NOISE ANALYSIS IN TOEPLITZ AND HANKEL KERNELS FOR ESTIMATING TIME-VARYING SPECTRA

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ABSTRACT

An analysis of noise influence to time-varying spectra estimators is presented. Hankel and Toeplitz kernel forms of distributions are considered. It is shown that these two forms of kernels have the same variance, if the windows have the same form. Since the Hankel kernel results in a highly concentrated distribution for a noisy, frequency-modulated signal, this form can be more reliable for the estimation of signal parameters in noise. The results are generalized to multiplicative noise; in this case, the same conclusions hold for frequency-modulated signals.

1. INTRODUCTION

The analysis of noise influence on time-frequency distributions is a theoretically and practically interesting research topic [1]-[8]. It has recently been shown that two large families of time-varying spectra of discrete-time random processes can be derived based on (i) a diagonal-Toeplitz-diagonal (dTd) factorization or (ii) a diagonal-Hankel-diagonal (dHd) factorization [3]. Since both families are based on the short-time Fourier transform, they have similar forms. However, they behave just opposite with respect to time-frequency concentration. The dTd factorization aligns the short-time Fourier transform, resulting in a lower time-frequency resolution than in the original spectrogram [3]. The dHd factorization misalignes the short-time Fourier transform, improving the time-frequency resolution. The misalignment may generate cross-terms between time-frequency nonoverlapping components, which are not present in the aligned forms [3].

When considering the estimators of spectra, it is necessary to know their performances with respect to noise. The original motivation for introducing the smoothed spectrogram, which results from the dTd factorization, was to reduce the estimation variance. In this paper we will show that the two families of time-varying spectra estimators that will be presented, although behaving quite differently from the point of view of signal concentration, have the same variances in the case of stationary and quasi-stationary white noise.

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2. ADDITIVE NOISE IN QUADRATIC TIME-FREQUENCY DISTRIBUTIONS

The discrete-time form of a quadratic distribution reads

$$P_s(t,\theta;q) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} s(t+m)e^{-j\theta m} \times q(m,n)e^{j\theta n}s^*(t+n), \qquad (1)$$

where $q(m,n)$ is the kernel function. When the signal $s(t)$ is corrupted by Gaussian, complex valued i.i.d., additive noise $\nu(t)$, hence, $x(t) = s(t) + \nu(t)$, then the estimator $P_x(t,\theta;q)$ of $P_s(t,\theta;q)$ has a bias and a variance. These bias and variance will be analyzed, with special attention to two families of the kernels [3], having the form:

$$\begin{array}{lrcl} \text{dTd:} & q_T(m,n) & = & w_1(m)w_0(m-n)w_1(n) \\ \text{dHd:} & q_H(m,n) & = & w_1(m)w_0(m+n)w_1(n) \end{array} \qquad (2)$$

Since the windows are commonly real-valued even functions, we have $q_T(m,n) = q_T^*(n,m)$ and $q_H(m,n) = q_H^*(n,m)$.

2.1. Mean value

The mean value of $P_x(t,\theta;q) = P_{s+\nu}(t,\theta;q)$ is

$$E\{P_x(t,\theta;q)\} = P_s(t,\theta;q) + \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q(m,n)R_{\nu\nu}(t+m,t+n)e^{-j\theta(m-n)}.$$

For white noise, for which the autocorrelation function has the form $R_{\nu\nu}(t+m,t+n) = \sigma_\nu^2\delta(n-m)$, the bias for $q_T(m,n)$ and $q_H(m,n)$ is

$$\left\{ \begin{array}{lcl} \text{bias}_T & = & \sigma_\nu^2 \sum_{m=-\infty}^{\infty} w_1^2(m)w_0(0), \\ \text{bias}_H & = & \sigma_\nu^2 \sum_{m=-\infty}^{\infty} w_1^2(m)w_0(2m), \end{array} \right. \qquad (3)$$

respectively. Note that the mean value is a signal independent constant. Since $w_0(0) \geq w_0(m)$, we can conclude that the bias will be smaller in the dHd form of the kernel.

2.2. Variance

The distribution variance $\sigma_P^2(t,\theta) = \mathrm{var}\{P_x(t,\theta;q)\} = \mathrm{var}\{P_{s+\nu}(t,\theta;q)\}$ consists of two components [1, 4]:

$$\sigma_P^2(t,\theta) = \sigma_{\nu\nu}^2(t,\theta) + \sigma_{s\nu}^2(t,\theta). \tag{4}$$

The first component depends on the noise only, while the second one is both signal and noise dependent [1, 5]. The noise-only dependent part of the variance has the form

$$\sigma_{\nu\nu}^2(t,\theta) = \sum_{m_1}\sum_{m_2}\sum_{n_1}\sum_{n_2} q(m_1,n_1)q^*(m_2,n_2)$$
$$\times R_{\nu\nu}(t+m_1,t+m_2)R_{\nu\nu}(t+n_2,t+n_1)$$
$$\times e^{j\theta(m_2-n_2-m_1+n_1)}. \tag{5}$$

The signal-and-noise dependent part can be written as

$$\sigma_{s\nu}^2(t,\theta) = 2\sum_{m_1}\sum_{m_2}\sum_{n_1}\sum_{n_2} q(m_1,n_1)q^*(m_2,n_2)s(t+m_1)$$
$$\times s^*(t+m_2)R_{\nu\nu}(t+n_1,t+n_2)e^{j\theta(m_2-n_2-m_1+n_1)}$$
$$= 2\sum_{m_1}\sum_{m_2}\tilde{q}(m_1,m_2)[s(t+m_1)e^{-j\theta m_1}]$$
$$\times [s(t+m_2)e^{-j\theta m_2}]^* = 2P_s(t,\theta;\tilde{q}). \tag{6}$$

For $R_{\nu\nu}(t+n_1,t+n_2) = I(t+n_2)R_{\nu\nu}(n_1-n_2)$, the new kernel $\tilde{q}(m_1,m_2)$ in $\sigma_{s\nu}^2(t,\theta)$ takes the form

$$\tilde{q}(m_1,m_2) = \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} q(m_1,n_1)q^*(m_2,n_2)$$
$$\times e^{-j\theta(n_2-n_1)}I(t+n_2)R_{\nu\nu}(n_1-n_2). \tag{7}$$

Note that the signal dependent part of the variance is a quadratic distribution of the signal $s(t)$, with the new kernel $\tilde{q}(m_1,m_2)$.

2.3. Special cases

1. **Stationary white complex noise** with $I(t) = \sigma_\nu^2$ yields

$$\tilde{q}(m_1,m_2) = \sigma_\nu^2 \sum_{n=-\infty}^{\infty} q(m_1,n)q^*(m_2,n). \tag{8}$$

For finite limits, Eq. (8) represents a matrix multiplication, $\tilde{\mathbf{Q}} = \sigma_\nu^2\mathbf{Q}\cdot\mathbf{Q}^*$, which reduces to $\tilde{\mathbf{Q}} = \sigma_\nu^2\mathbf{Q}^2$, since $q^*(m_2,n) = q(n,m_2)$. Thus,

$$\begin{cases} \sigma_{\nu\nu}^2(t,\theta) &= \sigma_\nu^4 \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}|q(m,n)|^2 \\ \sigma_{s\nu}^2(t,\theta) &= 2P_s(t,\theta;\sigma_\nu^2\mathbf{Q}^2). \end{cases} \tag{9}$$

Proposition: For an even lag window $w_1(n)$, the dTd distribution [smoothed spectrogram $STFT_s(t,\theta)$] with kernel $q_T(m,n) = w_1(m)w_0(m-n)w_1(n)$,

$$P_s^T(t,\theta) = \sum_{i=-L}^{L} W_0(i)\,|STFT_s(t,\theta+i\Delta\theta)|^2, \tag{10}$$

and the dHd distribution (the S-method, see [6]) with kernel $q_H(m,n) = w_1(m)w_0(m+n)w_1(n)$,

$$P_s^H(t,\theta) = \sum_{i=-L}^{L} W_0(i)$$
$$\times STFT_s(t,\theta+i\Delta\theta)STFT_s^*(t,\theta-i\Delta\theta), \tag{11}$$

have the same variance; W_0 is the Fourier transform of w_0.

Proof: The noise-only dependent part of the variance $\sigma_{\nu\nu}^2(t,\theta)$ is the same, since $\sum_m\sum_n|q_T(m,n)|^2 = \sum_m\sum_n|q_H(m,n)|^2$ for $w_1(n) = w_1(-n)$. The kernels $\tilde{q}_{T,H}(m_1,m_2)$, corresponding to the dTd form and the dHd form with $q_{T,H}(m,n) = w_1(m)w_0(m\mp n)w_1(n)$, respectively, now take the form

$$\tilde{q}_{T,H}(m_1,m_2) = \sigma_\nu^2\sum_{n=-\infty}^{\infty} w_1(m_1)w_0(m_1\mp n)$$
$$\times w_1(n)w_1(m_2)w_0(m_2\mp n)w_1(n).$$

For an even lag window, $w_1(n) = w_1(-n)$, we get $\tilde{q}_T(m_1,m_2) = \tilde{q}_H(m_1,m_2)$, or $\mathbf{Q}_T^2 = \mathbf{Q}_H^2$, which proves that the $\sigma_{s\nu}^2(t,\theta)$ parts of the variances are the same.

2. For **nonstationary white complex noise**, the noise-only dependent variance takes the form

$$\begin{aligned}\sigma_{\nu\nu}^2(t,\theta) &= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}|q(m,n)|^2 I(t+m)I(t+n) \\ &= P_I(t,0;|q|^2).\end{aligned} \tag{12}$$

The kernel for $\sigma_{s\nu}^2(t,\theta)$ now reads

$$\tilde{q}(m_1,m_2) = \sum_{n=-\infty}^{\infty} q(m_1,n)I(t+n)q^*(m_2,n) = \mathbf{Q}\mathbf{I_t}\mathbf{Q}^*, \tag{13}$$

where $\mathbf{I}_t$ is a diagonal matrix with elements $I(t+n)$. For the quasistationary case, $I(t+n)\delta(n-m) \approx I(t)\delta(n-m)$, we get

$$\tilde{q}(m_1,m_2) = I(t)\sum_{n=-\infty}^{\infty} q(m_1,n)q^*(m_2,n),$$

and we conclude that $\tilde{q}_T(m_1,m_2) = \tilde{q}_H(m_1,m_2)$, according to the Proposition.

3. MULTIPLICATIVE NOISE

We now consider deterministic signals $s(t)$, corrupted by zero-mean multiplicative noise: $x(t) = s(t)[1 + \mu(t)]$.

3.1. Mean value

For multiplicative noise, the mean value of $P_x(t, \theta; q)$ reads

$$E\{P_x(t,\theta;q)\} = P_s(t,\theta;q) + \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} s(t+m) e^{-j\theta m}$$

$$\times\, q(m,n) e^{j\theta n} s^*(t+n) R_{\mu\mu}(t+m, t+n).$$

For nonstationary white complex noise, with autocorrelation function $R_{\mu\mu}(m,n) = I(n)\delta(m-n)$, the mean value takes the form

$$E\{P_x(t,\theta;q)\} = P_s(t,\theta;q)$$

$$+ \sum_{m=-\infty}^{\infty} I(t+m) \left|s(t+m)\right|^2 q(m,m),$$

and we thus get for the dTd and the dHd bias [cf. Eq. (3)]:

$$\left\{ \begin{array}{lcl} \text{bias}_T & = & \displaystyle\sum_{m=-\infty}^{\infty} I(t+m) \left|s(t+m)\right|^2 w_1^2(m) w_0(0), \\ \text{bias}_H & = & \displaystyle\sum_{m=-\infty}^{\infty} I(t+m) \left|s(t+m)\right|^2 w_1^2(m) w_0(2m). \end{array} \right. \tag{14}$$

3.2. Variance

The variance has two parts here, as well; both of them are signal dependent. However, one of them, denoted by $\sigma^2_{\mu\mu}$, is a function of the fourth power of the noise, and the other, denoted by $\sigma^2_{s\mu}$, is a function of the second power, cf. Eqs. (9). For nonstationary white complex noise we get [cf. Eq. (12)]

$$\sigma^2_{\mu\mu}(t,\theta) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left|q(m,n)\right|^2 \left|s(t+m)\right|^2$$

$$\times\, I(t+m) \left|s(t+n)\right|^2 I(t+n) = P_{I_s}(t, 0; |q|^2), \tag{15}$$

where $I_s(t) = |s(t)|^2 I(t)$. The stationary case follows with $I(t) = \sigma^2_\mu$. For an FM signal $s(t) = A\exp[j\phi(t)]$ and stationary noise, the variance $\sigma^2_{\mu\mu}(t,\theta)$ is equal for both the dHd and the dTd distribution kernel.

The second part of the variance for nonstationary white complex noise reads

$$\sigma^2_{s\mu}(t,\theta) = 2 \sum_{m_1} \sum_{m_2} \sum_{n} q(m_1,n) q^*(m_2,n) s(t+m_1)$$

$$\times s^*(t+m_2) I(t+n) \left|s(t+n)\right|^2 e^{j\theta(m_2 - m_1)},$$

which results in [cf. Eq. (6)]

$$\sigma^2_{s\mu}(t,\theta) = 2P_s(t,\theta;\tilde{\mathbf{Q}}), \tag{16}$$

where the elements of the matrix $\tilde{\mathbf{Q}}$ are

$$\tilde{q}(m_1,m_2) = \sum_{n=-\infty}^{\infty} \left|s(t+n)\right|^2 I(t+n)$$

$$\times\, q(m_1,n) q^*(m_2,n). \tag{17}$$

For finite limits, Eq. (17) represents a matrix multiplication, $\tilde{\mathbf{Q}} = \mathbf{Q}\mathbf{I}_t\mathbf{Q}^*$,where $\mathbf{I}_t$ is a diagonal matrix, with elements $\left|s(t+n)\right|^2 I(t+n)$.

Again, for stationary noise and an FM signal $s(t) = A\exp[j\phi(t)]$, both dHd and dTd have the same variance, according to the Proposition. It is equal to [cf. Eq. (9)]

$$\sigma^2_{s\mu}(t,\theta) = 2A^2 P_s(t,\theta;\tilde{\mathbf{Q}}). \tag{18}$$

4. NUMERICAL EXAMPLE

Consider a noisy multicomponent signal

$$x(t) = \exp[j1200(t+0.1)^2]$$

$$+\ \exp[-25(t-0.25)^2] \exp[j1000(t+0.75)^2]$$

$$+\ \exp[-25(t-0.67)^2] \exp[j1000(t-0.4)^2] + \nu(t)$$

within the interval $[0,1]$, sampled at $\Delta t = 1/1024$, where $\nu(t)$ is assumed to be stationary white complex noise with variance $\sigma^2_\nu = 2$. A Hanning lag window of width $T_w = 1/4$ is used. The normalized values of the spectrogram, the dTd distribution (smoothed spectrogram), the dHd distribution (S-method), and the (pseudo) Wigner distribution (WD) of the non-noisy signal are presented in Figs. 1a), b), c), and d), respectively. The variances are calculated statistically from 1000 simulations. Their normalized values are presented in Figs. 2a), b), c), and d). They fully correspond to the derived expressions for this kind of noise, and the Proposition.

5. CONCLUSION

It has been shown that the Hankel and Toeplitz factorization based time-frequency kernels result in the same distribution variance. For stationary white noise it is proportional to the smoothed spectrogram of the original signal. Since the Hankel kernel produces a higher concentration of a frequency modulated signal, this form of kernel can produce better estimation results. The same conclusions hold for multiplicative white stationary noise and frequency modulated signals.

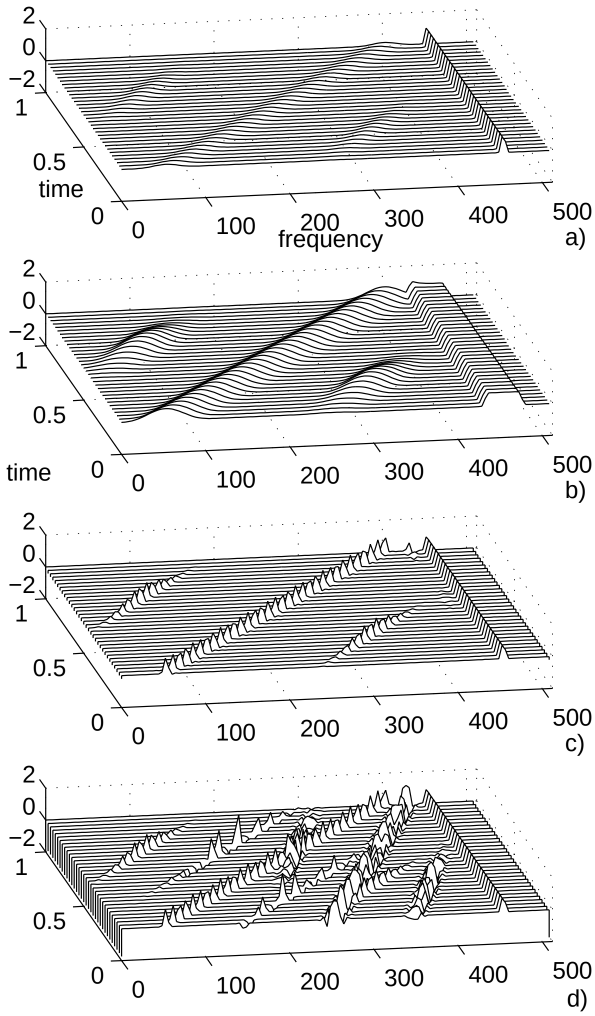

Figure 1: The normalized values of a) the spectrogram, b) the dTd distribution (smoothed spectrogram), c) the dHd distribution (S-method), and d) the WD of the non-noisy signal

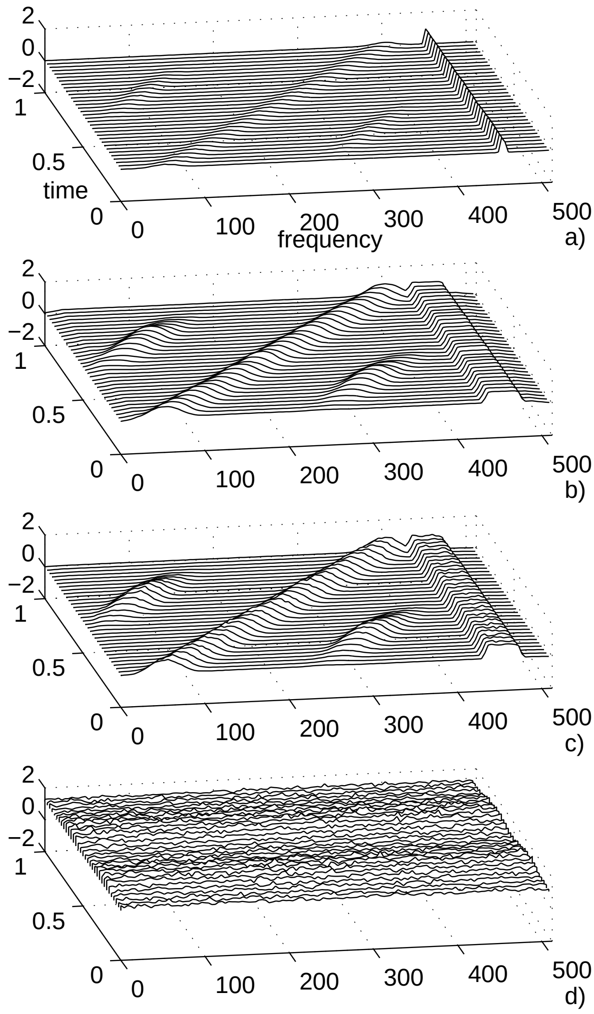

Figure 2: The normalized values of the variances in a) the spectrogram, b) the dTd distribution (smoothed spectrogram), c) the dHd distribution (S-method), and d) the WD of signal in white stationary Gaussian noise.

6. REFERENCES

[1] M. G. Amin, "Minimum-variance time-frequency distribution kernels for signals in additive noise," *IEEE Trans. Signal Process.*, vol. 44, 1996, pp. 2352-2356.

[2] S. B. Hearon and M. G. Amin, "Minimum-variance time-frequency distributions kernels," *IEEE Trans. Signal Process.*, vol. 43, 1995, pp. 1258-1262.

[3] B. Friedlander and L. L. Scharf, "Toeplitz and Hankel Kernels for Esimating time-varying spectra of discrete-time random processes," *IEEE Trans. Signal Process.*, vol. 49, 2001, pp. 179-189.

[4] LJ. Stanković and S. Stanković, "On the Wigner distribution of discrete-time noisy signals with application to the study of quantization effects," *IEEE Trans. Signal Process.*, vol. 42, 1994, pp. 1863-1867.

[5] LJ. Stanković and V. Ivanović, "Further results on the minimum variance time-frequency distributions kernels," *IEEE Trans. Signal Process.*, vol. 45, 1997, pp. 1650-1655.

[6] LJ. Stanković and V. Katkovnik, "The Wigner distribution of noisy signals with adaptive time-frequency varying window," *IEEE Trans. Signal Process.*, vol. 47, 1999, pp. 1099-1108.

[7] LJ. Stanković, "An analysis of noise in time-frequency distributions," *IEEE Signal Process. Lett.*, submitted.

[8] P. Duvaut and D. Declerq, "Statistical properties of the pseudo Wigner-Ville representation of normal random processes," *Signal Process.*, vol. 75, 1999, pp. 93-98.