ON THE NON-SEPARABLE DISCRETE GABOR SIGNAL EXPANSION AND THE ZAK TRANSFORM

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ABSTRACT

The discrete Gabor signal expansion on a lattice that is obtained by linear combinations of two independent vectors, and its relation with the discrete Zak transform are presented. It is shown how the Zak transform can be helpful in determining Gabor's signal expansion coefficients and how it can be used in finding the dual window that corresponds to a given window for this (generally non-separable) lattice.

1. INTRODUCTION

Gabor's discrete signal expansion is a useful tool to generate a local-frequency spectrum of a signal. Traditionally, the Gabor signal expansion is formulated on a rectangular lattice [1, 2], i.e., an expansion of a signal into a discrete set of properly shifted and modulated versions of an elementary signal. It has been shown that the Zak transform is very useful to determine Gabor's signal coefficients and to determine the dual window [3, 4]. Some preliminary work to extend the Gabor signal expansion on a rectangular lattice to an expansion on a (non-separable) lattice that is an additive subgroup of the time-frequency domain can be found in [5]. In that paper the dual window is determined by the use of the frame operator. Recently, the Gabor expansion is extended to the multi-window Gabor signal expansion [6]. A special case of this is the Gabor expansion on a quincunx lattice. Gabor's signal expansion on a quincunx lattice and its relation with the Zak transform have been shown in [7]. In the present paper we extend the idea that is used for the quincunx lattice to a lattice that can be obtained by linear combinations of two independent vectors.

2. NON-SEPARABLE GABOR TRANSFORM

In this paper we consider Gabor's discrete signal expansion on a lattice that is obtained by linear combinations of two independent vectors. Thus we consider the lattice Λ

$$\Lambda = \{n_0\boldsymbol{v}_0 + n_1\boldsymbol{v}_1 | n_0, n_1 \in \mathbb{Z}\}$$

where $\boldsymbol{v}_0$ and $\boldsymbol{v}_1$ are two independent vectors:

$$\boldsymbol{v}_0 = [aN,\ c/DK]^T, \quad \boldsymbol{v}_1 = [bN,\ d/DK]^T$$

with a, b, c and d integers, N and K integers, and $D = ad - bc$. Each point $\boldsymbol{\lambda} \in \Lambda$ in the time-frequency plane can be obtained by a matrix-vector product

$$\forall_{\boldsymbol{\lambda}\in\Lambda}\exists_{\boldsymbol{n}\in\mathbb{Z}^2}\ \ \boldsymbol{\lambda} = \boldsymbol{U}\boldsymbol{L}\boldsymbol{n},$$

with

$$\boldsymbol{U} = \frac{1}{DK}\begin{bmatrix} NDK & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{L} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Conversely, a point $\boldsymbol{\eta}' = \boldsymbol{U}^{-1}\boldsymbol{\eta}$ is a part of the lattice Λ if the vector $\boldsymbol{x}$

$$\boldsymbol{x} = \boldsymbol{L}^{-1}\boldsymbol{\eta}' = \frac{1}{D}\begin{bmatrix} d\eta'_0 - b\eta'_1 \\ -c\eta'_0 + a\eta'_1 \end{bmatrix}$$

contains only integers. Or worded differently, a point $\boldsymbol{\eta}' \in \mathbb{Z}^2$ is a part of the lattice Λ if D is a divisor of $(d\eta'_0 - b\eta'_1)$ and $(-c\eta'_0 + a\eta'_1)$. From the set of shifted and modulated versions of the window $g[n]$

$$\{g'_{mk}[n] = g[n - mN]\exp(j2\pi kn/DK)\}$$

with $m \in \mathbb{Z}$ and $k = 0 \ldots DK - 1$, we only need those on the lattice Λ. By using the Poisson summation formula

$$\frac{1}{D}\sum_{\ell=0}^{D-1} e^{j2\pi\ell n/D} = \sum_{\ell}\delta[n - \ell D],$$

where $\delta[k]$ is a Kronecker delta, with $\delta[0] = 1$ and $\delta[k] = 0$ for $k \neq 0$, we obtain the shifted and modulated versions of the window $g[n]$ on the lattice Λ

$$g_{mk}[n] = \frac{1}{D^2}\sum_{\ell_0=0}^{D-1} e^{j2\pi\ell_0(dm - bk)/D} \sum_{\ell_1=0}^{D-1} e^{-j2\pi\ell_1(cm - ak)/D} g[n - mN]e^{j2\pi kn/DK}. \tag{1}$$

We assume that the greatest common divisors $\gcd(a,b)=1$ and $\gcd(c,d)=1$, and the determinant $D=\det(\boldsymbol{L})>0$. A possible common divisor can be unified with N and K.

It will be clear that there are a lot of matrices that generate the same lattice Λ. One form, the Hermite Normal Form [8], is very interesting:

$$\boldsymbol{L}'=\begin{bmatrix} 1 & 0 \\ -r & D \end{bmatrix},$$

where $-r=h_0c+h_1d$ with integers h_0 and h_1 such that $h_0a+h_1b=1$. Note that these integers h_0 and h_1 exist, since $\gcd(a,b)=1$, and can be obtained by the Euclidean algorithm. This leads to the simplified shifted and modulated versions $g_{mk}[n]$ [see (1)]

$$g_{mk}[n]=\frac{1}{D}\sum_{\ell_1=0}^{D-1} e^{j2\pi\ell_1(rm+k)/D}$$
$$g[n-mN]e^{j2\pi kn/DK}. \tag{2}$$

The shifted and modulated versions $\gamma_{mk}[n]$ are similar.

Gabor's expansion of the signal $\varphi[n]$ on the lattice Λ looks like

$$\varphi[n]=\sum_{k=0}^{DK-1}\sum_{m} a_{mk}g_{mk}[n], \tag{3}$$

where

$$a_{mk}=<\varphi,\gamma_{mk}>=\sum_{\ell}\varphi[\ell]\gamma^*_{mk}[\ell] \tag{4}$$

with $\gamma_{mk}[n]$ the shifted and modulated versions of the dual window $\gamma[n]$ [cf. (1)].

The area of a cell (a parallelogram) in the time-frequency plane equals the determinant of $\boldsymbol{UL}$, which is equal to N/K. In this paper we will show how to find the dual window $\gamma[n]$ and the expansion coefficients a_{mk} of the Gabor expansion on the lattice Λ where the parameters N and K satisfy the relation $N/K=q/p\leq 1$, where the coprimes p and q are positive integers, $p\geq q\geq 1$.

3. NON-SEPARABLE GABOR SIGNAL EXPANSION AND THE ZAK TRANSFORM

We consider signals $\varphi[n]$ and windows $\gamma[n]$ that have a finite support N_φ and N_γ, respectively. Under these conditions of finite support, the array a_{mk} has a finite support M in the m variable, where the support M satisfies the condition $MN\geq N_\varphi+N_\gamma-1$. By periodizing the signal $\varphi[n]$ and the array a_{mk},

$$\Phi[n]=\sum_{\ell}\varphi[n+\ell MN],\ A_{mk}=\sum_{\ell}a_{m+\ell M,k},$$

and the windows $\gamma[n]$ and $g[n]$,

$$\Gamma[n]=\sum_{\ell}\gamma[n+\ell MN],\ G[n]=\sum_{\ell}g[n+\ell MN],$$

we get, under the assumption that DK is a divisor of MN and D is a divisor M, the relationships

$$\Phi[n]=\sum_{m=<M>}\sum_{k=<DK>}A_{mk}G_{mk}[n] \tag{5}$$
$$A_{mk}=\sum_{n=<MN>}\Phi[n]\Gamma^*_{mk}[n], \tag{6}$$

which are the periodized versions of Gabor's signal expansion (3) and the Gabor transform (4), respectively. The expression $m=<M>$ throughout denotes a finite interval of M successive integers m. For convenience, we introduce two integers p and q ($p\geq q\geq 1$) that do not have common factors and for which the relationship $pN=qK$ holds; note that $K/N=p/q\geq 1$ represents the degree of oversampling. From the assumption that DK is a divisor of MN and D is a divisor M and from the relationship $pN=qK$, it follows that $M=pLD$ and $MN=qLDK$ for an integer L.

It can be shown that combining (5) and (6) yields the following condition

$$\sum_{m=<M>}e^{-j2\pi m\ell r/D}\Gamma^*[n-\ell K-mN]$$
$$\times G[n-mN]=\frac{1}{K}\sum_{k}\delta[\ell-kqLD], \tag{7}$$

for $\ell=<qLD>$, with r as defined in (2). It can be shown that this condition (7) can be written as a sum-of-products form

$$\sum_{\ell=0}^{f_qp-1}\sum_{m=<M>}G[n+iK+mN]$$
$$\times e^{-j2\pi m(u+\ell M/f_qp-irM/D)/M}$$
$$\times\sum_{k=<M>}\Gamma^*[n+sK+kN]$$
$$\times e^{j2\pi k(u+\ell M/f_qp-srM/D)/M}=\frac{f_qp}{K}\delta[i-s] \tag{8}$$

with the integer f_q such that $D=\gcd(D,q)f_q$, and where $i=0\ldots f_qq-1$ and $s=0\ldots f_qq-1$. By using the discrete Zak transform $\tilde{\varphi}[n,\ell,N,M]$ of a periodized signal $\Phi[n]$, which is defined according to

$$\tilde{\varphi}[n,\ell,N,M]=\sum_{m=<M>}\Phi[n+mN]e^{-j2\pi m\ell/M},$$

Eq. (8) becomes

$$\sum_{k=0}^{f_q p-1} \tilde{g}[n+iK, \ell+kM/f_q p - irM/D, N, M]$$
$$\times \tilde{\gamma}^*[n+sK, \ell+kM/f_q p - srM/D, N, M]$$
$$= \frac{f_q p}{K}\delta[i-s]. \tag{9}$$

In matrix notation this becomes

$$\boldsymbol{G}\boldsymbol{\Gamma}^* = \frac{f_q p}{K}\boldsymbol{I}_{f_q q}, \tag{10}$$

with $\boldsymbol{G}$ an $f_q q \times f_q p$ matrix with elements

$$G_{ik} = \tilde{g}[n+iK, \ell+kM/f_q p - irM/D, N, M]$$

and $\boldsymbol{\Gamma}$ an $f_q q \times f_q p$ matrix with elements

$$\Gamma_{ik} = \tilde{\gamma}[n+iK, \ell+kM/f_q p - irM/D, N, M],$$

and where $\boldsymbol{I}_{f_q q}$ is the $f_q q \times f_q q$ identity matrix.

By using the Fourier transform $\overline{a}[n, \ell, K, M]$ of the periodic array A_{mk}, which is defined according to

$$\overline{a}[n, \ell, K, M] =$$
$$\sum_{m=<M>} \sum_{k=<K>} A_{mk} e^{-j2\pi(m\ell/M - kn/K)},$$

and the discrete Zak transform, it can be shown that the following sum-of-products form holds

$$\overline{a}[n, \ell+sM/f_q p, DK, M] =$$
$$K \sum_{i=0}^{f_q q-1} \tilde{\gamma}^*[n+iK, \ell+sM/f_q p - irM/D, N, M]$$
$$\times \tilde{\varphi}[n+iK, \ell - irM/D, f_q pN, M/f_q p]$$

with $s = 0 \ldots f_q p - 1$. In matrix notation this becomes

$$\boldsymbol{a} = K\boldsymbol{\Gamma}^*\boldsymbol{\phi}, \tag{11}$$

where $\boldsymbol{a}$ is an $f_q p$-dimensional column vector of functions

$$\boldsymbol{a} = \left(a_0[n,\ell], a_1[n,\ell], \ldots, a_{f_q p}[n,\ell]\right)^T$$

with

$$a_s[n,\ell] = \overline{a}[n, \ell+sM/f_q p, DK, M],$$

and where $\boldsymbol{\phi}$ is an $f_q q$-dimensional column vector of functions

$$\boldsymbol{\phi} = \left(\varphi_0[n,\ell], \varphi_1[n,\ell], \ldots, \varphi_{f_q q-1}[n,\ell]\right)^T$$

with

$$\varphi_i[n,\ell] = \tilde{\varphi}[n+iK, \ell - irM/D, f_q pN, M/f_q p].$$

The relation (10) applied to the arbitrary vector $\boldsymbol{\phi}$ leads to the condition

$$\boldsymbol{G}\boldsymbol{\Gamma}^*\boldsymbol{\phi} = \frac{f_q p}{K}\boldsymbol{\phi}. \tag{12}$$

Substitution of (11) into (12) yields

$$\boldsymbol{\phi} = \frac{1}{f_q p}\boldsymbol{G}\boldsymbol{a}. \tag{13}$$

Eq. (11) represents $f_q p$ equations and $f_q q$ unknowns. Eq. (13) represents $f_q q$ equations and $f_q p$ unknowns. In the case of oversampling ($p > q \geq 1$) the latter set of equations is thus under-determined.

Note that in the case of a rectangular lattice ($D = 1$), the integer $f_q = 1$, and (9) reduces to the well-known sum-of-products form [3]. Another special case is the quincunx lattice ($D = 2$). Then (9) falls apart into two cases; the case that q is even ($f_q = 1$) and odd ($f_q = 2$), as showed in [7].

4. EXAMPLE

In this section we take as an example the Gabor expansion on a lattice with matrix

$$\boldsymbol{L} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}.$$

We determine some dual windows $\gamma[n]$ for different values of oversampling for the given Gaussian window

$$g[n] = 2^{\frac{1}{4}} T^{-\frac{1}{2}} e^{-\pi((n-(NM/2-1))/T)^2}$$

with $T = 3N/(3\sqrt{8}q/p)^{\frac{1}{2}}$ for this particular lattice. The width of the Gaussian windows is, roughly, T. It will be clear that when we truncate the window function to an interval of N_w where N_w is much larger than T, the discrete Zak transform of this truncated window function will be almost equal to the one of the untruncated window function. The dual windows are depicted in Fig. 1 on the next page. Note that the dual windows $\gamma[n]$ are not real valued. This is the result of the non-symmetrical lattice with respect to the time-axis ($g_{mk}[n] \neq g^*_{m,-k}[n]$).

As a measure we take the ℓ_2 norm of the difference of the dual window $\gamma[n]$ and the optimum dual window $\gamma_{opt}[n]$ which is proportional to the window $g[n]$.

5. CONCLUSIONS

We presented Gabor's discrete signal expansion on a lattice that is obtained by linear combinations of two independent vectors and its relation with the discrete Zak transform. It is shown that the Zak transform can be helpful in determining Gabor's expansions coefficients and how it can be used in finding the dual window that corresponds to a given window for this lattice.

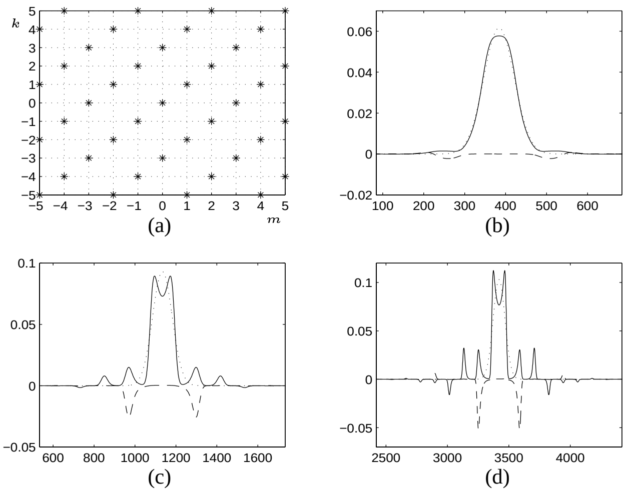

Figure 1: (a) The (normalized) lattice Λ. The dual windows (b), (c), and (d) (solid line the real part, dashed line the imaginary part) of a Gaussian elementary signal $g[n] = 2^{\frac{1}{4}} T^{-\frac{1}{2}} \exp(-\pi(n - (NM/2 - 1)/T)^2)$ and the optimum windows $\gamma_{opt}[n]$ (dotted line) for different values of oversampling, and the difference of the dual window and the optimum dual window in the ℓ_2 norm sense. (b) L=2, K=128, $p/q = 2/1$, $||\gamma - \gamma_{opt}|| = 0.0363$, (c) L=1, K=126, $p/q = 7/6$, $||\gamma - \gamma_{opt}|| = 0.3270$ (d) L=1, K=120, $p/q = 20/19$, $||\gamma - \gamma_{opt}|| = 0.5691$.

6. REFERENCES

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