

ON ROTATIONALLY SYMMETRIC PARTIALLY COHERENT LIGHT AND THE MOMENTS OF ITS WIGNER DISTRIBUTION

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ABSTRACT

The Wigner distribution of rotationally symmetric partially coherent light is considered and the constraints for its moments are derived. While all odd-order moments vanish, these constraints lead to a drastic reduction in the number of parameters that we need to describe all even-order moments. A way to measure the moments as intensity moments in the output planes of (generally anamorphic but separable) first-order optical systems is presented.

1. INTRODUCTION

After the introduction of the Wigner distribution [1] for the description of coherent and partially coherent optical fields [2, 3], it became an important tool for optical signal/image analysis and beam characterization [4, 5, 6, 7, 8]. The Wigner distribution completely describes the complex amplitude of a coherent optical field (up to a constant phase factor), or the two-point correlation function (or cross-spectral density) of a partially coherent field. Since the Wigner distribution of a two-dimensional optical field is a function of four variables, it is difficult to analyze. Therefore, the optical field is often represented not by the Wigner distribution itself, but by its global moments. Beam characterization based on the second-order moments of the Wigner distribution, for instance, thus became the basis of an International Organization for Standardization standard. [9]

In previous papers, the special but important case of rotational symmetry has been studied extensively; we mention twisted Gaussian-Schell model light [10] and the characterization of rotationally symmetric light in terms of second-order moments [8, 11]. In this paper we study the particular case of rotationally symmetric partially coherent light, and the constraints that this kind of symmetry imposes

on the second- and higher-order moments. We will see that only even-order moments remain, and that the number of moments reduces drastically compared to the general case.

2. WIGNER DISTRIBUTION

The Wigner distribution [1] of partially coherent light can be defined in terms of the cross-spectral density [12, 13] $\Gamma(x_1, y_1, x_2, y_2)$ as

$$W(x, y, u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Gamma(x + \frac{1}{2}x', y + \frac{1}{2}y', x - \frac{1}{2}x', y - \frac{1}{2}y') e^{-j2\pi(ux' + vy')} dx' dy'. \quad (1)$$

A distribution function according to definition (1) was first introduced in optics by Walther [2], who called it the generalized radiance. The (real-valued) Wigner distribution $W(x, y, u, v)$ represents partially coherent light in a combined space/spatial-frequency domain, the so-called phase space, where u is the spatial-frequency variable associated to the space variable x , and v the spatial-frequency variable associated to the space variable y .

3. ROTATIONAL SYMMETRY

To formulate rotational symmetry for the Wigner distribution $W(x, y, u, v)$, we express the space variables x and y in polar coordinates, $x = \rho \cos \phi$ and $y = \rho \sin \phi$, and, with the angle ϕ as an offset, we do the same with the spatial-frequency variables u and v , hence:

$$\begin{cases} x = \rho \cos \phi \\ y = \rho \sin \phi \end{cases} \quad \begin{cases} u = \zeta \cos(\phi + \theta) \\ v = \zeta \sin(\phi + \theta). \end{cases} \quad (2)$$

We can then formulate an expression for $W(x, y, u, v)$ in terms of the four variables ρ , ϕ , ζ , and θ ; for rotational

symmetry we require that this expression does not depend on the angle ϕ :

$$W(\rho \cos \phi, \rho \sin \phi, \zeta \cos(\phi + \theta), \zeta \sin(\phi + \theta)) \\ = W_{\circ}(\rho, \zeta, \theta). \quad (3)$$

A different way to define rotational symmetry, but leading to the same result, is by requiring that the cross-spectral density be independent of rotation.

4. WIGNER DISTRIBUTION MOMENTS

With $E = \iiint W(x, y, u, v) dx dy du dv$ the total energy of the light, we define the normalized moments μ_{pqrs} of the Wigner distribution as

$$\mu_{pqrs} E = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(x, y, u, v) \\ \times x^p u^q y^r v^s dx dy du dv. \quad (4)$$

In general there are $(N+1)(N+2)(N+3)/6$ moments [14] of order $N = p + q + r + s$. In the case of rotational symmetry, however, the number of parameters that we need to describe all even-order moments is reduced drastically to $(1 + N/2)^2$. This can easily be seen from Eq. (3), from which we conclude that the four-dimensional Wigner distribution $W(x, y, u, v)$ is completely determined by the three-dimensional function $W(x, 0, u, v)$, where, moreover, $W(x, 0, u, v)$ is an even function of x ; and this three-dimensional function has $(1 + N/2)^2$ different nonvanishing moments of even order N .

Using the symmetry condition (3), we can write

$$\mu_{pqrs} E \\ = \int_0^{\infty} \int_0^{\infty} \int_0^{2\pi} W_{\circ}(\rho, \zeta, \theta) \rho^{p+r+1} \zeta^{q+s+1} d\rho d\zeta d\theta \\ \times \int_0^{2\pi} (\cos \phi)^p (\cos(\phi + \theta))^q (\sin \phi)^r (\sin(\phi + \theta))^s d\phi. \quad (5)$$

From the special form of the integral over ϕ ,

$$I_{pqrs}(\theta) = \frac{1}{\pi} \int_0^{2\pi} (\cos \phi)^p (\cos(\phi + \theta))^q \\ \times (\sin \phi)^r (\sin(\phi + \theta))^s d\phi, \quad (6)$$

we immediately get the following symmetry relations for the moments

$$\mu_{pqrs} = (-1)^{p+q} \mu_{rspq} = (-1)^{r+s} \mu_{rspq} \\ = (-1)^{p+q+r+s} \mu_{pqrs} \quad (7)$$

and we conclude that all odd-order moments (i.e., $N = p + q + r + s$ is odd) are zero. For even-order moments (i.e.,

$N = p + q + r + s$ is even) we have two possibilities: $p + q$ and $r + s$ are both even or both odd; in the case that they are even we have $\mu_{pqrs} = \mu_{rspq}$, and in the case that they are odd we have $\mu_{pqrs} = -\mu_{rspq}$. Note, moreover, that $\mu_{p0r0} = 0$ unless both p and r are even, and that, similarly, $\mu_{0q0s} = 0$ unless both q and s are even.

It is advantageous to write the moments as $\mu_{p,q,m-p,n-q}$ where $p + r = m$ and $q + s = n$, and to group those moments together that have identical m and n . For easy reference, $I_{pqrs}(\theta)$ has been tabulated below for second-order moments [Table 1: $(m, n) = (2, 0), (1, 1),$ and $(0, 2)$] and fourth-order moments [Table 2: $(m, n) = (4, 0), (3, 1), (2, 2), (1, 3),$ and $(0, 4)$] in such a way that equal m and n (with $m + n = 2$ and $m + n = 4$, respectively) are grouped together. Identical values of $I_{pqrs}(\theta)$ in the same block then lead to companion moments μ_{pqrs} . For different choices of p and q (but with constant $m = p + r$ and $n = q + s$) we can easily find relations between the different moments μ_{pqrs} . In particular, we find that in any (m, n) block, the number of nonvanishing parameters equals $1 + \min(m, n)$, leading to a total of $(1 + N/2)^2$ parameters to describe the moments of order $N = m + n$.

Table 1: Second-order moments

m	n	$I_{pqrs}(\theta)$	μ_{pqrs}	companion
2	0	1	μ_{2000}	
2	0	0	μ_{1010}	—
2	0	1	μ_{0020}	μ_{2000}
1	1	$\cos \theta$	μ_{1100}	
1	1	$\sin \theta$	μ_{1001}	
1	1	$-\sin \theta$	μ_{0110}	$-\mu_{1001}$
1	1	$\cos \theta$	μ_{0011}	μ_{1100}
0	2	1	μ_{0200}	
0	2	0	μ_{0101}	—
0	2	1	μ_{0002}	μ_{0200}

Let us consider the second-order moments, $N = m + n = 2$, which can be represented elegantly in the usual form of a real, symmetric 4×4 matrix. As a consequence from the moment relations, this matrix takes the special form [10]

$$\begin{bmatrix} \mu_{2000} & 0 & \mu_{1100} & \mu_{1001} \\ 0 & \mu_{2000} & -\mu_{1001} & \mu_{1100} \\ \mu_{1100} & -\mu_{1001} & \mu_{0200} & 0 \\ \mu_{1001} & \mu_{1100} & 0 & \mu_{0200} \end{bmatrix}. \quad (8)$$

We conclude that the symmetric matrix of second-order moments of rotationally symmetric light is determined by four parameters instead of the ten parameters in the general case. In particular we observe that three moments appear in pairs with a positive companion (μ_{2000} , μ_{1100} , and μ_{0200})

and one moment forms a pair with a negative companion (μ_{1001}); moreover, two moments vanish (μ_{1010} and μ_{0101}). Note that the additional requirement that $W_o(\rho, \zeta, \theta)$ is an even function of θ , $W_o(\rho, \zeta, \theta) = W_o(\rho, \zeta, -\theta)$, leads to the vanishing of the moment with the negative companion: $\mu_{1001} = 0$.

Table 2: Fourth-order moments

m	n	$I_{pqrs}(\theta)$	μ_{pqrs}	companion
4	0	3/4	μ_{4000}	
4	0	0	μ_{3010}	—
4	0	1/4	μ_{2020}	$\mu_{4000}/3$
4	0	0	μ_{1030}	—
4	0	3/4	μ_{0040}	μ_{4000}
3	1	$3 \cos \theta/4$	μ_{3100}	
3	1	$3 \sin \theta/4$	μ_{3001}	
3	1	$-\sin \theta/4$	μ_{2110}	$-\mu_{3001}/3$
3	1	$\cos \theta/4$	μ_{2011}	$\mu_{3100}/3$
3	1	$\cos \theta/4$	μ_{1120}	$\mu_{3100}/3$
3	1	$\sin \theta/4$	μ_{1021}	$\mu_{3001}/3$
3	1	$-3 \sin \theta/4$	μ_{0130}	$-\mu_{3001}$
3	1	$3 \cos \theta/4$	μ_{0031}	μ_{3100}
2	2	$(2 + \cos 2\theta)/4$	μ_{2200}	
2	2	$\sin 2\theta/4$	μ_{2101}	
2	2	$(2 - \cos 2\theta)/4$	μ_{2002}	
2	2	$-\sin 2\theta/4$	μ_{1210}	$-\mu_{2101}$
2	2	$\cos 2\theta/4$	μ_{1111}	$(\mu_{2200} - \mu_{2002})/2$
2	2	$\sin 2\theta/4$	μ_{1012}	μ_{2101}
2	2	$(2 - \cos 2\theta)/4$	μ_{0220}	μ_{2002}
2	2	$-\sin 2\theta/4$	μ_{0121}	$-\mu_{2101}$
2	2	$(2 + \cos 2\theta)/4$	μ_{0022}	μ_{2200}
1	3	$3 \cos \theta/4$	μ_{1300}	
1	3	$\sin \theta/4$	μ_{1201}	
1	3	$\cos \theta/4$	μ_{1102}	$\mu_{1300}/3$
1	3	$3 \sin \theta/4$	μ_{1003}	$3\mu_{1201}$
1	3	$-3 \sin \theta/4$	μ_{0310}	$-3\mu_{1201}$
1	3	$\cos \theta/4$	μ_{0211}	$\mu_{1300}/3$
1	3	$-\sin \theta/4$	μ_{0112}	$-\mu_{1201}$
1	3	$3 \cos \theta/4$	μ_{0013}	μ_{1300}
0	4	3/4	μ_{0400}	
0	4	0	μ_{0301}	—
0	4	1/4	μ_{0202}	$\mu_{0400}/3$
0	4	0	μ_{0103}	—
0	4	3/4	μ_{0004}	μ_{0400}

Let us now consider the fourth-order moments, $N = m + n = 4$. From the moment relations we conclude that the moments are determined by nine parameters, whereas in the general case we would need 35 parameters. In particular we observe that two moments appear in pairs with a positive

companion (μ_{2200} and μ_{2002}); two moments appear in triples with positive companions (μ_{4000} and μ_{0400}); five moments appear in quadruples: two moments with positive companions (μ_{3100} and μ_{1300}) and three moments with one positive and two negative companions (μ_{3001} , μ_{2101} , and μ_{1201}); moreover, four moments vanish (μ_{3010} , μ_{1030} , μ_{0301} , and μ_{0103}) and the moment μ_{1111} follows from the relation $\mu_{1111} = (\mu_{2200} - \mu_{2002})/2$. Note that the additional requirement that $W_o(\rho, \zeta, \theta)$ is an even function of θ again leads to the vanishing of the moments with the negative companions: $\mu_{3001} = \mu_{2101} = \mu_{1201} = 0$.

5. MEASUREMENT OF THE MOMENTS

It is well-known that the input-output relationship between the Wigner distribution $W_{\text{in}}(x, y, u, v)$ at the input plane and the Wigner distribution $W_{\text{out}}(x, y, u, v)$ at the output plane of a separable first-order optical system reads [4, 5, 6]

$$W_{\text{out}}(x, y, u, v) = W_{\text{in}}(d_x x - b_x u, d_y y - b_y v, -c_x x + a_x u, -c_y y + a_y v). \quad (9)$$

The coefficients a_x, b_x, c_x, d_x and a_y, b_y, c_y, d_y are the matrix entries of the symplectic ray transformation matrix [15] that relates the position x, y and direction u, v of an optical ray in the input and the output plane of the first-order optical system:

$$\begin{bmatrix} x_{\text{out}} \\ u_{\text{out}} \\ y_{\text{out}} \\ v_{\text{out}} \end{bmatrix} = \begin{bmatrix} a_x & b_x & 0 & 0 \\ c_x & d_x & 0 & 0 \\ 0 & 0 & a_y & b_y \\ 0 & 0 & c_y & d_y \end{bmatrix} \begin{bmatrix} x_{\text{in}} \\ u_{\text{in}} \\ y_{\text{in}} \\ v_{\text{in}} \end{bmatrix}. \quad (10)$$

For separable systems, symplecticity reads simply $a_x d_x - b_x c_x = 1$ and $a_y d_y - b_y c_y = 1$. Note that in a first-order optical system, with such a symplectic ray transformation matrix, the total energy E is invariant.

Extending the procedure described in [14] for fractional Fourier transform systems to more general, separable systems, the input moments μ_{pqrs} can be determined from measurement of the intensity distribution $\Gamma(x, y, x, y) = \iint W(x, y, u, v) du dv$ in the output plane of some (possibly anamorphic but separable) first-order optical systems for appropriately chosen values of a_x, b_x, c_x, d_x and a_y, b_y, c_y, d_y . In the output plane we then measure the intensity moments μ_{p0r0}^{out} , cf. Eq. (4) with $q = s = 0$, which are completely determined by the output intensity distribution. The general relationship between the output intensity moments and the moments in the input plane reads

$$\mu_{p0r0}^{\text{out}}(\alpha, \beta) = \sum_{k=0}^p \sum_{m=0}^r \binom{p}{k} \binom{r}{m} \times a_x^{p-k} b_x^k a_y^{r-m} b_y^m \mu_{p-k, k, r-m, m}. \quad (11)$$

In the case of second-order moments, and following the procedure outlined in [14], we have a set of equations in which the intensity moments μ_{2000}^{out} , μ_{1010}^{out} , and μ_{0020}^{out} at the output of a separable first-order optical system are expressed in terms of the input moments. With the help of the moment relations, this set can be reduced to the following two equations:

$$\mu_{2000}^{\text{out}} = a_x^2 \mu_{2000} + 2a_x b_x \mu_{1100} + b_x^2 \mu_{0200}, \quad (12)$$

$$\mu_{1010}^{\text{out}} = (a_x b_y - b_x a_y) \mu_{1001}. \quad (13)$$

To measure the moment μ_{1001} from the intensity moment μ_{1010}^{out} , we clearly need an anamorphic system: $a_x b_y \neq b_x a_y$. Together with two additional isotropic systems we can then construct four equations from measurements of the intensity distributions in the three output planes, with which the four moments can be determined. We would not need the anamorphic system if the rotationally symmetric light satisfies the additional condition that $W_o(\rho, \zeta, \theta)$ is an even function of θ ; in that case the moment μ_{1001} vanishes, Eq. (13) is no longer necessary, and the three remaining moments can be determined with three isotropic systems by using only Eq. (12).

In the case of fourth-order moments, we get the following set of relevant equations for the output intensity moments:

$$\mu_{4000}^{\text{out}} = a_x^4 \mu_{4000} + 4a_x^3 b_x \mu_{3100} + 6a_x^2 b_x^2 \mu_{2200} + 4a_x b_x^3 \mu_{1300} + b_x^4 \mu_{0400}, \quad (14)$$

$$\mu_{3010}^{\text{out}} = (a_x^2 \mu_{3001} + 3a_x b_x \mu_{2101} + 3b_x^2 \mu_{1201}) \times (a_x b_y - b_x a_y), \quad (15)$$

$$3\mu_{2020}^{\text{out}} = a_x^2 a_y^2 \mu_{4000} + 2a_x a_y (a_x b_y + b_x a_y) \mu_{3100} + 6a_x b_x a_y b_y \mu_{2200} + 2b_x b_y (a_x b_y + b_x a_y) \mu_{1300} + b_x^2 b_y^2 \mu_{0400} + 3(a_x b_y - b_x a_y)^2 \mu_{2002}. \quad (16)$$

To determine the moments μ_{3001} , μ_{2101} , and μ_{1201} from the intensity moment μ_{3010}^{out} , and the moment μ_{2002} from the intensity moment μ_{2020}^{out} , we need anamorphic systems; and obviously we need three of them. Together with two additional isotropic systems, we can then construct nine equations from measurements of the intensity distributions in the five output planes, with which the nine moments can be determined. If we have the additional condition that $W_o(\rho, \zeta, \theta)$ is an even function of θ again, the moments μ_{3001} , μ_{2101} , and μ_{1201} vanish, and Eq. (15) is no longer necessary. But even in this highly symmetric case, we still need one anamorphic system to determine the moment μ_{2002} (and its companions μ_{0220} and μ_{1111}).

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