that the sampling distances T and Ω satisfied the relation $\Omega T = 2\pi$.

The coefficients in Gabor's signal expansion can be determined by using an analysis window $w(t)$ [2, 3]. In the case of critical sampling, i.e., $\Omega T = 2\pi$, the analysis window $w(t)$ follows uniquely from the given synthesis window $g(t)$. However, such a unique analysis window appears to have some mathematically very unattractive properties. For this reason, the expansion should be formulated on a denser lattice, $\Omega T < 2\pi$. This makes the analysis window no longer unique and thus allows for finding an analysis window that is optimal in some way. We can, for instance, look for the analysis window that resembles best the synthesis window; a better resemblance can then be reached for a higher degree of oversampling.

A better resemblance can also be reached if we adapt the structure of the lattice to the form of the window as it is represented in the time-frequency domain. For the Gaussian window, for instance, its time-frequency representation has circular contour lines, and it is well known that circles are better packed on a hexagonal lattice than on a rectangular lattice. Gabor's signal expansion on such a hexagonal, non-orthogonal lattice then leads to a better resemblance between the window functions $g(t)$ and $w(t)$ than the expansion on a rectangular, orthogonal lattice does.

2. GABOR'S SIGNAL EXPANSION

Gabor's expansion of a signal $\varphi(t)$ into a set of shifted and modulated versions $g_{mk}(t) = g(t - mT)\exp(jk\Omega t)$ of a synthesis window $g(t)$ with $\Omega T \le 2\pi$,

$$\varphi(t) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_{mk} g_{mk}(t), \tag{1}$$

and the Gabor transform

$$a_{mk} = \int_{-\infty}^{\infty} \varphi(t) w_{mk}^*(t) dt \tag{2}$$

form a transform pair, if the synthesis window $g(t)$ and the analysis window $w(t)$ are related to each other in such a way that their shifted and modulated versions constitute two sets of functions that are biorthogonal:

$$\sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} g_{mk}(t_1)w_{mk}^*(t_2)=\delta(t_1-t_2). \quad (3)$$

3. NON-ORTHOGONAL SAMPLING

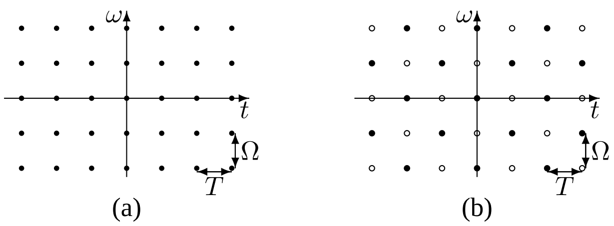
The rectangular (or orthogonal) lattice that we considered above, can be obtained by integer combinations of two orthogonal vectors $[T,0]^t$ and $[0,\Omega]^t$, see Fig. 1a.

We now consider a time-frequency lattice that is no longer orthogonal. Such a lattice is obtained by integer combinations of two linearly independent, but no longer orthogonal vectors, which we express in the forms $[aT,c\Omega]^t$ and $[bT,d\Omega]^t$, with a,b,c and d integers. Note that we only consider lattices that have samples on the time and frequency axes and that are therefore suitable for a discrete-time approach, as well.

There are a lot of lattice generator matrices that generate the same lattice. We will use the one that is based on the Hermite normal form, unique for any lattice,

$$\begin{bmatrix} aT & bT \\ c\Omega & d\Omega \end{bmatrix} = \begin{bmatrix} T & 0 \\ R\Omega & D\Omega \end{bmatrix} = \begin{bmatrix} T & 0 \\ 0 & \Omega \end{bmatrix}\begin{bmatrix} 1 & 0 \\ R & D \end{bmatrix},$$

where R and D are relatively prime integers and $0 \le |R| < D$. Sampling then occurs on the lattice points $(\tau = mT, \omega = [mR+nD]\Omega)$, and it is evident that these points of the non-orthogonal lattice form a subset of the points $(\tau = mT, \omega = k\Omega)$ of the orthogonal lattice: the non-orthogonal lattice is formed by those points of the rectangular (orthogonal) lattice for which $k-mR$ is an integer multiple of D. Note that the original rectangular lattice arises for $R=0$ and $D=1$ and that a hexagonal lattice occurs for $R=1$ and $D=2$, see Fig. 1b.



(a) (b)

Fig. 1. A rectangular (a) and a hexagonal (b) lattice.

4. NON-ORTHOGONAL GABOR LATTICE

If we define the two-dimensional array λ_{mk} as

$$\lambda_{mk} = \sum_{n=-\infty}^{\infty} \delta_{k-mR-nD}, \quad (4)$$

Gabor's signal expansion on a non-orthogonal lattice can be expressed as [cf. Eq. (1)]

$$\varphi(t)=\sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\lambda_{mk}a_{mk}g_{mk}(t), \quad (5)$$

while – with a different analysis window $w(t)$, though! – the expansion coefficients a_{mk} are still determined by the Gabor transform (2). Of course, since we only need the limited array $a'_{mk}=\lambda_{mk}a_{mk}$, we need only calculate the coefficients a_{mk} for those values of m and k for which $k-mR$ is an integer multiple of D.

The biorthogonality condition now takes the form

$$\sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\lambda_{mk}g_{mk}(t_1)w_{mk}^*(t_2)=\delta(t_1-t_2) \quad (6)$$

and leads to the equivalent but simpler expression

$$\frac{2\pi}{D\Omega}\sum_{m=-\infty}^{\infty} g(t-mT)w^*\left(t-\left[mT+n\frac{2\pi}{D\Omega}\right]\right) \times e^{j2\pi mnR/D}=\delta_n. \quad (7)$$

5. CONCLUSION

Since we have related Gabor's signal expansion on a non-orthogonal lattice to sampling on a denser but orthogonal lattice, followed by restriction to a sub-lattice that corresponds to the non-orthogonal lattice, we can still use all the techniques that have developed for rectangular lattices, in particular the technique of determining Gabor's expansion coefficients via the Zak transform [2, 3]. We also note that if everything remains to be based on a rectangular sampling geometry, it will be easier to extend the theory of the Gabor scheme to higher-dimensional signals; see, for instance, [4], where the multi-dimensional case is treated for continuous-time as well as discrete-time signals.

6. REFERENCES

[1] D. Gabor, "Theory of communication," *J. Inst. Elec. Eng.*, vol. 93 (III), pp. 429–457, 1946.

[2] M. J. Bastiaans, *Gabor's expansion and the Zak transform for continuous-time and discrete-time signals: Critical sampling and rational oversampling*, Eindhoven University of Technology Research Report 95-E-295, Eindhoven, Netherlands, 1995.

[3] H. G. Feichtinger and T. Strohmer, Eds., *Gabor Analysis and Algorithms: Theory and Applications*, Birkhäuser, Berlin, Germany, 1998.

[4] A. J. van Leest, *Non-separable Gabor schemes: their design and implementation*, PhD thesis, Eindhoven University of Technology, Eindhoven, Netherlands, 2001.