PHASE RECONSTRUCTION FROM INTENSITY MEASUREMENTS IN ONE-PARAMETER CANONICAL-TRANSFORM SYSTEMS

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ABSTRACT

We consider the one-parameter canonical transform of an optical signal and derive the relationship between its local spatial frequency (signal phase derivative) and the derivative of the squared modulus of the one-parameter canonical transform with respect to the parameter. From this relationship we conclude that the phase of the signal can be reconstructed by letting the signal propagate in an appropriate one-parameter canonical-transform system and then measure two intensity profiles for two close values of the parameter.

1. INTRODUCTION

Phase retrieval and local frequency estimation of a signal from intensity profiles are important problems in radio location, optical signal processing, quantum mechanics, and other fields. Several successful iterative algorithms for phase reconstruction from the squared modulus of the signal and its power spectrum, or its Fresnel spectrum, have been proposed recently [1, 2, 3, 4], and related techniques are applied in various regions of the electromagnetic spectrum and in quantum mechanics [5, 6, 7]. The development of non-iterative procedures for generic systems remains an attractive research topic.

A non-iterative approach for phase retrieval, based on the so-called transport-of-intensity equation in optics, was proposed by Teague [8] and then further developed by others [9, 10, 11]. It was shown that the longitudinal derivative of the Fresnel spectrum is proportional to the transversal derivative of the product of the instantaneous power and the instantaneous frequency of the signal. A similar procedure was proposed for the fractional Fourier transform [12, 13].

In this paper we show that a non-iterative formulation applies for general one-parameter canonical transforms [14, 15]. We show that the local frequency (the first derivative of the phase of the signal) is directly related to the derivative of the squared modulus of the one-parameter canonical transform with respect to the parameter, and given by the evolution Hamiltonian of the optical medium. From this relationship we conclude that the phase of the signal can be reconstructed by letting it propagate in such systems, and measuring the intensity profiles of the signal for two close values of the parameter.

2. PHASE AND LOCAL FREQUENCY

Let us consider a one-dimensional, coherent optical signal with complex amplitude $\psi(x) = |\psi(x)| \exp[i\varphi(x)]$. Its local spatial frequency $\overline{p}(x)$ is defined as the derivative of the phase $\varphi(x)$:

$$\overline{p}(x) = \varphi'(x) = \frac{d\varphi(x)}{dx}. \tag{1}$$

To express the local frequency in terms of the signal $\psi(x)$ itself, we write

$$\begin{aligned}\overline{p}(x) &= \frac{d\varphi(x)}{dx} = \mathrm{Im}\,\frac{d\ln\psi(x)}{dx} = \mathrm{Im}\,\frac{\psi'(x)}{\psi(x)} \\ &= \tfrac{1}{2}i\left(\frac{\psi'^*(x)}{\psi^*(x)} - \frac{\psi'(x)}{\psi(x)}\right) \\ &= \tfrac{1}{2}i\,\frac{\psi'^*(x)\,\psi(x) - \psi^*(x)\,\psi'(x)}{\psi^*(x)\,\psi(x)}\end{aligned}$$

and get

$$\overline{p}(x)\,|\psi(x)|^2 = \tfrac{1}{2}i[\psi'^*(x)\,\psi(x) - \psi^*(x)\,\psi'(x)]. \tag{2}$$

3. ONE-PARAMETER CANONICAL OPERATORS

We consider the one-parameter canonical operator $\mathbb{C}(\alpha) = \exp(i\alpha\mathbb{H})$, see [15, Eq. (9.73)]

$$\mathbb{C}(\alpha)\,\psi(x) = \psi_\alpha(x) = \int_{-\infty}^{\infty} C(\alpha; x, \xi)\,\psi(\xi)\,d\xi; \qquad (3)$$

note that $\mathbb{C}^\dagger(\alpha) = \mathbb{C}(-\alpha)$ and $C^*(\alpha; x, \xi) = C(-\alpha; \xi, x)$, see [15, Eq. (9.25)]. The Fresnel transformation with integral kernel

$$C(\alpha; x, \xi) = \frac{\exp(i\pi/4)}{\sqrt{2\pi\alpha}}\ \exp\left(-i\frac{x^2 - 2x\xi + \xi^2}{2\alpha}\right) \qquad (4)$$

and the fractional Fourier transformation with integral kernel

$$C(\alpha; x, \xi) = \frac{\exp(i\pi/4)}{\sqrt{2\pi\sin\alpha}} \times \exp\left(-i\frac{x^2\cos\alpha - 2x\xi + \xi^2\cos\alpha}{2\sin\alpha}\right) \qquad (5)$$

are two important representatives of such operators.

The self-adjoint generating operator $\mathbb{H} = \mathbb{H}^\dagger$ can be expressed in terms of the kernel $C(\alpha; x, \xi)$ via

$$\mathbb{H}\psi(x) = -i\frac{d}{d\alpha}\int_{-\infty}^{\infty} C(\alpha; x, \xi)\,\psi(\xi)\,d\xi\Big|_{\alpha=0}, \qquad (6)$$

see [15, Eq. (9.74)], and has the general form

$$\mathbb{H} = \tfrac{1}{2}A\mathbb{P}^2 + \tfrac{1}{2}B(\mathbb{QP} + \mathbb{PQ}) + \tfrac{1}{2}C\mathbb{Q}^2 = \mathbb{H}^\dagger \qquad (7)$$

with

$$\mathbb{P} = -i\frac{\partial}{\partial x}\,;\quad \mathbb{Q} = x\,;\quad \mathbb{P}^2 = -\frac{\partial^2}{\partial x^2}\,;$$
$$\mathbb{QP} + \mathbb{PQ} = -2i\left(x\frac{\partial}{\partial x} + \tfrac{1}{2}\right)\,;\quad \mathbb{Q}^2 = x^2.$$

In the case of the Fresnel transformation (the 'parabolic subgroup') and the fractional Fourier transformation (the 'elliptic subgroup') we have $A = 1$ and $B = C = 0$, and $A = C = 1$ and $B = 0$, respectively; see also [15, Eqs. (9.76)], where these and three other cases (the 'hyperbolic subgroup,' with $A = -C = 1$ and $B = 0$; scaling, with $A = C = 0$ and $B = 1$; and multiplication by a quadratic phase factor, with $A = B = 0$ and $C = 1$) are treated.

We now differentiate $|\psi_\alpha(x)|^2 = |\exp(i\alpha\mathbb{H})\,\psi(x)|^2$ with respect to α

$$\begin{aligned}
\frac{\partial|\psi_\alpha(x)|^2}{\partial\alpha} \\
&= \frac{\partial\psi_\alpha^*\psi_\alpha}{\partial\alpha} = \frac{\partial\psi_\alpha^*}{\partial\alpha}\psi_\alpha + \psi_\alpha^*\frac{\partial\psi_\alpha}{\partial\alpha} \\
&= [i\mathbb{H}\psi_\alpha]^*\psi_\alpha + \psi_\alpha^*[i\mathbb{H}\psi_\alpha] \\
&= -i\left[\left(\tfrac{1}{2}A\mathbb{P}^2 + \tfrac{1}{2}B(\mathbb{QP}+\mathbb{PQ}) + \tfrac{1}{2}C\mathbb{Q}^2\right)\psi_\alpha\right]^*\psi_\alpha \\
&\quad + i\psi_\alpha^*\left[\left(\tfrac{1}{2}A\mathbb{P}^2 + \tfrac{1}{2}B(\mathbb{QP}+\mathbb{PQ}) + \tfrac{1}{2}C\mathbb{Q}^2\right)\psi_\alpha\right] \\
&= -i\left[-\tfrac{1}{2}A\psi_\alpha''^* + iB(x{\psi'_\alpha}^* + \tfrac{1}{2}\psi_\alpha^*) + \tfrac{1}{2}Cx^2\psi_\alpha^*\right]\psi_\alpha \\
&\quad + i\psi_\alpha^*\left[-\tfrac{1}{2}A\psi_\alpha'' - iB(x\psi_\alpha' + \tfrac{1}{2}\psi_\alpha) + \tfrac{1}{2}Cx^2\psi_\alpha\right] \\
&= \tfrac{1}{2}iA\left(\psi_\alpha''^*\psi_\alpha - \psi_\alpha^*\psi_\alpha''\right) + Bx\left({\psi'_\alpha}^*\psi_\alpha + \psi_\alpha^*\psi_\alpha'\right) \\
&\quad + B\psi_\alpha^*\psi_\alpha
\end{aligned}$$

and arrive at

$$\frac{\partial|\psi_\alpha(x)|^2}{\partial\alpha} = \tfrac{1}{2}iA\frac{d}{dx}\left[{\psi'_\alpha}^*(x)\,\psi_\alpha(x) - \psi_\alpha^*(x)\,\psi_\alpha'(x)\right] + B\frac{d}{dx}[x\,\psi_\alpha^*(x)\,\psi_\alpha(x)]. \qquad (8)$$

4. RECONSTRUCTION OF THE PHASE

If we combine Eq. (2) with Eq. (8) we get

$$\frac{\partial|\psi_\alpha(x)|^2}{\partial\alpha}\Big|_{\alpha=0} = \frac{d\,[A\,\overline{p}(x) + Bx]\,|\psi(x)|^2}{dx}, \qquad (9)$$

which shows a relationship between longitudinal derivatives (i.e., with respect to the canonical parameter α) on the one-hand and transversal derivatives (i.e., with respect to the space variable x) on the other. After integration, Eq. (9) leads to

$$\int_{-\infty}^{x}\frac{\partial|\psi_\alpha(\xi)|^2}{\partial\alpha}\Big|_{\alpha=0}d\xi = [A\,\overline{p}\,(x) + Bx]\,|\psi(x)|^2, \quad (10)$$

where we have assumed that $[A\,\overline{p}\,(x) + Bx]\,|\psi(x)|^2 \to 0$ for $x \to -\infty$. If we introduce the unit step function $\mathrm{u}(x)$, with $\mathrm{u}(x) = 1$ for $x > 0$ and $\mathrm{u}(x) = 0$ for $x < 0$, Eq. (10) can be expressed in the form of a convolution of $\partial|\psi_\alpha(x)|^2/\partial\alpha\big|_{\alpha=0}$ with $\mathrm{u}(x)$:

$$[A\,\overline{p}\,(x) + Bx]\,|\psi(x)|^2 = \int_{-\infty}^{\infty}\frac{\partial|\psi_\alpha(\xi)|^2}{\partial\alpha}\Big|_{\alpha=0}\mathrm{u}(x-\xi)\,d\xi. \qquad (11)$$

Since a real canonical transformation satisfies Parseval's relation [15, Eq. (9.11)],

$$\int_{-\infty}^{\infty} f_\alpha(x)\,g_\alpha^*(x)\,dx = \int_{-\infty}^{\infty} f(x)\,g^*(x)\,dx,$$

we have

$$\int_{-\infty}^{\infty} \frac{\partial|\psi_\alpha(\xi)|^2}{\partial\alpha}\bigg|_{\alpha=0} d\xi = \frac{\partial}{\partial\alpha}\int_{-\infty}^{\infty}|\psi_\alpha(\xi)|^2\, d\xi\bigg|_{\alpha=0} = \frac{\partial}{\partial\alpha}\int_{-\infty}^{\infty}|\psi(\xi)|^2\, d\xi\bigg|_{\alpha=0} = 0; \quad (12)$$

note that this relationship is in accordance with the assumption that $[A\,\overline{p}\,(x) + Bx]\,|\psi(x)|^2 \to 0$ not only for $x \to -\infty$ but also for $x \to +\infty$. Equation (11) can then as well be expressed in the form

$$[A\,\overline{p}\,(x) + Bx]\,|\psi(x)|^2 = \tfrac{1}{2}\int_{-\infty}^{\infty} \frac{\partial|\psi_\alpha(\xi)|^2}{\partial\alpha}\bigg|_{\alpha=0} \mathrm{sgn}(x-\xi)\, d\xi, \quad (13)$$

where we have introduced the signum function $\mathrm{sgn}(x) = 2\mathrm{u}(x) - 1$. With $A = 1$ and $B = 0$ we thus get the result derived in the case of the fractional Fourier transform [12], where the local frequency is expressed as a convolution of $\partial|\psi_\alpha(x)|^2/\partial\alpha\big|_{\alpha=0}$ with $\mathrm{sgn}(x)$.

We conclude that the convolution relations (11) and (13) are not restricted to the fractional Fourier transform, but hold as well for other one-parameter canonical transforms, like the Fresnel transform. In general these convolutions allow us to reconstruct the local frequency (and thus the phase) of the signal from knowledge of two intensity profiles: one for $\alpha = 0$ [corresponding to the signal $\psi(x) = \psi_0(x)$ itself] and one for a small value $\Delta\alpha$ [corresponding to the canonical transform $\psi_{\Delta\alpha}(x)$]. We use the difference of the two intensity profiles divided by the parameter difference, $\left(|\psi_{\Delta\alpha}(x)|^2 - |\psi(x)|^2\right)/\Delta\alpha$, as an approximation for the differential quotient $\partial|\psi_\alpha(x)|^2/\partial\alpha\big|_{\alpha=0}$, and determine the expression $[A\,\overline{p}\,(x) + Bx]\,|\psi(x)|^2$ via the convolutions (11) or (13). Calculation of the local frequency $\overline{p}(x)$ and the subsequent determination of the phase $\varphi(x)$ as the integral of this local frequency is then straightforward.

The one-parameter canonical transformation should be such that its parameter A does not vanish. This condition is satisfied by the three first-order optical systems described by their ray-transformation matrices as, see [15, Eq. (9.75)],

$$\begin{bmatrix} 1 & -\alpha \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} \cosh\alpha & -\sinh\alpha \\ -\sinh\alpha & \cosh\alpha \end{bmatrix},$$

corresponding to the 'parabolic subgroup' (Fresnel transformation: free space propagation under the paraxial approximation), the 'elliptic subgroup' (fractional Fourier transformation), and the 'hyperbolic subgroup,' respectively. Note that for these three canonical transformations the parameter B vanishes.

We note the absence of C, the Hamiltonian parameter associated to the refractive index, in the left-hand side of Eqs. (11) and (13); comparison with the classical form of the Hamiltonian function

$$h(x,k) = \frac{1}{2n_\circ}\,k^2 + \nu x^2, \quad (14)$$

which characterizes a quasi-homogeneous medium of refractive index $n(x) \approx n_\circ - \nu x^2$ in the paraxial regime of geometric optics, shows that the determination of the signal phase is the same whether the medium be harmonic ($\nu > 0$, the 'elliptic subgroup'), repulsive ($\nu < 0$, the 'hyperbolic subgroup'), or free ($\nu = 0$, the 'parabolic subgroup'). The parameter B of the scaling term of the Hamiltonian does appear in that left-hand side; but when present, it will only chirp the local frequency $\overline{p}(x)$ by $-Bx/A$. Thus we see that to determine the local frequency of a signal from two intensities it is only necessary that the canonical transform be generated by a Hamiltonian with a nonzero Ak^2 term.

We remark that the extension to two-dimensional signals is rather straightforward. In that case we would need a two-dimensional, separable canonical transformation, operating in the x-direction with parameter α_x and in the y-direction with parameter α_y. The general relationship (9) now splits up into two relations:

$$\frac{\partial|\psi_{\alpha_x,\alpha_y}(x,y)|^2}{\partial\alpha_x}\bigg|_{\alpha_x=\alpha_y=0} = \frac{d\,[A_x\,\overline{p_x}(x,y) + B_x x]\,|\psi(x,y)|^2}{dx}$$

$$\frac{\partial|\psi_{\alpha_x,\alpha_y}(x,y)|^2}{\partial\alpha_y}\bigg|_{\alpha_x=\alpha_y=0} = \frac{d\,[A_y\,\overline{p_y}(x,y) + B_y y]\,|\psi(x,y)|^2}{dy}, \quad (15)$$

and to reconstruct the phase, we would need three intensity profiles: $|\psi(x,y)|^2 = |\psi_{0,0}(x,y)|^2$, $|\psi_{\Delta\alpha_x,0}(x,y)|^2$, and $|\psi_{0,\Delta\alpha_y}(x,y)|^2$.

5. CONCLUSION

In this paper we have established the relationship between the derivative of the squared modulus of a general, one-parameter, canonical transform of an optical signal and the signal's local frequency (or phase derivative). This relationship allows us to reconstruct the local frequency (and thus the phase) of the signal by letting the signal propagate in an appropriate one-parameter canonical-transform system and then measure two intensity profiles for two close values of the parameter.

6. REFERENCES

[1] Z. Zalevsky, D. Mendlovic, and R. G. Dorsch, “Gerchberg-Saxton algorithm applied in the fractional Fourier or the Fresnel domain,” *Opt. Lett.*, vol. 21, pp. 842–844, 1996.

[2] H. M. Ozaktas, Z. Zalevsky, and M. A. Kutay, *The Fractional Fourier Transform*, Wiley, Chichester, 2001.

[3] W. X. Cong, N. X. Chen, and B. Y. Gu, “Recursive algorithm for phase retrieval in the fractional Fourier-transform domain,” *Appl. Opt.*, vol. 37, pp. 6906–6910, 1998.

[4] W. X. Cong, N. X. Chen, and B. Y. Gu, “Phase retrieval in the Fresnel transform system – a recursive algorithm,” *J. Opt. Soc. Am. A*, vol. 16, pp. 1827–1830, 1999.

[5] K. A. Nugent and D. Paganin, “Matter-wave phase measurement: A noninterferometric approach,” *Phys. Rev. A*, vol. 61, pp. 063614, 2000.

[6] D. Paganin, S. C. Mayo, T. E. Gureyev, P. R. Miller, and S. W. Wilkins, “Simultaneous phase and amplitude extraction from a single defocused image of a homogeneous object,” *J. Microscopy*, vol. 206, pp. 33–40, 2002.

[7] D. Paganin and K.A. Nugent, “Phase measurement of waves that obey nonlinear equations,” *Opt. Lett.*, vol. 27, pp. 622–624, 2002.

[8] M. R. Teague, “Deterministic phase retrieval: a Green function solution,” *J. Opt. Soc. Am.*, vol. 73, pp. 1434–1441, 1983.

[9] N. Streibl, “Phase imaging by the transport equation of intensity,” *Opt. Commun.*, vol. 49, pp. 6–10, 1984.

[10] K. Ichikawa, A. W. Lohmann, and M. Takeda, “Phase retrieval based on the Fourier transport method: experiments,” *Appl. Opt.*, vol. 27, pp. 3433–3436, 1988.

[11] T. E. Gureev, A. Roberts, and K. A. Nugent, “Partially coherent fields, the transport-of-intensity equation, and phase uniqueness,” *J. Opt. Soc. Am. A*, vol. 12, pp. 1942–1946, 1995.

[12] T. Alieva and M. J. Bastiaans, “On fractional Fourier transform moments,” *IEEE Signal Process. Lett.*, vol. 7, pp. 320–323, 2000.

[13] T. Alieva, M. J. Bastiaans, and LJ. Stanković, “Signal reconstruction from two close fractional Fourier power spectra,” *IEEE Trans. Signal Process.*, vol. 51, pp. 112–123, 2003.

[14] M. Moshinsky and C. Quesne, “Oscillator Systems”. In *Proceedings of the 15th Solvay Conference in Physics (1970)*, Gordon and Breach, New York, 1974; M. Moshinsky and C. Quesne, “Linear canonical transformations and their unitary representation”, *J. Math. Phys.*, vol. 12, pp. 1772–1780, 1971; C. Quesne and M. Moshinsky, “Canonical transformations and matrix elements”, *J. Math. Phys.*, vol. 12, pp. 1780–1783, 1971.

[15] K. B. Wolf, *Integral Transforms in Science and Engineering*, Chap. 9, Plenum Press, New York, 1979.