Fractional transforms in optical information processing

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Abstract

In this paper we review the progress achieved in optical information processing during the last decade by applying fractional linear integral transforms. The fractional Fourier transform and its applications for phase retrieval, beam characterization, space-variant pattern recognition, adaptive filter design, encryption, watermarking, etc., is discussed in detail.

A general algorithm for the fractionalization of linear cyclic integral transforms is introduced and it is shown that they can be fractionalized in an infinite number of ways. Basic properties of fractional cyclic transforms are considered. The implementation of some fractional transforms in optics, such as fractional Hankel, sine, cosine, Hartley, and Hilbert transforms, is discussed.

New horizons of the application of fractional transforms for optical information processing are underlined.

1 Introduction

During the last decades, optics is playing an increasingly important role in computing technology: data storage (CD-ROM) and data communication (optical fibres). In the area of information processing optics also has certain advantages with respect to electronic computing, thanks to its massive parallelism, operating with continuous data, etc. [1–3]. Moreover, the modern trend from binary logic to fuzzy logic, which is now used in several areas of science and technology such as control and security systems, robotic vision, industrial inspection, etc., opens up new perspectives for optical information processing. Indeed, typical optical phenomena such as diffraction and interference, inherit fuzziness and therefore permit an optical implementation of fuzzy logic [4].

The first and highly successful configuration for optical data processing – the optical correlator – was introduced by Van der Lugt more than 30 years ago [5]. It is based on the ability of a thin lens to produce the two-dimensional Fourier transform (FT) of an image in its back focal plane. This invention led to further creation of a great variety of optical and optoelectronic processors such as joint correlators, adaptive filters, optical differentiators, etc. [6]. More sophisticated tools such as wavelet transforms [7] and bilinear distributions [8–14], which are actively used in digital data processing, have been implemented in optics.

Nowadays, fractional transforms play an important role in information processing [15–31], and the obvious question is: Why do we need fractional transformations if we successfully apply the ordinary ones? First, because they naturally arise under the consideration of different problems, for example in optics and quantum mechanics, and secondly, because fractionalization gives us a new degree of freedom (the fractional order) which can be used for more complete characterization of an object (a signal, in general) or as an additional encoding parameter. The canonical fractional FT, for instance, is used for phase retrieval [32–42], signal characterization [43–56], space-variant filtering [29, 57–77], encryption [78–85], watermarking [86, 87], creation of neural networks [88–93], etc., while the fractional Hilbert transform was found to be very promising for selective edge detection [94–96]. Several fractional transforms can be performed by simple optical configurations.

In this paper we review the progress achieved in optical information processing during the last decade by application of fractional transforms. We will start from the definition of a fractional transformation in Section 2. Then we consider, in Section 3, the fractionalization in paraxial optics described by the canonical integral transformation. Two fractional canonical transforms, the Fresnel transform and the fractional FT, are commonly used in optical information processing. The fractional FT, which is a generalization of the ordinary FT with an additional parameter α that can be interpreted as a rotational angle in the phase plane, is considered in more detail.

Since the convolution operation is fundamental in information processing, there were several proposals to generalize it to the fractional case. In Section 4 we define the generalized fractional convolution, and in the subsequent Sections 5-8, we consider its application for information processing: phase retrieval, signal characterization, filtering, noise reduction, encryption, and watermarking.

The second part of the paper will be devoted to the fractionalization procedure of other important transforms. We will restrict ourselves to the consideration of cyclic transforms, which produce the identity transform when they act an integer number of times N. In Sections 9-11, we will show that there are different ways for the construction of a fractional transform for a given cyclic transform. In Section 12 we briefly mention the common properties of fractional cyclic transforms.

The fractional Hankel, Hartley, sine and cosine, and Hilbert transforms, which can all be implemented in optics, will be considered in Section 13. Finally, we discuss the main lines of future development of fractional optics in Section 14 and make some conclusions.

2 Fractional transform: a general definition

The word 'fraction' is nowadays very popular in different fields of science. We recall fractional derivatives in mathematics, fractal dimension in geometry, fractal noise, fractional transformations in signal processing, etc. In general, 'fractional' means that some parameter has no longer a integer value.

To define the fractional version of a given linear integral transform, let us consider the operator $\mathcal{R}$ of such a transform, acting on a function $f(x)$,

$$\mathcal{R}\,[f(x)]\,(u) = \int_{-\infty}^{\infty} K(x,u)\; f(x)\,dx, \qquad (1)$$

with $K(x,u)$ the operator kernel. As an example we mention the Fourier transformation, for which the kernel reads $K(x,u) = \exp(-i2\pi ux)$. The fractional transform operator is denoted by R^p, where p is the parameter of fractionalization:

$$\mathcal{R}^p\,[f(x)]\,(u) = \int_{-\infty}^{\infty} K(p,x,u)\; f(x)\,dx. \qquad (2)$$

We will formulate some desirable properties of this fractional transform first.

The fractional transform has to be continuous for any real value of the parameter p, and additive with respect to this parameter: $\mathcal{R}^{p_1+p_2} = \mathcal{R}^{p_2}\,\mathcal{R}^{p_1}$. Moreover it has to reproduce the ordinary transform and powers of it for integer values of p. In particular, for $p = 1$ we should get the ordinary transform $\mathcal{R}^1 = \mathcal{R}$, and for $p = 0$ the identity transform $\mathcal{R}^0 = I$. From the additivity property it follows that $\int_{-\infty}^{\infty} K(p_1,x,u)\,K(p_2,u,y)\,du = K(p_1+p_2,x,y)$. Note that the parameter p, as we will see further, may be given by a matrix, and the additivity property is then formulated easily as the product of the corresponding matrices.

As we have mentioned in the Introduction, some fractional transforms arise under consideration of different problems: description of paraxial diffraction in free space and in a quadratic refractive index medium, resolution of the non-stationary Schrödinger equation in quantum mechanics, phase retrieval, etc. Other fractional transforms can be constructed for their own sake, even if their direct application may not be obvious yet. In particular, in Section 9 we consider a general algorithm for the fractionalization of a given linear cyclic integral transform. The application of a particular fractional transform for optical information processing then depends on its properties and on the possibility of its experimental realization in optics.

3 Fractionalization in paraxial optics: the canonical integral transform

Analog optical signal processing systems are often described in the framework of paraxial scalar diffraction theory. A typical subset of such a system is displayed in Fig. 1 and contains a thin lens with focal distance f, preceded and followed by two sections of free space with distances z_1 and z_2, respectively. Note that the conventional Van der Lugt correlator [5, 6], mentioned in the Introduction, is constructed by a cascade of two such subsets, with each subset forming an FT system ($z_1 = z_2 = f$) and with a filter mask inserted between them. A monochromatic optical field in a transversal

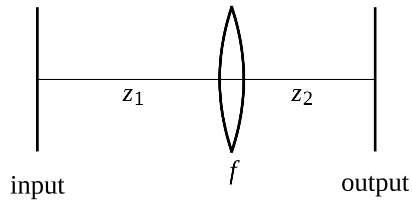

Figure 1: A typical optical information processing system.

plane (x, y) is then described either by a complex field amplitude $f(x, y)$ for the coherent case, or by the two-point correlation function $\Gamma(x_1, x_2; y_1, y_2) = < f(x_1, y_1)\, f^*(x_2, y_2) >$ for the partially coherent case, where the asterisk denotes complex conjugation and $<>$ indicates ensemble averaging; note that these cases correspond to a deterministic or a stochastic signal description in signal theory, respectively.

Under the paraxial approximation of scalar diffraction theory, the complex amplitude $f(x_{\rm in}, y_{\rm in})$ of a monochromatic coherent optical field at the input plane of the setup depicted in Fig. 1 and the complex amplitude $F_M(x_{\rm out}, y_{\rm out})$ at the output plane of it, are related by the input-output relationship [97]

$$F_M(x_{\mathrm{out}}, y_{\mathrm{out}}) = \mathcal{R}^M\left[f(x_{\mathrm{in}}, y_{\mathrm{in}})\right](x_{\mathrm{out}}, y_{\mathrm{out}})$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_{M_x}(x_{\mathrm{in}}, x_{\mathrm{out}})\, K_{M_y}(y_{\mathrm{in}}, y_{\mathrm{out}}) \times\ f(x_{\mathrm{in}}, y_{\mathrm{in}})\, dx_{\mathrm{in}}\, dy_{\mathrm{in}}, \quad (3)$$

where the kernel $K_{M_x}(x_{\mathrm{in}}, x_{\mathrm{out}})$ takes the form

$$K_{M_x}(x_{\mathrm{in}}, x_{\mathrm{out}}) = \begin{cases} \dfrac{1}{\sqrt{ib_x}} \exp\left(i\pi\, \dfrac{a_x x_{\mathrm{in}}^2 + d_x x_{\mathrm{out}}^2 - 2x_{\mathrm{out}}x_{\mathrm{in}}}{b_x}\right), & b_x \neq 0, \\ \dfrac{1}{\sqrt{|a_x|}} \exp\left(i\pi\, \dfrac{c_x x_{\mathrm{out}}^2}{a_x}\right) \delta\left(x_{\mathrm{in}} - \dfrac{x_{\mathrm{out}}}{a_x}\right), & b_x = 0, \end{cases} \quad (4)$$

with

$$M_x = \begin{pmatrix} a_x & b_x \\ c_x & d_x \end{pmatrix} = \begin{pmatrix} 1 - z_2/f_x & \lambda(z_1 + z_2 - z_1 z_2/f_x) \\ -1/\lambda f_x & 1 - z_1/f_x \end{pmatrix} \quad (5)$$

and λ the optical wavelength, and where similar expressions, with x replaced by y, hold for the kernel $K_{M_y}(y_{\mathrm{in}}, y_{\mathrm{out}})$ and the matrix M_y. Note that the optical wavelength λ enters the expressions for b and c as a mere scaling factor; very often, we like to work with reduced, dimensionless coordinates, in which case b and c take a form that would also be achieved by assigning an appropriate value to λ. We remark that the application of cylindrical lenses, $f_x \neq f_y$, permits to perform anamorphic transformations.

The coefficients a_x, b_x, c_x, and d_x that arise in the kernel (4), are entries of the general, symplectic ray transformation matrix [98] that relates the position (x, y) and direction (ξ, η) of an optical ray in the input and the output plane of a so-called first-order optical system, and we have

$$\begin{pmatrix} x_{\mathrm{out}} \\ \xi_{\mathrm{out}} \end{pmatrix} = \begin{pmatrix} a_x & b_x \\ c_x & d_x \end{pmatrix} \begin{pmatrix} x_{\mathrm{in}} \\ \xi_{\mathrm{in}} \end{pmatrix} = M_x \begin{pmatrix} x_{\mathrm{in}} \\ \xi_{\mathrm{in}} \end{pmatrix} \quad (6)$$

and a similar relation for the other dimension, with x and ξ replaced by y and η, respectively. For separable systems, to which we restrict ourselves throughout, symplecticity reads simply $a_x d_x - b_x c_x = 1$ and $a_y d_y - b_y c_y = 1$. The transform described by Eq. (3) is known by such names as canonical integral transform and generalized Fresnel transform [97–100].

Special cases of canonical integral transform systems include

- an imaging system ($1/z_1 + 1/z_2 = 1/f$, and hence $ad = 1$ and $b = 0$);
- a simple lens ($z_1 = z_2 = 0$, and hence $a = d = 1$ and $b = 0$);
- a section of free space ($f \to \infty$, and hence $a = d = 1$ and $c = 0$), which is also known as a parabolic system [97] and which in the paraxial approximation performs a Fresnel transformation;
- a FT system ($z_1 = z_2 = f$, and hence $a = d = 0$ and $bc = -1$), and more generally, a fractional FT system [15–18] ($z_1 = z_2 = 2f\sin^2(\alpha/2)$ [22], and hence $a = d = \cos\alpha$ and $bc = -\sin^2\alpha$), which is also known as an elliptic system [97]; the common case for which $b = -c = \sin\alpha$, follows when we normalize x/ξ with respect to $\lambda f \sin\alpha$, and can also be achieved by formally choosing $\lambda f \sin\alpha = 1$;
- a hyperbolic system [97], with $a = d = \cosh\alpha$ and $bc = \sinh^2\alpha$.

To treat the propagation of partially coherent light through first-order optical systems, it is advantageous to describe such light not by its two-point correlation function $\Gamma(x_1, x_2; y_1, y_2)$ as mentioned before, but by the related Wigner distribution (WD) [101, Chapter 12]; of course, the coherent case considered in Eq. (3), is just a special case of this more general, partially coherent case. The Wigner distribution of partially coherent light is defined in terms of the two-point correlation function by

$$W(x, \xi; y, \eta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Gamma(x + x'/2, x - x'/2; y + y'/2, y - y'/2) \times\ \exp[-i2\pi(\xi x' + \eta y')]\, dx'\, dy'. \quad (7)$$

A distribution function according to definition (7) was first introduced in optics by Walther [8, 9], who called it the generalized radiance. The WD $W(x, \xi; y, \eta)$ represents partially coherent light in a combined space/spatial-frequency domain, the so-called phase plane, where ξ, η are the spatial-frequency variables associated to the positions x, y, respectively.

The WD is closely related to another bilinear distribution, the ambiguity function (AF) [101, Chapter 12], which was also applied to the description of optical fields [10] and which is related to the WD by a combined FT/inverse FT. Note that the introduction of the WD and the AF in optics [8–14] has allowed to describe – through the same function – both coherent and partially coherent optical fields, and to unify approaches for optical and digital information processing.

It is well known that the input-output relationship between the WDs $W_{\mathrm{in}}(x, \xi; y, \eta)$ and $W_{\mathrm{out}}(x, \xi; y, \eta)$ at the input and the output plane of a separable first-order optical system, respectively, reads [12–14]

$$W_{\mathrm{out}}(x, \xi; y, \eta) = W_{\mathrm{in}}(d_x x - b_x \xi, -c_x x + a_x \xi; d_y y - b_y \eta, -c_y y + a_y \eta), \quad (8)$$

which elegant expression can be considered as the counterpart of the canonical integral transform (3) in the phase plane, valid for partially coherent and completely coherent light. A similar relation holds for the AF [10].

Every separable, first-order optical system is described by a set of 2×2 matrices M, one for each transversal coordinate, whose entries are real-valued and whose determinants equal 1, and we have the important symmetry property $K_M^*(x_{\mathrm{in}}, x_{\mathrm{out}}) = K_{M^{-1}}(x_{\mathrm{out}}, x_{\mathrm{in}})$. The cascade of two such systems is characterized by the matrix product $M_3 = M_2 M_1$, which expresses the additivity of first-order optical systems. We might say that each separate subsystem performs a separate fraction of the total canonical integral transform that corresponds to the system as a whole. We may demand that in distributing the total canonical transform over the separate subsystems, certain rules of the dividing procedure should hold, for example, that all fractional subsets should be identical and be defined by the same matrix [102]. It is often possible to separate the original setup into equal subsets characterized by a one-parameter matrix; this is in particular the case for one-parameter systems like the parabolic, the elliptic and the hyperbolic system.

It is easy to see from Eq. (4) that two canonical systems whose parameters are related as $b_1/a_1 = b_2/a_2$, produce the same transformation of the complex amplitude of the input field, and differ only in a scaling (determined by b_2/b_1) and an additional quadratic phase shift [51, 103]:

$$\mathcal{R}^{M_1}\left[f(x_{\mathrm{in}})\right](x_{\mathrm{out}}) = \frac{b_2}{b_1} \exp\left[\frac{i x^2}{2b_1^2}\,(d_1 b_1 - d_2 b_2)\right] \times \mathcal{R}^{M_2}\left[f(x_{\mathrm{in}})\right]\left(\frac{b_2}{b_1} x_{\mathrm{out}}\right). \quad (9)$$

In this sense the elliptic (fractional FT), parabolic (Fresnel transform), and hyperbolic systems with the same b/a, determined by the angle α or the propagation distance z, behave similarly.

The fractional FT and the Fresnel transform are usually applied in optical information processing due to their simple analog realizations. Since both of them belong to the class of canonical integral transforms, we summarize the main theorems for the canonical transform in Table 1. For simplicity, we consider only the one-dimensional case, and we will do the same in the rest of the paper if the generalization to the two-dimensional case is straightforward. The eigenfunctions of the linear canonical transform were considered in [99, 104].

1. linearity
   $$R^M\left[\textstyle\sum_j \mu_j f_j(x)\right](u) = \textstyle\sum_j \mu_j R^M\left[f_j(x)\right](u)$$
2. Parseval's equality
   $$\int_{-\infty}^{\infty} f(x)\, g^*(x)\, dx = \int_{-\infty}^{\infty} F_M(u)\, G_M^*(u)\, du$$
3. shifting
   $$R^M\left[f(x - x_\circ)\right](u) = \exp[i\pi(2ux_\circ - a x_\circ^2)c]\, R^M\left[f(x)\right](u - a x_\circ)$$
4. scaling
   $$R^M\left[f(\mu x)\right](u) = (1/\mu) R^{M_\mu}\left[f(x)\right](u)$$
   $$\text{with } M_\mu = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1/\mu & 0 \\ 0 & \mu \end{pmatrix}$$
5. differentiation
   $$R^M\left[\frac{d^n f(x)}{dx^n}\right](u) = (2\pi i)^n \left[-cu + \frac{a}{2\pi i}\frac{d}{du}\right]^n R^M\left[f(x)\right](u)$$

Table 1: Canonical integral transform: main theorems

4 Fractional Fourier transform and generalized fractional convolution

Since the FT plays an important role in data processing, its generalization – the fractional FT – was probably the most intensively studied among all fractional transforms. Although the FT can be divided into fractions in different ways, the canonical fractional FT certainly has advantages for application in optical information processing. First, because this fractional FT can easily be realized experimentally by using simple optical setups [22], and secondly, because it produces a mere rotation of the two fundamental phase-space distributions: the WD and the AF.

The canonical fractional FT was introduced more than 60 years ago in the mathematical literature [19]; after that, it was reinvented for applications in quantum mechanics [20, 21], optics [15, 16, 18], and signal processing [23]. After the main properties of the fractional FT were established, the perspectives for its implementations in filter design, signal analysis, phase retrieval, watermarking, etc., became clear. Moreover, the use of refractive optics for analog realizations of the fractional FT opened a way for fractional Fourier optical information processing. In this section we will point out the basic properties of the fractional FT and its applications in optics.

In the one-dimensional case we define the fractional FT of a signal $f(x)$ as

$$F_\alpha(u) = \mathcal{R}^\alpha\left[f(x)\right](u) = \int_{-\infty}^{\infty} K(\alpha, x, u)\, f(x)\, dx, \quad (10)$$

where the kernel $K(\alpha, x, u)$ is given by

$$K(\alpha, x, u) = \frac{\exp(i\alpha/2)}{\sqrt{i \sin\alpha}} \exp\left[i\pi\,\frac{(x^2 + u^2)\cos\alpha - 2ux}{\sin\alpha}\right]. \quad (11)$$

Here we use reduced, dimensionless variables x and u. Note the slight change in notation in comparison to Section 2; it will soon be clear that in the case of the fractional FT we prefer to use the fractional angle $\alpha = p\,(\pi/2)$.

The fractional FT can be considered as a generalization of the ordinary FT for the parameter α, which may be interpreted as a rotation angle in the phase plane [22]. This can easily be seen by considering the WD (or the AF) and by noting that a fractional FT system is a special case of a first-order optical system with $a = d = \cos\alpha$ and $b = -c = \sin\alpha$. If $f_{\rm out}(u) = \mathcal{R}^{\alpha}\,[\,f_{\rm in}(x)]\,(u)$ is the fractional FT of $f_{\rm in}(x)$, then the WD $W_{\rm in}(x,\xi)$ of $f_{\rm in}(x)$ and the WD $W_{\rm out}(u,\upsilon)$ of $f_{\rm out}(u)$ are related as $W_{\rm in}(x,\xi) = W_{\rm out}(u,\upsilon)$, see Eq. (8), where x and ξ are related to u and υ by the rotation operation

$$\begin{pmatrix} u \\ \upsilon \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix}. \qquad (12)$$

A detailed analysis of the fractional FT can be found in [24, 25, 29–31]. From its properties we mention that for $\alpha = \pm\pi/2$, we have the normal FT and its inverse [and also $F_{\alpha+\pi}(u) = F_{\alpha}(-u)$], while for $\alpha \to 0$ we have the identity transformation: $F_0(x) = f(x)$. Note also the symmetry properties $K(\alpha,x,u) = K(\alpha,u,x)$ and $K^*(\alpha,x,u) = K(-\alpha,u,x)$, and the reversion property $R^{\alpha}[\,f(-x)](u) = R^{\alpha}[\,f(x)](-u)$. The analysis and synthesis of eigenfunctions of the fractional FT for a given angle were discussed in [105–109].

Besides the optical realization of a fractional FT system mentioned before in Section 3, other optical schemes have been proposed [22, 110–113]. In particular, the complex amplitudes at two spherical surfaces of given curvature and spacing are related by a fractional FT, where the angle is proportional to the Gouy phase shift between the two surfaces [110–112]. This relationship can be helpful for the analysis of quasi-confocal resonators and data transmission between a spherical emitter and receiver.

In the sequel, optical systems performing a fractional FT will be called fractional FT systems. As we have mentioned before, the use of cylindrical refractive index media allows to perform a separable, two-dimensional fractional FT for different angles in the two dimensions [114, 115].

One of the most important properties of the FT is related to the convolution operation on two signals $f(x)$ and $g(x)$,

$$h_{f,g}(x) = \int_{-\infty}^{\infty} f(x')\,g(x-x')\,dx', \qquad (13)$$

which in the spectral domain takes the form

$$\mathcal{R}^{\pi/2}\left[h_{f,g}(x)\right] = \left\{\mathcal{R}^{\pi/2}\,[\,f(x)]\right\}\left\{\mathcal{R}^{\pi/2}\,[g(x)]\right\}. \qquad (14)$$

After the introduction of the fractional FT, several kinds of fractional convolution and correlation operations were proposed [57–70]. These operations can be expressed in the form of a generalized fractional convolution (GFC) $H_{f,g}(x,\alpha,\beta,\gamma)$, defined by [66]

$$\mathcal{R}^{\alpha}\left[H_{f,g}(x,\alpha,\beta,\gamma)\right] = \left\{\mathcal{R}^{\beta}\,[\,f(x)]\right\}\left\{\mathcal{R}^{\gamma}\,[g(x)]\right\}, \qquad (15)$$

cf. Eq. (14), or equivalently by

$$\mathcal{R}^{\alpha-\pi/2}\left[H_{f,g}(x,\alpha,\beta,\gamma)\right](u) = \int_{-\infty}^{\infty} F_{\beta-\pi/2}(u')\,G_{\gamma-\pi/2}(u-u')\,du', \qquad (16)$$

cf. Eq. (13).

It is easy to see that the GFC includes as particular cases almost all definitions of the fractional convolution and correlation operations proposed before [57–70]. Also the expressions for the cross-WD and cross-AF can easily be given in terms of the GFC; for the cross-WD and cross-AF expressed in polar coordinates [34],

$$W_{f,g}(r,\phi) = 2\int_{-\infty}^{\infty} F_{\phi+\pi/2}(u)\,G^*_{\phi+\pi/2}(-u)\,\exp[i2\pi u(2r)]\,du, \qquad (17)$$

$$A_{f,g}(r,\phi) = \int_{-\infty}^{\infty} F_{\phi+\pi/2}(u)\,G^*_{\phi+\pi/2}(u)\,\exp(i2\pi ur)\,du, \qquad (18)$$

we thus have

$$W_{f,g}(r,\phi) = 2H_{f,g^*}(2r,\pi/2,\phi+\pi/2,-\phi+\pi/2), \qquad (19)$$
$$A_{f,g}(r,\phi) = H_{f,g^*}(r,\pi/2,\phi+\pi/2,-\phi-\pi/2), \qquad (20)$$

respectively. The GFC system is represented schematically in Fig. 2, indicating a general procedure to obtain the GFC.

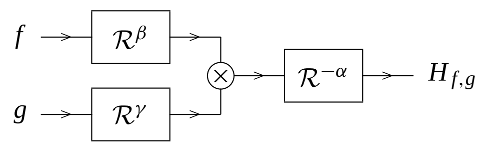

Figure 2: Schematic representation of the generalized fractional convolution system.

In view of the canonical integral transform, a further generalization of the convolution operation $H_{f,g}(x,M_1,M_2,M_3)$ can be proposed as [69]

$$\mathcal{R}^{M_1}\left[H_{f,g}(x,M_1,M_2,M_3)\right] = \left\{\mathcal{R}^{M_2}\,[\,f(x)]\right\}\left\{\mathcal{R}^{M_3}\,[g(x)]\right\}, \qquad (21)$$

where the kernels of the three canonical integral transforms are parameterized by a matrix M, see Eq. (6). This definition corresponds to the nonconventional convolution that is used in real optical systems under the paraxial approximation of the scalar diffraction theory, where the image and filter planes are shifted from their conventional positions [68, 71]. As particular cases, the GFC and the Fresnel convolution can thus be realized. The introduction of the canonical convolution operation permits to find features similar to the ones of the fractional Fourier correlators and the Fresnel correlator, proposed several years ago in [71], and to treat easily the fractional correlator based on the modified fractional FT [68].

Note that the GFC of a one-dimensional signal is a function of four variables: x, α, β, and γ. The angle variables are often considered as parameters, and the function becomes one-dimensional. As we will see below in Sections 5 and 6, optical signal processing allows to treat the GFC as a two-dimensional function, where one of the parameters is considered as the second coordinate. The choice of the parameters and the number of variables of the GFC depends on the particular application. In the following Sections 5-8, we will consider the applications of the GFC for phase retrieval, signal characterization, pattern recognition, and filtering tasks, respectively.

5 Fractional power spectra for phase retrieval

Phase retrieval from intensity information is an important problem in many areas of science, including optics, quantum mechanics, X-ray radiation, etc. In particular non-interferometric techniques have attracted considerable attention recently. In this section we consider the application of fractional FT systems for the phase retrieval problem.

The squared moduli of the fractional FT, also called fractional power spectra, correspond to the projection of the WD upon the direction at an angle α in the phase plane. Note also that the fractional power spectrum is the particular case of the GFC

$$|F_\alpha(u)|^2 = H_{f,f^*}(u, 0, \alpha, -\alpha). \tag{22}$$

Fractional power spectra play an important role in fractional optics: they are related to the intensity distributions at the output plane of a fractional FT system and therefore can be easily measured in optics. The set of fractional power spectra for $\alpha \in [0, \pi]$ is called the Radon-Wigner transform [116], because it defines the Radon transform of the WD. The WD can be obtained from the Radon-Wigner transform by applying the inverse Radon transform [101, Chapter 8]. This is a basis for phase-space tomography [32], a method for experimental determination of the complex field amplitude in the coherent case or the two-point correlation function for partially coherent fields, from the measurements of only intensity distributions. Application of cylindrical lenses allows the reconstruction of two-dimensional optical fields.

In the case of coherent optical signals, other methods for phase retrieval based on the measurements of fractional power spectra have been proposed. One of them is related to the estimation of the instantaneous spatial frequency $\Xi(x)$ from two close fractional power spectra. It was shown that the instantaneous frequency is related to the convolution of the angular derivative of the fractional power spectrum and the signum function [33],

$$\begin{aligned}
&\Xi_{F_\beta}(x) \\
&\quad= \frac{\int_{-\infty}^{\infty} \xi\, W_f(x\cos\beta - \xi\sin\beta, x\sin\beta + \xi\cos\beta)\, d\xi}{\int_{-\infty}^{\infty} W_f(x\cos\beta - \xi\sin\beta, x\sin\beta + \xi\cos\beta)\, d\xi} \\
&\quad= \frac{1}{2\left|F_\beta(x)\right|^2} \int_{-\infty}^{\infty} \left.\frac{\partial \left|F_\alpha(x')\right|^2}{\partial\alpha}\right|_{\alpha=\beta} \operatorname{sgn}(x - x')\, dx',
\end{aligned} \tag{23}$$

where $\operatorname{sgn}(x) = \pm 1$ for $x \gtrless 0$. Moreover, since the instantaneous frequency is the phase derivative of the fractional FT of a signal,

$$2\pi\ \Xi_{F_\beta}(x) = d\varphi_\beta(x)/dx \tag{24}$$

where $\varphi_\beta(x) = \arg F_\beta(x)$, the complex field amplitude up to a constant phase factor can be reconstructed from only two close fractional power spectra [33–35]. This method has been demonstrated on different examples of multicomponent and noisy signals and exhibits high quality of phase reconstruction [35]. Note that a similar method of phase retrieval can be applied for any one-parameter canonical transform [36]. Thus, in the case of the Fresnel transform we can mention a non-iterative approach for phase retrieval in free space, based on the so-called transport-of-intensity equation in optics, proposed by Teague [37] and then further developed by others.

In the case that two fractional power spectra are known for angles which are not close to each other, iterative methods of phase retrieval can be applied [38–40]. These methods are a generalization of the iterative Gerchberg-Saxton algorithm, designed for the recovery of a complex signal from its intensity distribution and power spectra.

Another method for phase retrieval is based on a signal decomposition as a series of orthogonal Hermite-Gauss modes [41]. It has been shown that if a coherent optical signal contains only a finite number of Hermite-Gauss modes N, then it can be reconstructed from the knowledge of its $2N$ fractional power spectra – associated with the intensity distribution in a fractional FT system – at only two transversal points. Note that this method can be generalized to the case of other fractional optical systems to be discussed below, such as for example the fractional Hankel one.

A further method for phase retrieval is based on filtering of the optical field in fractional Fourier domains [42]. Indeed, the phase derivative $d\varphi/dx$, and therefore the phase $\varphi(x)$ up to a constant term, can be reconstructed from the knowledge of the intensity $|f(x)|^2$ and the intensity distributions at the output of two fractional FT filters with mask u

$$\frac{d\varphi(x)}{dx} = \pi \frac{\left|\mathcal{R}^{-\alpha}\left[F_\alpha(u)\, u\right](x)\right|^2 - \left|\mathcal{R}^{\alpha}\left[F_{-\alpha}(u)\, u\right](x)\right|^2}{x\, |f(x)|^2\ \sin 2\alpha}. \tag{25}$$

The efficiency of this approach has been demonstrated by numerical simulations. A simple optical configuration for the experimental realization of the method was discussed in [42].

6 Fractional power spectra for optical beam characterization

Since the AF, the WD, and other bilinear distributions of two-dimensional optical signals are functions of four variables, their direct application for the analysis and characterization is limited. Mostly the moments of these distributions are used for beam characterization. The normalized moments μ_{pqrs} of the WD are defined by

$$\mu_{pqrs}\, E = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(x,\xi;y,\eta) \times x^p\,\xi^q\,y^r\,\eta^s\,dx\,d\xi\,dy\,d\eta \quad (p,q,r,s \geq 0), \quad (26)$$

where the normalization is with respect to the total energy E of the signal (and hence $\mu_{0000} = 1$). Note that in a first-order optical system, with a symplectic ray transformation matrix, the total energy E is invariant. The low-order moments represent the global features of the optical signal such as total energy, width, principal axes, etc. Thus the second-order moments of the WD ($p+q+r+s = 2$) are used as a basis of an International Organization for Standardization standard of beam quality. The combination of the second-order moments $(\mu_{1001} - \mu_{0110})\,E$, for instance, describes the orbital angular momentum of the optical beam, which is actively used for the description of vortex beams [117]. The moments of higher order are related to finer details of the optical signal.

Note that for $q = s = 0$ and for $p = r = 0$ we have the position and frequency moments, which can easily be obtained from measurements of the intensities in the signal and the Fourier domain, respectively:

$$\mu_{p0r0}\, E = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^p\,y^r\,|F_0(x,y)|^2\,dx\,dy, \quad (27)$$

$$\mu_{0q0s}\, E = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \xi^q\,\eta^s\,\left|F_{\pi/2}(\xi,\eta)\right|^2\,d\xi\,d\eta. \quad (28)$$

Since in optics only intensity distributions can be measured directly, it was proposed in [43] to apply fractional FT systems in order to calculate other moments from the intensity moments. It was shown that the moments at the output plane of a separable fractional FT system, with fractional angles α and β in the x- and the y-direction, respectively, are related to the input ones as

$$\mu^{\text{out}}_{pqrs} = \sum_{k=0}^{p}\sum_{l=0}^{q}\sum_{m=0}^{r}\sum_{n=0}^{s}\binom{p}{k}\binom{q}{l}\binom{r}{m}\binom{s}{n} \times (-1)^{l+n}\,(\cos\alpha)^{p-k+q-l}\,(\sin\alpha)^{k+l}\,(\cos\beta)^{r-m+s-n} \times (\sin\beta)^{m+n}\,\mu^{\text{in}}_{p-k+l,q-l+k,r-m+n,s-n+m}, \quad (29)$$

and for the intensity moments in particular we have

$$\mu^{\text{out}}_{p0r0} = \sum_{k=0}^{p}\sum_{m=0}^{r}\binom{p}{k}\binom{r}{m}(\cos\alpha)^{p-k}\,(\sin\alpha)^{k} \times (\cos\beta)^{r-m}\,(\sin\beta)^{m}\,\mu^{\text{in}}_{p-k,k,r-m,m}. \quad (30)$$

From Eq. (30) a set of fractional FT systems can be found for which the input moments can be derived from knowledge of the intensity moments in the output, i.e. from fractional power spectra for selected angles α and β. It was demonstrated [43] that in order to find all n-th order moments – and we have $(n+1)(n+2)(n+3)/6$ of such moments – we need N fractional power spectra, where $N = (n+2)^2/4$ for even n and $N = (n+1)(n+3)/4$ for odd n. Moreover $N-(n+1)$ spectra have to be anamorphic, i.e., spectra with non-equal fractional order for the two transversal coordinates ($\alpha \neq \beta$). In particular, we need 2 fractional spectra to find the 4 first-order moments, 4 fractional spectra (one of which has to be anamorphic) to find the 10 second-order moments, 6 fractional spectra (with 2 anamorphic ones) to find the 20 third-order moments, etc.

Regarding the evolution of the second-order moments in a fractional FT system, we can find the fractional domain where the signal has the best concentration or where it is the most widely spread, by calculating the zeros of the angular derivatives of the central moments $\mu_{p0r0}(\alpha,\beta)$. This analysis [33, 34] is helpful, for example, in search for an appropriate fractional domain to perform filtering operations [45]. Smoothing interferograms in the optimal fractional domain leads to a weighted WD with significantly reduced interference terms of multicomponent signals, while the auto terms remain almost the same as in the WD. In general, based on this approach optimal signal-adaptive distributions can be constructed with low cost [46].

The way to determine the moments from measurements of intensity distributions as described by Eq. (30), has been generalized to the case of arbitrary separable first-order optical systems [44]. Using an equation similar to Eq. (29) one can easily determine the evolution of these moments during propagation of the beam in any first-order optical system; in particular this was applied to the analysis of optical vortices [47].

In signal processing, the fractional FT spectra were primarily developed for detection and classification of multicomponent linear FM in noise [48, 49].

It was shown [50–56] that the fractional FT spectra as well as the Fresnel spectra are also useful for the analysis of fractal signals. Thus the hierarchical structure of the fractal fields and its main characteristics such as fractal dimension, Hurst exponent, scaling parameters, fractal level, etc., can be obtained from the analysis of the fractional spectra for the angular region from 0 to $\pi/2$ [50–53]. Since in this region the fractional FT spectra and the Fresnel transform spectra differ only by a scaling parameter, the Fresnel diffraction is applied for this task [51, 52, 55]. Recently the experimental fractal tree of triadic Cantor bars has been constructed from the observation of the evolution of diffraction patterns in free space [54]. The general properties of the Fresnel diffraction by structures constructed through the multiplicative iterative procedure have been studied in [56].

7 Generalized fractional convolution for pattern recognition

A great part of the proposed applications of the GFC is related to pattern recognition tasks [57, 60, 66–74]. It was shown [66, 67] that for this purpose the following relation between the angular parameters has to hold

$$\cot\alpha = \cot\beta + \cot\gamma. \tag{31}$$

Then the amplitude of the GFC is expressed in the form [66],

$$\begin{aligned}\left|H_{f,g^*}(x,\alpha,\beta,\gamma)\right| &= C\left|\int_{-\infty}^{\infty} f\left[\frac{\sin\beta}{\sin\gamma}\left(x\frac{\sin\gamma}{\sin\alpha}-y\right)\right] g^*(y)\right.\\ &\quad\times \exp\left[i\pi y^2\,\frac{\cot\alpha\,(1+\cot\gamma\,\cot\beta)}{1+\cot^2\beta}\right.\\ &\qquad\left.\left.-i\pi yx\,\frac{\sin 2\beta}{\sin\alpha\,\sin\gamma}\right]dy\right|, \end{aligned} \tag{32}$$

where C is a constant for fixed α, β, and γ. The quadratic phase factor under the integral vanishes – which brings the integral in the form of a windowed FT – if $\cot\alpha\,(1+\cot\gamma\,\cot\beta) = 0$. In the case $\cot\alpha = 0$ (and hence $\alpha = \pi/2$ and $\gamma = -\beta$) which is usually considered, $H_{f,g^*}(r,\pi/2,\beta,-\beta)$ corresponds to radial slices $A_{f,g}(r,\beta-\pi/2)$ of the cross-AF of the signals $f(x)$ and $g(x)$, cf. Eq. (20).

If the position and the size of the object is known, then the correlation operation $H_{f,g^*}(x,\alpha,\beta,-\beta)$ for pattern recognition can be performed in any fractional domain β, since the auto-AF has a maximum at the coordinate origin $r = 0$. Nevertheless, in spite of the fact that the magnitude of the correlation maximum is the same in any fractional domain, the forms of the correlation peaks are different. It was shown [70] on the example of a rectangular function that the narrowest correlation peak is observed in the fractional domain with fractional angle $\beta = 0$. Note also that the object is usually corrupted by noise, or is blurred. The characteristics of the noise (except white noise) in different fractional domains depend on the fractional angle [75]. The fractional correlation offers the flexibility to choose the domain where the effect of noise on the correlation operation is minimized. Moreover, for the recognition of complex or highly degraded objects, several fractional correlation operations for different angles can be performed in order to make the right decision.

On the other hand, if the position of the object is unknown, the choice of the fractional domain is related to the tolerance to a shift variance of the correlation operation. A shift of the signal leads to a shift and a modulation of the cross-AF:

$$A_{f(y-s),g(y)}(x,\xi) = A_{f(y),g(y)}(x-s,\xi)\,\exp(-i\pi s\xi) \tag{33}$$

Then the form of the AF radial slices of a shifted signal is changing except for the angle corresponding to the ordinary correlation (see Fig. 3).

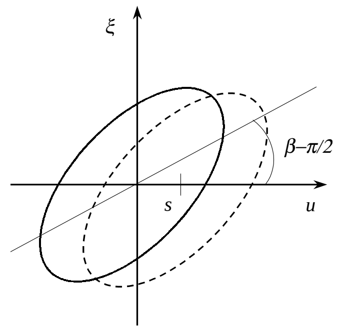

Figure 3: Schematic representation of the cross-AF of two signals, before (solid line) and after (dashed line) shifting of one of the signals.

Therefore fractional correlations are shift variant for $\beta \neq \pi/2 + n\pi$. Thus if in the conventional correlator a shift of the object results in a shift with opposite sign of the correlation peak at the output plane, the shape of the peak is also changed in the fractional correlator. This effect increases with decreasing parameter β from $\pi/2$ down to 0. For large β the fractional correlator is almost shift invariant, whereas for small β it becomes strongly shift variant. Note that there are applications, such as cryptography or image coding, where the location of the object can be as important as its form. In these cases fractional correlators with fractional parameter β, $0 < \beta < \pi/2$, must be used.

The shift tolerance condition is usually written in the form [29, 59, 60] $\pi s\sigma\cot\beta \ll 1$, where s is the signal shift and σ the signal width. More precisely the shift variance depends on the fractional order, the signal size, and also the form of the AF.

The tasks of pattern detection and recognition in optics are mostly related to two-dimensional signals (images). It is also possible to choose different fractional orders for the two orthogonal coordinates and thus to better control the shift variance. In order to recognize a letter on a certain line of the text, for example, one can choose the parameter $\beta_x = \pi/2$ and $\beta_y < \pi/2$ while the filter corresponds to the inverse fractional FT with parameters β_x, β_y of a letter situated on a given line. The exciting results demonstrating the efficiency of shift-variant pattern recognition in the fractional domain, can be found in [72–74].

The fractional correlation operation can be performed in optics by a fractional Van der Lugt correlator [72–74] or by a nonconventional joint transform correlator [118].

In order to maximize the Horner efficiency of the correlation operation, phase-only filters are often used. It was shown in [76] that in general the phase of the fractional FT for $\alpha \neq n\pi$ contains more information about the signal/image than the amplitude. Therefore the phase-only filters can also be applied in the fractional Fourier domain. The development of liquid crystal spatial light modulators allows their relatively simple implementation in optics.

Another particular case of GFC which can be applied for recognition tasks, is related to the fractional FT of the ordinary correlation operation [23] $H_{f,g^*}(x, \alpha, \pi/2, -\pi/2)$. We believe that this type of operation can be useful for angles α at the region near $\pi/2$ in order to improve the performance of the conventional correlation operation. Thus it was shown [77] that for α slightly different from $\pi/2$, the performance of the joint-transform correlator improves and higher correlation peaks are observed. Efficient use of the light source and a larger joint transform spectrum were achieved. Moreover for these angles α the correlator still remains shift invariant. Nevertheless using angles α far from $\pi/2$, leads to confusing results for interpreting the correlation peaks. Indeed, if the conventional correlation operation does not produce clear local maximum and is almost constant, then a sharp peak in fractional correlation $H_{f,g^*}(x, \alpha \approx 0, \pi/2, -\pi/2)$ can appear.

8 Generalized fractional convolution for filtering and data protection

Let us consider now the filtering operation in the fractional domain. The parameters of the GFC in this case depends on the particular application of filtering. If the filter is used for improvement of image quality or for manipulation of the image f in order to extract its features (for example for edge detection or image deblurring), then we have to choose $\beta = \alpha$, in order to represent the result of filtering in the position domain. Since we are free to assign an arbitrary fractional domain for the filter function g, we can as well put $\gamma = \alpha$. Thus the complete operation leads to the $H_{f,g}(x, \alpha, \alpha, \alpha)$. The useful properties of this type of GFC,

$$\mathcal{R}^{\beta}\left[H_{f,g}(x, \alpha, \alpha, \alpha)\right](u) = H_{F_\beta, G_\beta}(u, \alpha-\beta, \alpha-\beta, \alpha-\beta),$$

$$H_{f,g}(x, \alpha, \alpha, \alpha) = H_{f,g}(x, \alpha+\pi, \alpha+\pi, \alpha+\pi),$$

were proved in [62]. Moreover this type of convolution operation is associative for a fixed parameter α.

The GFC $H_{f,g}(x, \alpha, \alpha, \alpha)$ has been found very powerful for noise reduction, if the noise is separable from the signal or very well concentrated in some fractional domain [57]. It was shown that in particular for chirp-like noise, the performance of filtering in a fractional domain is more relevant [24, 29]. Since the fractional FT of a chirp becomes proportional to a Dirac-delta function in an appropriate fractional domain, it can be detected as a local maximum on the Radon-Wigner transform map and then easily removed by a notch filter, which minimizes the signal information loss.

Several applications of fractional FT filtering systems for industrial devices have been proposed recently.

Chirp detection, localization, and estimation via the fractional FT formalism are applied now in different areas of science. Appropriate filtering in fractional domains, which allows to extract linear chirps out of a multicomponent and noisy signal, is used to analyze the propagation of acoustic waves in a dispersive medium [119]. In particular, the nonlinear effects due to the Helmholtz resonators are considered.

A new spatial filtering technique for partially coherent light in the fractional Fourier domain [120] was proposed to improve image contrast and depth of focus in projection photo lithography. Unlike the currently applied pupil method of filtering in the Fourier domain, the fractional filter can be placed at any location along the projection optical path other than the pupil plane. On the examples of designed phase filters for contact hole and line-space patterns, it was demonstrated that the fractional FT filtering technique can significantly improve image fidelity, reduce the optical proximity effect, and increase the depth of focus.

Optical technologies play an increasing role in securing information [121]. Also the GFC found its way into security protection: encryption and watermarking techniques originally proposed for the Fourier domain, were generalized to the fractional domain.

Optical image encryption by random phase filtering in the fractional Fourier domain was proposed in [78,79]. It can be described by the GFC $H_{f,g}(x, \alpha, \beta, \beta)$, where the phase mask G_β and the parameters α and β are the encryption codes. This procedure was further generalized by application of the cascaded fractional FT with random phase filtering [80]. In order to encode the image, the fractional transform is performed and random phase is introduced by means of a spatial light modulator. After repeating this procedure several times, the encrypted image is obtained. In order to decode it, not only the information about the used random phase masks has to be known, but also the parameters and the types of the fractional transforms. It was demonstrated that it is impossible to reconstruct the image using the correct masks but the wrong fractional orders. Without increasing the complexity of the hardware, the fractional Fourier optical image encryption system has additional keys provided by the fractional order of the fractional convolution operation. Due to the double domain properties of the fractional FT the algorithm demonstrates the robustness to the blind deconvolution.

Recently, some modifications of the optical encryption procedures in the fractional Fourier domain were proposed. Thus in [81] the combination of a jigsaw transform and a localized fractional FT were applied. The image to be encrypted is divided into independent non-overlapping segments, and each segment is encrypted using different fractional parameters and two statistically independent random phase codes. The random phase codes, the set of fractional orders, and the jigsaw transform index, are the keys to the encrypted data. The encryption by juxtaposition of sections of the image in fractional Fourier domains without random phase screen keys, was proposed in [82].

Another encryption technique discussed in [83], is based on a method of phase retrieval using the fractional FT. The encrypted image consists of two intensity distributions, obtained in the output of two fractional FT systems of different fractional orders, where the input of each system is formed by the 2-D complex signal multiplied by a random phase mask. The two statistically independent random phase masks and the fractional orders form the encryption key. Decryption is based on the correlation property of the fractional FT, which allows to recover the signal recursively.

The implementation of a fully phase encryption system, using a fractional FT to encrypt and decrypt a 2-D phase image obtained from an amplitude image, was reported in [84]. A comparative analysis of the encryption techniques based on the implementation of the fractional FT has been done in [85].

Watermarking is another widely applied data protection operation. A watermarking technique in the fractional domain was proposed in [86, 87]. In this case, the GFC $H_{f,g}(x, \alpha, \alpha, \alpha)$ is commonly used. In order to include the watermark, the α-fractional FT of the image is performed. The signature has to be such a function which is spread in the image domain and well localized in the fractional domain α. Usually the chirp signal which becomes a δ-function in a certain fractional domain and spread in the image domain is used. Introducing the watermark and performing the inverse fractional FT finally we obtain the protected image. Usually several watermarks in the different fractional FT domains are introduced. Only the owner of the image, who knows the all fractional domains will be able to remove them. This watermarking technique is robust to translation, rotation, cropping and filtering [86, 87].

9 General algorithm for the fractionalization of cyclic transforms

We have considered the properties and application of the fractional FT. Now the following key questions arise :

- Is this fractional FT unique? Or is it possible to generate other fractional FTs?
- How can we generate the fractional version of other transformations, for example Hilbert, sine, cosine?
- Do fractional transforms have some common properties?

In order to answer these questions, we will consider the procedure of fractionalization of a given transform [27,28]. Similar approaches for fractionalization of the integral transform, and the FT in particular, were reported in [122] and [123], respectively. We will restrict ourselves to the consideration of cyclic transforms. There is a long list of linear transforms, actively used in optics and signal/image processing, which belong to this class of cyclic transforms. Thus, if $\mathcal{R}$ is an operator of a linear integral transform, see Eq. (1), this transform is a cyclic one, if it produces the identity transform when it acts an integer number of times N:

$$\mathcal{R}^N \left[f(x) \right](u) = f(u). \tag{34}$$

For example, the Fourier and Hilbert transforms are cyclic with a period $N = 4$, and the Hankel and Hartley transforms have a period $N = 2$. Cyclic canonical transforms of period N with kernel $K(x, u) = K_M(x, u)$, cf. Eq. (4),

$$K(x, u) = \frac{1}{\sqrt{ib}} \exp\left(i\pi \, \frac{ax^2 + du^2 - 2ux}{b} \right), \tag{35}$$

where $a + d = 2\cos(2\pi m/N)$ and m and N are integers, were mentioned in [124].

All cyclic transforms have some common properties. In particular, the eigenvalues of cyclic transforms can be represented as $A = \exp(i2\pi L/N)$, where L is an integer. Indeed, let $\Phi(x)$ be an eigenfunction of $\mathcal{R}$ with eigenvalue $A = |A| \exp(i\varphi)$; from Eq. (34) one gets that $A^N = 1$, and hence $|A| = 1$ and $\varphi = 2\pi L/N$.

In Section 2 we have formulated the requirements for the fractional $\mathcal{R}$-transform $\mathcal{R}^p$, where p is the parameter of the fractionalization: continuity of $\mathcal{R}^p$ for any real value p; additivity of $\mathcal{R}^p$ with respect to the parameter p; reproducibility of the ordinary transform for integer values of p: $\mathcal{R}^1 = R$ and $\mathcal{R}^0 = I$. In the case of cyclic transforms we obviously demand that $\mathcal{R}^N = I$.

Let us analyze the structure of the kernel $K(p, x, u)$ of a fractional $\mathcal{R}$-transform with period N. Due to its periodicity with respect to the parameter p, one can represent $K(p, x, u)$ in the form

$$K(p, x, u) = \sum_{n=-\infty}^{\infty} k_n(x, u) \, \exp(i2\pi pn/N), \tag{36}$$

where the coefficients $k_n(x, u)$ have to satisfy the system of N equations [27]

$$K(l, x, u) = \sum_{n=-\infty}^{\infty} k_n(x, u) \, \exp(i2\pi ln/N) \tag{37}$$

with $l = 0, \ldots, N - 1$. From the additivity property for the fractional transform it follows that the coefficients have to be orthonormal to each other [27,28],

$$\int_{-\infty}^{\infty} k_n(x, u) \, k_m(u, y) \, du = \delta_{n,m} \, k_n(x, y), \tag{38}$$

where $\delta_{n,m}$ denotes the Kronecker delta.

Note that all coefficients $k_{n+mN}(x,u)$ for fixed n and an arbitrary integer m, have the same exponent factor in the system of Eqs. (37). Therefore we can rewrite Eq. (37) as

$$K(l,x,u) = \sum_{n=0}^{N-1} \exp(i2\pi ln/N) \sum_{m=-\infty}^{\infty} k_{n+mN}(x,u). \quad (39)$$

If we introduce the new variables $C_n(x,u)$, which are the partial sums of the coefficients in the Fourier expansions (36) and (37),

$$C_n(x,u) = \sum_{m=-\infty}^{\infty} k_{n+mN}(x,u), \quad (40)$$

Eq. (39) reduces to a system of N linear equations with N variables. This system has a unique solution [27]

$$C_n(x,u) = \frac{1}{N} \sum_{l=0}^{N-1} \exp(-i2\pi ln/N)\, K(l,x,u). \quad (41)$$

It is easy to see that the variables C_n satisfy a condition similar to Eq. (38):

$$\int_{-\infty}^{\infty} C_n(x,u)\, C_m(u,y)\, du = \delta_{n,m}\, C_n(x,y). \quad (42)$$

Note that some partial sums for certain transforms may be equal to zero. As we will see further on, this is the case for the Hilbert transform, for instance.

So, if we find the coefficients $k_n(x,u)$ that satisfy the condition (38) and whose partial sums are given by Eq. (41), we can construct the fractional transform. In general, there are a number of sets $\{k_n(x,u)\}$ that generate fractional transforms of a given $\mathcal{R}$-transform.

10 N-periodic fractional transform kernels with N harmonics

Let us first construct the fractional transform kernel with N harmonics, where N is the period of the cyclic transform. Then every sum $C_n(x,u)$ ($n \in [0, N-1]$) contains only one element $k_{n+\varphi_n}(x,u) = C_n(x,u)$ from the decomposition (36), where $\varphi_n = mN$ and m is an arbitrary integer. Therefore, in the general case, the kernel of the fractional $\mathcal{R}$-transform with N harmonics can be written as

$$\begin{aligned} K(p,x,u) &= \sum_{n=0}^{N-1} k_{n+\varphi_n}(x,u)\, \exp[i2\pi p(n+\varphi_n)/N] \\ &= \frac{1}{N} \sum_{l=0}^{N-1} K(l,x,u) \sum_{n=0}^{N-1} \exp(-i2\pi ln/N) \\ &\qquad \times \exp[i2\pi p(n+\varphi_n)/N]. \quad (43) \end{aligned}$$

This equation provides a formula for recovering the continuous periodic function $K(p,x,u)$ from its N samples $K(l,x,u)$, under the assumption that the spectrum of $K(p,x,u)$ contains only N harmonics at the frequencies $\{\varphi_0, 1+\varphi_1, \ldots, n+\varphi_n, \ldots, N-1+\varphi_{N-1}\}$.

If we put $\varphi_n = 0$ ($n = 0, 1, \ldots, N-1$), we obtain the fractional transform with the kernel

$$\begin{aligned} K(p,x,u) &= \frac{1}{N} \sum_{l=0}^{N-1} \exp[i\pi(N-1)(p-l)/N] \\ &\qquad \times \frac{\sin[\pi(p-l)]}{\sin[\pi(p-l)/N]}\, K(l,x,u) \quad (44) \end{aligned}$$

proposed by Shih in [125]. In particular, this formula is used as the definition of a kind of fractional FT (for the continuous as well as the discrete case) [125, 126].

With N an odd integer and choosing N nonzero coefficients in the decomposition (36) with indices $j = -(N-1)/2, \ldots, 0, \ldots, (N-1)/2$ [corresponding to the indices $n+mN$ for $m = 0$ and $n = 0, 1, \ldots, (N-1)/2$, and $m = -1$ and $n = (N-1)/2+1, \ldots, N-1$], we obtain the kernel

$$K(p,x,u) = \frac{1}{N} \sum_{l=0}^{N-1} \frac{\sin[\pi(p-l)]}{\sin[\pi(p-l)/N]}\, K(l,x,u). \quad (45)$$

This equation corresponds to the recovering procedure of a band-limited periodic function from its values on equidistant sampling points [127]. In particular, if $K(l,x,u)$ is real for integer $l = 0, 1, \ldots, N-1$, then the kernel of the fractional transform determined by Eq. (45) is real, too. It also means that the Fourier spectrum of $K(p,x,u)$ with respect to the parameter p is symmetric: $|k_j| = |k_{-j}|$.

As an example, let us consider the general expression (43) for the kernel of the fractional $\mathcal{R}$-transform with period 4 (which is the case for the Fourier and Hilbert transforms):

$$K(p,x,u) = \frac{1}{4} \sum_{l=0}^{3} K(l,x,u)\, S(l) \quad (46)$$

with $S(l) = \sum_{n=0}^{3} \exp(-inl\pi/2)\, \exp[i(n+\varphi_n)p\pi/2]$.

Note that for the Hilbert transform, the number of harmonics reduces to two, because $C_0(x,u) = C_2(x,u) = 0$, which follows from $K(0,x,u) = -K(2,x,u)$ and $K(1,x,u) = -K(3,x,u)$, From Eq. (43) we then conclude that the fractional Hilbert transform kernel can be written as

$$\begin{aligned} K(p,x,u) &= \exp[i(m_1+m_3+1)p\pi] \\ &\quad \times \{K(0,x,u)\, \cos[(m_3-m_1+1/2)p\pi] \\ &\quad -K(1,x,u)\, \sin[(m_3-m_1+1/2)p\pi]\}, \quad (47) \end{aligned}$$

where m_1 and m_3 are integers. In particular, for the case $m_1 = m_3 = 0$ ($k_n = 0$ if $n \neq 1, 3$), one gets

$$\begin{aligned} &K(p,x,u) = \exp(ip\pi) \\ &\times [K(0,x,u)\, \cos(p\pi/2) - K(1,x,u)\, \sin(p\pi/2)], \quad (48) \end{aligned}$$

while for the case $m_1 = 0$ and $m_3 = -1$ ($k_n = 0$ if $n \neq -1, 1$), the common form for the fractional Hilbert transform [94] with a real kernel is obtained:

$$K(p,x,u) = K(0,x,u)\ \cos(p\pi/2) + K(1,x,u)\ \sin(p\pi/2). \quad (49)$$

Therefore, even for the same number of harmonics, there are several ways for the fractionalization of cyclic transforms.

11 Fractional transform kernels construction using eigenfunctions of cyclic transforms

In the case there exist the set of orthonormal eigenfunctions of the cyclic transform one can construct fractional kernels with a number of harmonics $M > N$, where N is the period of the cyclic transform [27, 28].

Suppose that there is a complete set of orthonormal eigenfunctions $\{\Phi_n\}$ of the operator $\mathcal{R}$ with eigenvalues $\{A_n = \exp(i2\pi L_n/N)\}$, $n = 0, 1, \ldots$ (see Section 9). Then we can represent a kernel of the $\mathcal{R}$-transform of the integer power q as

$$K(q,x,u) = \sum_{n=0}^{\infty} \Phi_n(x)\, A_n^q\, \Phi_n^*(u) = \sum_{n=0}^{\infty} \Phi_n(x)\ \exp(i2\pi q L_n/N)\ \Phi_n^*(u). \quad (50)$$

One of the possible series of kernels for the fractional $\mathcal{R}$-transform can then be written in the form

$$K(p,x,u) = \sum_{n=0}^{\infty} \Phi_n(x)\ \exp[i2\pi(L_n/N + l_n)p]\ \Phi_n^*(u), \quad (51)$$

where l_n is an integer and indicates the location of the harmonics. This kernel satisfies the additivity condition due to the orthonormality of the eigenfunctions $\Phi_n(x)$.

Note that not all cyclic operators have a complete set of orthonormal eigenfunctions, as it is the case, for example, for the Hilbert operator, whose eigenfunctions $\Phi(x)$ are self-orthogonal. Nevertheless, the majority of cyclic transforms of interest in optics, such as Fourier, Hartley, Hankel, etc., have this set. For the Fourier and Hartley transforms, $\Phi_n(x)$ are the Hermite-Gauss modes [15, 16]

$$\Phi_n(x) = 2^{1/4}\left(2^n n!\right)^{-1/2} H_n(x\sqrt{2\pi})\ \exp(-\pi x^2), \quad (52)$$

where $H_n(x)$ are the Hermite polynomials; for the Hankel transform of different orders, $\Phi_n(x)$ are the normalized Laguerre-Gauss functions [128, 129].

The canonical fractional FT kernel, discussed in the previous sections, can be obtained from Eq. (51) as a particular case: $L_n = -n$ and $l_n = 0$,

$$K_F(p,x,u) = \sum_{n=0}^{\infty} \Phi_n(x)\ \exp(-inp\pi/2)\ \Phi_n^*(u) = \frac{\exp(inp\pi/4)}{\sqrt{i\sin(p\pi/2)}}\ \exp\left[i\pi\ \frac{(x^2+u^2)\cos(p\pi/2) - 2ux}{\sin(p\pi/2)}\right], \quad (53)$$

cf. Eq. (11). The fractional Hankel transform, defined by Eq. (51) for $L_n = -n$ and $l_n = 0$ and $\Phi_n(x)$ being the normalized Laguerre-Gauss functions, describes the propagation of rotationally symmetric optical beams through a medium with a quadratic refractive index [128, 129]. The kernels of these transforms contain an infinite number of harmonics.

Let us rewrite Eq. (51) in the form

$$K(p,x,u) = \sum_{n=-\infty}^{\infty} z_n(x,u)\ \exp(i2\pi np/N). \quad (54)$$

Here $z_n(x,u)$ is a sum of the elements $\Phi_j(x)\,\Phi_j^*(u)$ over j, where $\Phi_j(x)$ is the eigenfunction of the $\mathcal{R}$-transform with eigenvalue $\exp(i2\pi n/N)$. Thus for the case of the canonical fractional FT,

$$K_F(p,x,u) = \sum_{n=0}^{\infty} \Phi_n(x)\ \exp(-inp\pi/2)\ \Phi_n^*(u) = \sum_{n=-\infty}^{0} z_n(x,u)\ \exp(inp\pi/2), \quad (55)$$

the coefficients $z_n(x,u)$ vanish for positive n and $z_n(x,u) = \Phi_n(x)\,\Phi_n^*(u)$ for $n \leq 0$. As we will see below, the fractional Hartley transform [27] can be represented in the form

$$\begin{cases} K(p,x,u) &= \displaystyle\sum_{n=0}^{\infty} \exp(-i\pi np) z_{-n}(x,u) \\ z_{-n}(x,u) &= \Phi_{2n}(x)\,\Phi_{2n}(u) + \Phi_{2n+1}(x)\,\Phi_{2n+1}(u). \end{cases} \quad (56)$$

It is easy to see from Eq. (54) that we can generate another kernel series with M harmonics,

$$K(p,x,u) = \sum_{n=0}^{M-1} \exp(i2\pi np/M) \sum_{m=-\infty}^{\infty} z_{n+mM}(x,u), \quad (57)$$

which satisfy the requirements for the fractional transforms. Here the sums of the elements $z_j(x,u)$

$$k_n(M,x,u) = \sum_{m=-\infty}^{\infty} z_{n+mM}(x,u) \quad (58)$$

are used as the coefficients $k_n(x,u)$ in Eq. (36). Note that the relationship (38) holds for the coefficients $k_n(M,x,u)$ and

$k_m(M, x, u)$, because they are constructed from the disjoint series of orthonormal elements.

One can prove that the kernel (57) for $p = 1$ reduces to (50). In particular, if $\{\Phi_n\}$ is the Hermite-Gauss mode set and $z_{-n}(x, u) = \Phi_n(x)\Phi_n^*(u)$ for $n = 0, 1, \ldots$ and $z_{-n}(x, u) = 0$ for negative n, then Eq. (57) corresponds to the series of the M-harmonic fractional FTs proposed in [130],

$$\begin{aligned} K(p, x, u) &= \sum_{n=0}^{M-1} \exp[-i2\pi np(1-M)/M] \\ &\quad \times \sum_{m=0}^{\infty} \Phi_{n+mM}(x)\, \Phi_{n+mM}^*(u) \\ &= \frac{1}{M} \sum_{n=0}^{M-1} \exp[i\pi(M-1)(pl-n)/M] \\ &\quad \times \frac{\sin[\pi(pl-n)]}{\sin[\pi(pl-n)/M]} K_F(n/l, x, u), \end{aligned} \quad (59)$$

where $K_F(n/l, x, u)$ is the kernel of the canonical fractional FT. Application of such types of fractional FTs for image encryption was reported in [80]. If $M = N$ ($l = 1$), we obtain that the kernel of the Shih fractional transform defined by Eq. (44) can also be represented as

$$\begin{aligned} K(p, x, u) &= \sum_{n=0}^{N-1} \exp[-i2\pi np(1-N)/N] \\ &\quad \times \sum_{m=0}^{\infty} \Phi_{n+mN}(x)\, \Phi_{n+mN}^*(u). \end{aligned} \quad (60)$$

Finally we can conclude that if a complete orthonormal set of eigenfunctions for a given cyclic transform exists, then an infinite number of fractional transform kernels with an arbitrary number of harmonics can be constructed using the procedure (51). Some examples of fractional FTs whose kernels contain different numbers of harmonics were considered in [27].

12 Some properties of fractional cyclic transforms

Although there is a variety of schemes for the construction of fractional transforms, all of them have some common properties.

If the coefficients $k_n(x, u)$ in the decomposition (36) are real, then the following relationship holds:

$$\left\{\mathcal{R}^p\left[f^*(x)\right](u)\right\}^* = \mathcal{R}^{-p}\left[f(x)\right](u). \quad (61)$$

This is the case for the canonical fractional FT, the related fractional sine, cosine, and Hartley transforms, and the canonical fractional Hankel transform.

Eigenfunctions of fractional transforms

By analogy with the analysis of the fractional FT eigenfunctions, made in [106, 107], the eigenfunction $\Psi_{1/M}(x)$ for the fractional transform $\mathcal{R}^p$ for $p = 1/M$ with eigenvalue $A = \exp(i2\pi L/M)$, can be constructed from the arbitrary generator function $g(u)$ by the following procedure:

$$\Psi_{1/M}(x) = \frac{1}{M} \sum_{n=0}^{M-1} \exp(-i2\pi nL/M)\, \mathcal{R}^{n/M}\left[g(u)\right](x). \quad (62)$$

In the limiting case $M \to \infty$, one gets the eigenfunction for any value p with eigenvalue $\exp(i2\pi pL)$:

$$\Psi_p^L(x) = \frac{1}{N} \int_0^N \exp(-i2\pi pL)\, \mathcal{R}^p\left[g(u)\right](x)\, dp. \quad (63)$$

In particular for fractional transforms generated by Eq. (51) (as it was shown by the example for the fractional FT [107]), the functions $\Psi_p^L(x)$ correspond to the elements of the orthogonal set $\{a_L \Phi_L\}$, where the constant factors depend on the generator function.

Complex and real fractional transform kernels

We have seen in the previous section that if there exists a complete orthonormal set of eigenfunctions $\{\Phi_n\}$ for the $\mathcal{R}$-transform, then any coefficient in the harmonic decomposition of the fractional kernel $k_n(x, u)$ (36) can be expressed as a linear composition of the elements $\Phi_j(x)\, \Phi_j^*(u)$. For the kernel of the fractional transform to be real, the Fourier spectrum of the fractional kernel with respect to the parameter p has to be symmetric; this means that $|k_{-n}(x, u)| = |k_n(x, u)|$. Since the coefficients $k_n(x, u)$ with different indices n contain disjoint series of the orthogonal elements, their amplitudes cannot be equal. In the case that there exists a complete orthonormal set of eigenfunctions $\{\Phi_n\}$ for the $\mathcal{R}$-transform, the fractional kernel of the $\mathcal{R}^p$-transform cannot be real, even if the $\mathcal{R}$-transform kernel is real.

As we have seen above the fractional Hilbert kernel can be real, because there is no complete orthonormal set of eigenfunctions for the Hilbert transform.

13 Fractional cyclic transforms implemented in optics

Besides the canonical fractional FT discussed in Sections 4 and 11, other fractional cyclic transforms can be performed by optical setups. Thus the fractional FTs described in Sections 10 and 11 and represented as a sum of the weighted canonical fractional FTs for the corresponding parameters $\{\alpha_n\}$ [see for instance Eqs. (44) and (59)] can be obtained as an interference of optical beams at the output of the related canonical fractional FT optical systems. In general the most fractional cyclic transforms proposed for optical implementation are closely connected to the canonical fractional FT.

The two-dimensional fractional FT of a rotationally symmetric function leads to the fractional Hankel transform, analogous to the fact that its two-dimensional FT produces the Hankel transform [128,129]. The fractional Hankel transform of a function $f(r)$ is defined as

$$\mathcal{R}^{\alpha}\left[f(r)\right](u) = H_{\alpha}(u) = \int_0^{\infty} K(\alpha, r, u)\; f(r)\, r\, dr, \quad (64)$$

where the kernel $K(\alpha, r, u)$ is given by

$$K(\alpha, r, u) = \frac{\exp(i\alpha)}{i\sin\alpha}\; \exp\left[i\pi(r^2+u^2)\cot\alpha\right] \times J_0(2\pi r u/\sin\alpha) \quad (65)$$

with J_0 the first-type, zero-order Bessel function. One can represent the fractional Hankel kernel in the form (51), where $L_n = -n$, $l_n = 0$, and $\Phi_n(x)$ are the Laguerre-Gauss functions, which are the eigenfunctions of the fractional Hankel transform.

The fractional Hankel transform inherits the main properties of the fractional FT [103,128] and can be performed by the fractional FT setups described in Sections 3 and 4, if the input optical field is rotationally symmetric. The fractional Hankel transform can substitute the fractional FT in many optical signal processing tasks where rotationally symmetric beams are used.

Since the FT is closely related to sine, cosine, and Hartley transforms, which are cyclic ones with period $N = 2$, several attempts to introduce the fractional sine, cosine, and Hartley transforms were made in [25, 26], where the authors supposed that the kernels of these transforms are the real part of the kernel of the optical fractional FT, the imaginary part of this kernel, or the sum of these two parts, respectively. Nevertheless, they have mentioned that the transforms defined in such a manner, are not angle additive, and therefore, in our view, cannot be interpreted as fractional transforms. The kernels K_S, K_C, and K_H of the fractional sine, cosine, and Hartley transforms (ST, CT, HT) [27, 28, 131], respectively, which are closely related to the canonical fractional FT with kernel K_F and which are indeed angle additive, are defined as

$$\begin{aligned} ie^{-i\alpha}\, K_S(\alpha, x, u) &= 2\, k_{\alpha}(x, u)\; \sin(2\pi u x/\sin\alpha), \\ K_C(\alpha, x, u) &= 2\, k_{\alpha}(x, u)\; \cos(2\pi u x/\sin\alpha), \\ K_H(\alpha, x, u) &= k_{\alpha}(x, u)\, \mathrm{cas}\,(2\pi u x/\sin\alpha), \\ K_F(\alpha, x, u) &= k_{\alpha}(x, u)\; \exp(i2\pi u x/\sin\alpha), \end{aligned} \quad (66)$$

where

$$k_{\alpha}(x, u) = \frac{\exp(i\alpha/2)}{\sqrt{i\sin\alpha}}\; \exp\left[i\pi(x^2+u^2)\cot\alpha\right], \quad (67)$$

where, on the analogy of $\exp(i\varphi) = \cos\varphi + i\sin\varphi$, we have introduced $\mathrm{cas}\,\varphi = \cos\varphi + \sin\varphi$, and where, for easy reference, we have repeated the expression of the canonical fractional FT kernel K_F.

Since the fractional ST, CT, and HT can easily be expressed in terms of the fractional FT, and since optical realizations of the fractional FT [25] are well known, optical realizations of the fractional ST, CT, and HT can easily be constructed. One of the possible schemes for the fractional HT, based on [27]

$$\mathcal{R}_H^{\alpha} = \exp(i\alpha/2)\, \mathcal{R}_F^{\alpha}\left[\cos(\alpha/2) - i\,\sin(\alpha/2)\mathcal{R}_F^{\pi}\right], \quad (68)$$

is given in Fig. 4.

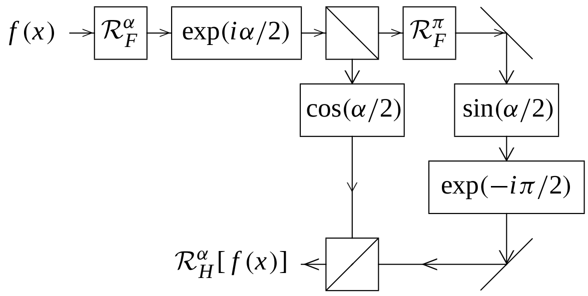

Figure 4: Schematic representation of a fractional Hartley transformer. The setup consists of two fractional FTs $\mathcal{R}_F^{\alpha}$ and $\mathcal{R}_F^{\pi}$, two beam splitters, two mirrors, two absorbing plates $\cos(\alpha/2)$ and $\sin(\alpha/2)$, and two phase plates $\exp(i\alpha/2)$ and $\exp(-i\pi/2)$.

As the ST, CT, and HT are widely used in signal processing, the application of their fractional versions in signal/image processing is very promising.

Since, as we have seen in Section 10, the kernel of the fractional Hilbert transform has only two harmonics, the number of possible fractionalization procedures is significantly reduced. The real kernel of the fractional Hilbert transform introduced in [94,95] and described by Eq. (49) is commonly used. Optical setups performing this transform were proposed in [94, 95]. As the fractional Hilbert transform is a weighted mixture of the optical field $f(u)$ itself and its Hilbert transform $H(u)$,

$$\mathcal{R}^{\alpha}\left[f(x)\right](u) = f(u)\; \cos\alpha + H(u)\; \sin\alpha, \quad (69)$$

an optical scheme performing the ordinary Hilbert transform [see Fig. 5, with $G(\nu) = i\,\mathrm{sgn}(\nu)$] can easily be adapted to perform a fractional Hilbert transform, by having the filter function $G(\nu)$ now taking the more general form $\exp[i\alpha\,\mathrm{sgn}(\nu)] = \cos\alpha + i\,\mathrm{sgn}(\nu)\sin\alpha$.

The Hilbert transform can be considered as a convolution of a function with a step function, which is a model for a perfect edge. Therefore the Hilbert transform produces edge enhancement. It was shown that the fractional Hilbert transform stresses the right-hand and the left-hand slopes unequally [94–96] and that variation of the fractional order changes the nature of the edge enhancement. Thus, for $\alpha \approx \pi/4, \pi/2, 3\pi/4$, the right-hand edges, both edges, and the left-hand edges of the input object are emphasized, respectively. In general we can conclude that the fractional Hilbert

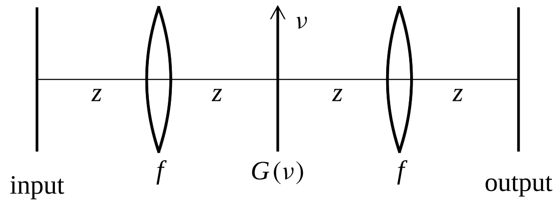

Figure 5: Schematic representation of a (fractional) Hilbert transformer: $z = f$, $G(\nu) = \exp[i\alpha\,\mathrm{sgn}(\nu)]$.

transform produces an output image that is selectively edge enhanced. This property of the fractional Hilbert transform makes it a perspective tool for image processing and pattern recognition.

14 New horizons of fractional optics

Fractional optics is a rapidly developing research area. Novel applications of the fractional transforms for motion detection and analysis, holographic data storage, optical neural networking, and optical security (see Section 8) have been proposed recently. In this section we give a short overview of the main directions of development of fractional optics.

Fractional Fourier transformers

Significant work has been done to improve fractional transformers.

The effect of the spherical aberration of a lens on the performance of the fractional Fourier transformation in the optical systems proposed by Lohmann in [22], was analyzed in [132]. It was shown that the effect of spherical aberration on the output intensity distribution of the fractional FT system depends on the sign and the absolute value of the aberration coefficient. Moreover, Lohmann's two types of optical setups for implementing the fractional FT, are no longer equivalent if the lenses suffer from spherical aberration.

In the optical systems proposed in [22], the fractional order is fixed by the ratio between the focal length of the lens and the distance of free space preceding and following the lens. This fact introduces a difficulty in the design of fractional Fourier transformers with a variable order. Fractional FT systems with a fixed optical setup but with different fractional orders, can be obtained by the implementation of programmable lenses, written onto a liquid-crystal spatial light modulator [133].

A one-dimensional, variable fractional Fourier transformer, based on the application of a reconfigurable electro-optical waveguide, was proposed in [134]. In general, this device produces a variable canonical transformation, with a ray-transformation matrix for which $a = d$ and for which the matrix entry b is controlled by the amplitude of an electric field.

A quantum circuit for the calculation of a fractional FT whose kernel contains four harmonics, was proposed in [135].

Propagation through a fractional FT system

The evolution during propagation through fractional FT systems of different types of beams frequently used in modern optics, such as flattened Gaussian [136, 137], elliptical Gaussian [138], and partially coherent and partially polarized Gaussian-Shell beams [139], has been studied. In particular, it was shown that the intensity distribution and polarization properties in the fractional FT plane are closely related to the fractional order of the fractional FT system and the initial coherence of the partially coherent beam [139–141].

Several devices for manipulation of optical beams based on the fractional FT have been proposed recently.

The fractional FT is applied in the $\pi/2$ converter, which is used to obtain focused Laguerre-Gaussian beams from Hermite-Gaussian radiation modes [142].

The design of a diffractive optical element for beam smoothing in the fractional Fourier domain was described in [143].

An iterative method for the reconstruction of a wave field or a beam profile from measurements obtained using low-resolution amplitude and phase sensors in several fractional Fourier domains, was proposed in [144].

Motion analysis

Several applications of the fractional FT for motion analysis have been proposed.

A method for the independent estimation of both surface tilting and translational motion using the speckle photographic technique by capturing consecutive images in two different fractional Fourier domains, has been proposed in [145].

In [146] the fractional FT is applied to airborne, synthetic aperture radar, slow-moving target detection. Since the echo from a ground moving target can be approximated as a chirp signal, the fractional FT is used to concentrate its energy. An iterative detection of strong moving targets and weak ones, based on filtering in the fractional Fourier domain, has been proposed.

The application of fractional FT correlators to control movements in a specific range, has been considered in [147]. Based on the controllable shift variance of fractional correlations, only the movements limited to a specific range are determined. Fractional FT correlators operating with a log-polar representation of two dimensional images (fractional Mellin-based correlator) allow to control the similarity of objects under rotation and scale transformations. Optically implemented fractional FT and Mellin correlators, providing correlation images directly at image acquisition time, have been proposed to be used in detecting or controlling a specific range of movements in navigation tasks.

Beamforming is another application of the fractional FT indirectly related to motion analysis. Beamforming is widely used in sensor arrays, signal processing for signal enhancement, direction of arrival and velocity estimation, etc. The conventional minimum-mean-squared error beamforming in the frequency domain or the spatial domain has been generalized to the fractional Fourier domain case [148]. It is especially useful for radar problems where chirp signals are encountered. Note that acceleration of the sinusoidal signal source yields that, due to the Doppler effect, a chirp signal arrives at the sensor. Such a chirp signal is often transmitted in active radar systems.

Neural networks implemented fractional FT

Several neural network schemes have been proposed recently, in which the canonical fractional FT was implemented.

An optical neural network based on the fractional correlation realized by a Van der Lugt correlator that employs fractional FTs, was proposed in [88]. The error back-propagation algorithm was used to provide the learning rule by which the filter values are changed iteratively to minimize the error function.

The replacement of the mean square error with the log-likelihood and the introduction of parallelism to this network significantly improve its learning convergence and recall rate [89]

It was demonstrated in [90] that, due to the shift variance of the fractional convolution, the fractional Van der Lugt correlator is more suitable than the conventional one for classification tasks. For a phase modulation filter, the optimal learning rate to improve the learning convergence and the classification performance, can quickly be found by Newton's method.

Besides these static networks with fixed weights and the learning based on the adjustment of the filter coefficients, another type of neural networks to implement the fractional FT has been proposed [91]. In this scheme the fractional FT is used for pre-processing of input signals to neural networks. Adjusting the fractional order of the fractional FT of the input signal leads to an overall improvement of the neural network performance, as has been demonstrated on the example of recognition and position estimation of different objects from their sonar returns. In [92] a comparative analysis has been made of different approaches of target differentiation and localization, including the target differentiation algorithm, Dempster-Shafer evidential reasoning, different kinds of voting schemes, statistical pattern recognition techniques (the k-nearest neighbor classifier, the kernel estimator, the parameterized density estimator, linear discriminant analysis, and the fuzzy c-means clustering algorithm), as well as artificial neural networks, trained with different input signal representations obtained using pre-processing techniques such as discrete ordinary and fractional Fourier, Hartley and wavelet transforms, and Kohonen's self-organizing feature map. It has been shown that the use of neural networks trained by the back-propagation algorithm with fractional FT pre-processing, results in near-perfect differentiation, around 85% correct range estimation, and around 95% correct azimuth estimation.

The potential application of a spatially varying, fractional correlation in implementing parallel fuzzy association, has been explored in [93].

Fresnel and fractional FT holograms

Holographic recording/reconstruction techniques are very well established for image, Fourier, and Fresnel holograms. In particular, since the fractional FT and the Fresnel transform belong to the class of canonical integral transforms, see Eq. (3), one can analyze the feasibility of fractional Fourier holograms in relation to Fresnel holograms properties. The fundamentals of Fresnel holograms have been known for about four decades. In 1965, Armstrong [149] published a general contribution on the basic formulation for describing the recording and reconstruction of two and three-dimensional Fresnel holograms using paraxial approximation. The total complex amplitude at recording step was formulated as a Fresnel integral. Assuming linearity conditions on the hologram development, the final field for the reconstructed image was formulated. This general analytical expression contains the basic contributions to the final field, namely: illumination conditions of the object, hologram diffraction properties, and illumination conditions of the hologram. To this respect, one of the major keys for Fresnel holograms with high reconstruction fidelity is resolution, which is determined by the hologram aperture. If the effect of a finite aperture on the complex amplitude field is not taken into account the hologram diffraction modulation turns out to be represented by a Dirac-delta function. Assuming that under a fractional FT regime we are dealing with a particular scaling defined by the order of the transformation, one can assert that resolution plays a similar role as in a standard Fresnel hologram. Taking into account the finite aperture implies that an intrinsic dependence of the hologram quality (fidelity) on the aberrated reconstructed wavefront has also to be considered.

Recently, several contributions on these subjects have been published. An algorithm for digital holography based on the so-called Fresnelets, which arise when the Fresnel transform is applied to a wavelet basis, has been developed in [150]. An experimental digital holography setup was shown, as well as results for Fresnelet holograms. An interesting result of this paper is related to the uncertainty relation for the Fresnel transform as a condition for signal localization. The condition suggests that Gaussian and Gabor-like functions, modulated with a Fresnel kernel, optimize the processing and reconstruction of Fresnel holograms. This is related to the illumination conditions of the object and the behavior of the hologram aperture as an apodizing optical pupil.

Other experiments for implementing Fresnel holograms on LCDs have been reported [151]. The hologram is obtained by back-propagating the object function applying an inverse Fresnel transform. Results indicate the presence of noise in the reconstruction due to the limited number of amplitude levels of the signal (8 bits).

Other authors [152] claim the implementation of multiple fractional FT holograms. Nevertheless, the lack of information related to critical parameters as real illumination conditions, object size, hologram size, type of holographic material, and minimum distance between objects for avoiding aliasing, makes it difficult to arrive to a precise conclusion on the actual proposed technique.

From the above mentioned results we can assert that Fresnel and fractional FT holograms show great practical interest for signal localization, data coding and decoding, and optical security systems.

Novel fractional transforms

Another direction for further research is the investigation of novel types of fractional transforms: their properties, applications, and possible implementations in optics.

The fractional cosine, sine, and Hartley transforms and their digital implementations were discussed in [27, 28, 131, 153]. It was shown that the fractional cosine and sine transforms are useful for processing one-sided signals, i.e., the independent variable is an element of $[0,\infty)$. From our point of view, the different types of fractional FT, ST, CT, and HT, constructed by the general fractionalization algorithm (see Sections 10 and 12) may be suitable for signal/image encryption and watermarking. Thus an image watermarking scheme based on different types of fractional discrete Fourier, Hartley, cosine, and sine transforms was proposed in [154]. To remove the watermarks in this case, the type of the used fractional transform and the orders of the fractional domains where signatures were introduced, have to be known.

Several other fractional transforms have been introduced recently.

The fractional FT of log-polar representation of a two dimensional image, generates the fractional Mellin transform [122, 147].

Another fractional transform, the complex fractional FT, closely related to the canonical fractional FT has been introduced in [155]. With $\xi = \xi_1 + i\xi_2$ and $\eta = \eta_1 + i\eta_2$, the kernel of the complex fractional FT takes the form, cf. Eq. (11)

$$\frac{\exp(i\alpha)}{i\sin\alpha}\exp\left[i\pi\,\frac{(|\xi|^2+|\eta|^2)\cos\alpha+\xi\eta^*-\xi^*\eta}{2\sin\alpha}\right].$$

Based on the approach of eigenfunction kernel decomposition similar to [27, 28], some new fractional integral transforms, including the fractional Mellin transform, a fractional transform associated with the Jacobi polynomials, a Riemann-Liouville fractional derivative operator, and a fractional Riemann-Liouville integral, have been proposed in [122]. In the analogy with canonical fractional Fourier and Hankel transforms the fractional Laplace and Barut-Girardello transforms have been introduced in [156].

The applications of these transforms in science and engineering is still subject of research.

15 Conclusions

We have reviewed the fractional transformations implemented in paraxial optics and their applications for optical information processing: phase retrieval, signal/image characterization, optical beam manipulation, pattern recognition and classification, adaptive filter design, encryption, watermarking, motion detection, holography etc.. A general algorithm of fractionalization, which allows to construct various fractional transforms related to a given cyclic transform, has been discussed. The usefulness of a specific fractional transform is related to its optical feasibility, as well as to its possible application in signal/image processing. The analysis of the harmonic contents for various types of fractional transforms offers a procedure for their experimental realization. It seems, that the fractional sine, cosine, Hartley, and Hankel transforms, discussed in Sections 13 and 14, due to their similarity to the canonical fractional FT, may act as a substitute for it in many tasks. The usage of the fractional Hilbert transform for selective edge enhancement produces very promising results. The exploration of other recently proposed fractional transforms is expected in near future.

Beside the theoretical and numerical simulation works demonstrating an important impact of the optical implementation of the fractional FT, the experimental realization of the corresponding devices and techniques takes up a significant place of research.

We believe that fractional optics significantly increases the importance of analog optical information processing. The design of new devices based on fractional optics, will lead to unified approaches of signal/image processing used in optics and electrical engineering, which will significantly enrich the fields of optoelectronics, optical security technology, and optical computing.

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References

[1] D. L. Flannery and J. L. Horner, "Fourier optical signal processors," *Proc. IEEE*, vol. 77, no. 10, pp. 1511–1527, 1989.

[2] B. Javidi and J. L. Horner, eds., *Real-Time Optical Information Processing*, Academic Press, New York, 1994.

[3] H. J. Caulfield, "Perspectives in optical computing," *Computer*, vol. 31, no. 2, pp. 22–25, 1998.

[4] A. V. Pavlov, "Mathematical models of optical methods in data processing," *J. Comput. Syst. Sci. Int.*, vol. 39, no. 3, pp. 441–447, 2000.

[5] A. Van der Lugt, ed., *Optical Signal Processing, Wiley series in Pure and Applied Optics*, John Wiley, New York, 1992.

[6] G. O. Reynolds, J. B. DeVelis, G. B. Parrent, and B. J. Thompson, *Physical Optical Notebook: Tutorials in Fourier Optics*, SPIE Optical Engineering Press, New York, 1989.

[7] Y. Li, H. H. Szu, Y. L. Sheng, and H. J. Caulfield, "Wavelet processing and optics," *Proc. IEEE*, vol. 84, no. 5, pp. 720–732, 1996.

[8] A. Walther, "Radiometry and coherence," *J. Opt. Soc. Am.*, vol. 58, no. 9, pp. 1256–1259, 1968.

[9] A. Walther, "Propagation of the generalized radiance through lenses," *J. Opt. Soc. Am.*, vol. 68, no. 11, pp. 1606–1610, 1978.

[10] A. Papoulis, "Ambiguity function in Fourier optics," *J. Opt. Soc. Am.*, vol. 64, no. 6, pp. 779–788, 1974.

[11] M. J. Bastiaans, "The Wigner distribution function applied to optical signals and systems," *Opt. Commun.*, vol. 25, no. 1, pp. 26–30, 1978.

[12] M. J. Bastiaans, "Wigner distribution function and its application to first-order optics," *J. Opt. Soc. Am.*, vol. 69, no. 12, pp. 1710–1716, 1979.

[13] M. J. Bastiaans, "Application of the Wigner distribution function to partially coherent light," *J. Opt. Soc. Am. A*, vol. 3, no. 8, pp. 1227–1238, 1986.

[14] W. Mecklenbräuker and F. Hlawatsch, eds., *The Wigner Distribution – Theory and Applications in Signal Processing*, Elsevier, Amsterdam, 1997.

[15] D. Mendlovic and H. M. Ozaktas, "Fractional Fourier transforms and their optical implementation: I," *J. Opt. Soc. Am. A*, vol. 10, no. 9, pp. 1875–1881, 1993.

[16] H. M. Ozaktas and D. Mendlovic, "Fractional Fourier transforms and their optical implementation. II," *J. Opt. Soc. Am. A*, vol. 10, no. 12, pp. 2522–2531, 1993.

[17] H. M. Ozaktas and D. Mendlovic, "Fourier transforms of fractional order and their optical interpretation," *Opt. Commun.*, vol. 101, no. 3–4, pp. 163–169, 1993.

[18] T. Alieva, V. Lopez, F. Agullo-Lopez, and L. B. Almeida, "The fractional Fourier transform in optical propagation problems," *J. Mod. Opt.*, vol.41, no. 5, pp. 1037–1044, 1994.

[19] H. Kober, "Wurzeln aus der Hankel-, Fourier- und aus anderen stetigen Transformationen," *Quart. J. Math. Oxford. Ser.*, vol. 10, pp. 45–59, 1939.

[20] V. Namias, "The fractional order Fourier transform and its applications to quantum mechanics," *J. Inst. Math. Appl.*, vol. 25, pp. 241–265, 1980.

[21] A. C. McBride and F. H. Kerr, "On Namias' fractional Fourier transforms," *IMA J. Appl. Math.*, vol. 39, no. 2, pp. 159–175, 1987.

[22] A. W. Lohmann, "Image rotation, Wigner rotation, and the fractional Fourier transform," *J. Opt. Soc. Am. A*, vol. 10, no. 10, pp. 2181–2186, 1993.

[23] L. B. Almeida, "The fractional Fourier transform and time-frequency representations," *IEEE Trans. Signal Process.*, vol. 42, no. 11, pp. 3084–3091, 1994.

[24] H. M. Ozaktas, M. A. Kutay, and D. Mendlovic, "Introduction to the fractional Fourier transform and its applications," in *Advances in Imaging and Electron Physics*, vol. 106, ed. P. W. Hawkes, Academic Press, London, pp. 239–291, 1999.

[25] A. W. Lohmann, D. Mendlovic, and Z. Zalevsky, "Fractional transformation in optics," in *Progress in Optics*, vol. XXXVIII, ed. E. Wolf, North-Holland, Amsterdam, pp. 263–342, 1998.

[26] A. W. Lohmann, D. Mendlovic, Z. Zalevsky, and R. G. Dorch, "Some important fractional transformations for signal processing," *Opt. Commun.*, vol. 125, no. 1–3, pp. 18–20, 1996.

[27] T. Alieva and M. L. Calvo, "Fractionalization of the linear cyclic transforms," *J. Opt. Soc. Am. A*, vol. 17, no. 12, pp. 2330–2338, 2000.

[28] T. Alieva, M. J. Bastiaans, and M. L. Calvo, "Fractional cyclic transforms in optics: theory and applications," *Recent Research and Developments in Optics*, vol. 1, ed. S. G. Pandalai, Research Signpost, Trivandrum, India, pp. 105–122, 2001.

[29] H. M. Ozaktas, Z. Zalevsky, and M. A. Kutay, *The Fractional Fourier Transform with Applications in Optics and Signal Processing*, Wiley, New York, 2001.

[30] A. Torre, "The fractional Fourier transform and some of its applications to optics," in *Progress in Optics*, vol. 43, ed. E. Wolf, North-Holland, Amsterdam, pp. 531–596, 2002.

[31] T. Alieva and M. J. Bastiaans, "Wigner distribution and fractional Fourier transform," in *Time-Frequency Signal Analysis and Processing: A Comprehensive Reference*, ed. B. Boashash, Elsevier, Oxford, UK, pp. 145–152, 2003.

[32] M. G. Raymer, M. Beck, and D. F. McAlister, "Complex wave-field reconstruction using phase-space tomography," *Phys. Rev. Lett.*, vol. 72, no. 8, pp. 1137–1140, 1994.

[33] T. Alieva and M. J. Bastiaans, "On fractional Fourier transform moments," *IEEE Signal Process. Lett.*, vol. 7, no. 11, pp. 320–323, 2000.

[34] T. Alieva and M. J. Bastiaans, "Phase-space distributions in quasi-polar coordinates and the fractional Fourier transform," *J. Opt. Soc. Am. A*, vol. 17, no. 12, pp. 2324–2329, 2000.

[35] T. Alieva, LJ. Stanković, and M. J. Bastiaans, "Signal reconstruction from two close fractional Fourier power spectra," *IEEE Trans. Signal Process.*, vol. 51, no. 1, pp. 112–123, 2003.

[36] M. J. Bastiaans and K. B. Wolf, "Phase reconstruction from intensity measurements in linear systems," *J. Opt. Soc. Am. A*, vol. 20, no. 6, pp. 1046–1049, 2003.

[37] R. M. Teague, "Deterministic phase retrieval: a Green function solution," *J. Opt. Soc. Am.*, vol. 73, no. 11, pp. 1434–1441, 1983.

[38] Z. Zalevsky, D. Mendlovic, and R. G. Dorsch, "Gerchberg-Saxton algorithm applied in the fractional Fourier or the Fresnel domain," *Opt. Lett.*, vol. 21, no. 12, pp. 842–844,

1996.
[39] B. Z. Dong, Y. Zhang, B. Y. Gu, and G. Z. Yang, "Numerical investigation of phase retrieval in a fractional Fourier transform," *J. Opt. Soc. Am. A*, vol. 14, no. 10, pp. 2709–2714, 1997.
[40] W. X. Cong, N. X. Chen, and B. Y. Gu, "Recursive algorithm for phase retrieval in the fractional Fourier transform domain," *Appl. Opt.*, vol. 37, no. 29, pp. 6906–6910, 1998.
[41] T. Alieva and M. J. Bastiaans, "Finite-mode analysis by means of intensity information in fractional optical systems," *J. Opt. Soc. Am. A*, vol. 19, no. 3, pp. 481–484, 2002.
[42] T. Alieva, M. L. Calvo, and M. J. Bastiaans, "Power filtering of *n*th order in the fractional Fourier domain," *J. Phys. A, Math. Gen.*, vol. 35, no. 36, pp. 7779–7785, 2002.
[43] M. J. Bastiaans and T. Alieva, "Wigner distribution moments in fractional Fourier transform systems," *J. Opt. Soc. Am. A*, vol. 19, no. 9, pp. 1763–1773, 2002.
[44] M. J. Bastiaans and T. Alieva, "Wigner distribution moments measured as intensity moments in separable first-order optical systems," submitted to *EURASIP J. Appl. Signal Process.*, 2004.
[45] LJ. Stanković, T. Alieva, and M. J. Bastiaans, "Time-frequency signal analysis based on the windowed fractional Fourier transform," *Signal Process.*, vol. 83, no. 11, pp. 2459–2468, 2003.
[46] M. J. Bastiaans, T. Alieva, and LJ. Stanković, "On rotated time-frequency kernels," *IEEE Signal Process. Lett.*, vol. 9, no. 11, pp. 378–381, 2002.
[47] T. Alieva and M. J. Bastiaans, "Evolution of the vortex and the asymmetrical parts of the orbital angular momentum in separable first-order optical systems," *Opt. Lett.*, vol. 29, no. 14, pp. 1587–1589, 2004.
[48] J. C. Wood and D. T. Barry, "Tomographic time-frequency analysis and its application toward time-varying filtering and adaptive kernel design for multicomponent linear-FM signals," *IEEE Trans. Signal Process.*, vol. 42, no. 8, pp. 2094–2104, 1994.
[49] J. C. Wood and D. T. Barry, "Linear signal synthesis using the Radon-Wigner transform," *IEEE Trans. Signal Process.*, vol. 42, no. 8, pp. 2105–2111, 1994.
[50] T. Alieva, "Fractional Fourier transform as a tool for investigation of fractal objects," *J. Opt. Soc. Am. A*, vol. 13, no. 6, pp. 1189–1192, 1996.
[51] T. Alieva and F. Agullo-Lopez, "Optical wave propagation of fractal fields," *Opt. Commun.*, vol. 125, no. 4–6, pp. 267–274, 1996.
[52] T. Alieva and F. Agullo-Lopez, "Diffraction analysis of random fractal fields," *J. Opt. Soc. Am. A*, vol. 15, no. 3, pp. 669–674, 1998.
[53] T. Alieva and A. Barbé, "Fractional Fourier analysis of objects with scaling symmetry," in *Fractals in Engineering*, eds. J. Levy Vehel, E. Lutton, and C. Tricot, Springer-Verlag, London, pp. 252–265, 1997.
[54] D. Rodriguez Merlo, J. A. Rodrigo Martin Romo, T. Alieva, and M. L. Calvo, "Fresnel diffraction by deterministic fractal gratings: an experimental study," *Opt. Spectrosc.*, vol. 95, no. 1, pp. 131–133, 2003.
[55] J. Uozumi and T. Asakura, "Fractal optics," in *Current Trends in Optics*, ed. J. C. Dainty, Academic Press, London, pp. 83–94, 1994.
[56] T. Alieva and M. L. Calvo, "Paraxial diffraction on structures generated by multiplicative iterative procedures," *J. Opt. A, Pure Appl. Opt.*, vol. 5, no. 5, pp. S324–S328, 2003.
[57] H. M. Ozaktas, B. Barshan, D. Mendlovic, and L. Onural, "Convolution, filtering, and multiplexing in fractional Fourier domains and their relation to chirp and wavelet transforms," *J. Opt. Soc. Am. A*, vol. 11, no. 2, pp. 547–559, 1994.
[58] L. B. Almeida, "Product and convolution theorems for the fractional Fourier transform," *IEEE Signal Process. Lett.*, vol. 4, no. 1, pp. 15–17, 1997.
[59] D. Mendlovic, H. M. Ozaktas, and A. W. Lohmann, "Fractional correlation," *Appl. Opt.*, vol. 34, no. 2, pp. 303–309, 1995.
[60] D. Mendlovic, Y. Bitran, R. G. Dorsch, and A. W. Lohmann, "Optical fractional correlation: experimental results," *J. Opt. Soc. Am. A*, vol. 12, no. 8, pp. 1665–1670, 1995.
[61] Z. Zalevsky and D. Mendlovic, "Fractional Wiener filter," *Appl. Opt.*, vol. 35, no. 20, pp. 3930–3936, 1996.
[62] D. J. Mustard, "Fractional convolution," *J. Australian Math. Soc. B – Appl. Math.*, vol. 40, part. 2, pp. 257–265, 1998.
[63] B. Ruiz and H. Rabal, "Fractional Fourier transform description with use of differential operators," *J. Opt. Soc. Am. A*, vol. 14, no. 11, pp. 2905–2913, 1997.
[64] A. I. Zayed, "A convolution and product theorem for the fractional Fourier transform," *IEEE Signal Process. Lett.*, vol. 5, no. 4, pp. 101–103, 1998.
[65] I. Raveh and D. Mendlovic, "New properties of the Radon transform of the cross Wigner/ambiguity distribution function," *IEEE Trans. Signal Process.*, vol. 47, no. 7, pp. 2077–2080, 1999.
[66] A. W. Lohmann, Z. Zalevsky, and D. Mendlovic, "Synthesis of pattern recognition filters for fractional Fourier processing," *Opt. Commun.*, vol. 128, no. 4–6, pp. 199–204, 1996.
[67] S. Granieri, R. Arizaga, and E. E. Sicre, "Optical correlation based on the fractional Fourier transform," *Appl. Opt.*, vol. 36, no. 26, pp. 6636–6645, 1997.
[68] A. M. Almanasreh and M. A. G. Abushagur, "Fractional correlations based on the modified fractional order Fourier transform," *Opt. Eng.*, vol. 37, no. 1, pp. 175–184, 1998.
[69] T. Alieva and M. L. Calvo, "Generalized fractional convolution," in *Perspectives in Modern Optics and Optical Instrumentation*, eds. J. Joseph, A. Sharma, and V. K. Rastogi, Anita Publications, New Delhi, pp. 163–173, 2002.
[70] O. Akay and G. F. Boudreaux-Bartels, "Fractional convolution and correlation via operator methods and an application to detection of linear FM signals," *IEEE Trans. Signal Process.*, vol. 49, no. 5, pp. 979–993, 2001.
[71] J. A. Davis, D. M. Cottrell, N. Nestorovic, and S. M. Highnote, "Space-variant Fresnel transform optical correlator," *Appl. Opt.*, vol. 31, no. 32, pp. 6889-6893, 1992.
[72] D. Mendlovic, Y. Bitran, R. G. Dorsch, C. Ferreira, J. García, H. M. Ozaktas, "Anamorphic fractional Fourier transform: optical implementation and applications," *Appl. Opt.*, vol. 34, no. 32, pp. 7451–7456, 1995.
[73] J. García, D. Mendlovic, Z. Zalevsky, and A. W. Lohmann, "Space-variant simultaneous detection of several objects by the use of multiple anamorphic fractional-Fourier-transform filters," *Appl. Opt.*, vol. 35, no. 20, pp. 3945–3952, 1996.
[74] J. Garcia, R. G. Dorsch, A. W. Lohmann, C. Ferreira,, and

Z. Zalevsky, “Flexible optical implementation of fractional Fourier transform processors. Applications to correlation and filtering,” *Opt. Commun.*, vol. 133, no. 1–6, pp. 393–400, 1997.
[75] Y. Bitran, Z. Zalevsky, D. Mendlovic, and R. G. Dorsch, “Fractional correlation operation: Performance analysis,” *Appl. Opt.*, vol. 35, no. 2, pp. 297–303, 1996.
[76] T. Alieva and M. L. Calvo, “Importance of the phase and amplitude in the fractional Fourier domain,” *J. Opt. Soc. Am. A*, vol. 20, no. 3, pp. 533–541, 2003.
[77] F. T. S. Yu, C. Zhang, Y. Jin, and D. A. Gregory, “Nonconventional joint-transform correlator,” *Opt. Lett.*, vol. 14, no. 17, pp. 922–924, 1989.
[78] G. Unnikrishnan and K. Singh, “Optical encryption using quadratic phase systems,” *Opt. Commun.*, vol. 193, no. 1–6, pp. 51–67, 2001.
[79] B. H. Zhu and S. T. Liu, “Optical image encryption based on the generalized fractional convolution operation,” *Opt. Commun.*, vol. 195, no. 5–6, pp. 371–381, 2001.
[80] B. H. Zhu, S. T. Liu, and Q. W. Ran, “Optical image encryption based on multifractional Fourier transforms,” *Opt. Lett.*, vol. 25, no. 16, pp. 1159–1161, 2000.
[81] N. K. Nishchal, G. Unnikrishnan, J. Joseph, and K. Singh, “Optical encryption using a localized fractional Fourier transform,” *Opt. Eng.*, vol. 42, no. 12, pp. 3566–3571, 2003.
[82] B. Hennelly and J. T. Sheridan, “Optical image encryption by random shifting in fractional Fourier domains,” *Opt. Lett.*, vol. 28, no. 4, pp. 269–271, 2003.
[83] B. Hennelly and J. T. Sheridan, “Fractional Fourier transform-based image encryption: phase retrieval algorithm,” *Opt. Commun.*, vol. 226, no. 1–6, pp. 61–80, 2003.
[84] N. K. Nishchal, J. Joseph, and K. Singh, “Fully phase encryption using fractional Fourier transform, ” *Opt. Eng.*, vol. 42, no. 6, pp. 1583–1588, 2003.
[85] B. M. Hennelly and J. T. Sheridan, “Image encryption and the fractional Fourier transform,” *Optik*, vol. 114, no. 6, pp. 251–265, 2003.
[86] S. Stanković, I. Djurović, and I. Pitas, “Watermarking in the space/spatial-frequency domain using two-dimensional Radon-Wigner distribution,” *IEEE Trans. Image Process.*, vol. 10, no. 4, pp. 650–658, 2001.
[87] I. Djurović, S. Stanković, and I. Pitas, “Digital watermarking in the fractional Fourier transformation domain,” *J. Netw. Comput. Appl.*, vol. 24, no. 2, pp. 167–173, 2001.
[88] S. Y. Lee and H. H. Szu, “Fractional Fourier transforms, wavelet transforms, and adaptive neural networks,” *Opt. Eng.*, vol. 33, no. 7, pp. 2326–2330, 1994.
[89] S. G. Shin, S. I. Jin, S. Y. Shin, and S. Y. Lee, “Optical neural network using fractional Fourier transform, log-likelihood, and parallelism,” *Opt. Commun.*, vol. 153, no. 4–6, pp. 218–222, 1998.
[90] S. I. Jin, Y. S. Bae, and S. Y. Lee, “Generalized Vander Lugt correlator as an optical pattern classifier and its optimal learning rate,” *Opt. Commun.*, vol. 206, no. 1–3, pp. 19–25, 2002.
[91] B. Barshan and B. Ayrulu, “Fractional Fourier transform pre-processing for neural networks and its application to object recognition,” *Neural Netw.*, vol. 15, no. 1, pp. 131–140, 2002.
[92] B. Barshan and B. Ayrulu, “Comparative analysis of different approaches to target differentiation and localization with sonar,” *Pattern Recognit.*, vol. 36, no. 5, pp. 1213–1231, 2003.
[93] S. Q. Zhang and M. A. Karim, “Fractional correlation filter for fuzzy associative memories,” *Opt. Eng.*, vol. 41, no. 1, pp. 126–129, 2002.
[94] A. W. Lohmann, D. Mendlovic and Z. Zalevsky, “Fractional Hilbert transform,” *Opt. Lett.*, vol. 21, no. 4, pp. 281–283, 1996.
[95] A. W. Lohmann, E. Tepichin, and J. G. Ramirez, “Optical implementation of the fractional Hilbert transform for two-dimensional objects,” *Appl. Opt.*, vol. 36, no. 26, pp. 6620–6626, 1997.
[96] J. A. Davis, D. E. McNamara, and D. M. Cottrell, “Analysis of the fractional Hilbert transform,” *Appl. Opt.*, vol. 37, no. 29, pp. 6911–6913, 1998.
[97] K. B. Wolf, *Integral Transforms in Science and Engineering*, Plenum Press, New York, Chapter 9, 1979.
[98] R. K. Luneburg, *Mathematical Theory of Optics*, University of California Press, Berkeley and Los Angeles, 1996.
[99] D. F. V. James and G. S. Agarwal, “The generalized Fresnel transform and its applications to optics,” *Opt. Commun.*, vol. 126, no. 4–6, pp. 207–212, 1996.
[100] S. Abe and J. T. Sheridan, “Almost-Fourier and almost-Fresnel transformations,” *Opt. Commun.*, vol. 113, no. 4–6, pp. 385–388, 1995.
[101] A. D. Poularikas, ed., *The Transforms and Applications Handbook*, CRC Press, Alabama, 1996: Chapter 8, S. R. Deans, “Radon and Abel transforms,” and Chapter 12, G. F. Boudreaux-Bartels, “Mixed time-frequency signal transformations.”
[102] J. Shamir and N. Cohen, “Root and power transformations in optics,” *J. Opt. Soc. Am. A*, vol. 12, no. 11, pp. 2415–2423, 1995.
[103] C. J. R. Sheppard and K. G. Larkin, “Similarity theorems for fractional Fourier transforms and fractional Hankel transforms,” *Opt. Commun.*, vol. 154, no. 4, pp. 173–178, 1998.
[104] S. C. Pei and J. J. Ding, “Eigenfunctions of linear canonical transform,” *IEEE Trans. Signal Process.*, vol. 50, no. 1, pp. 11–26, 2002.
[105] D. Mendlovic, H. M. Ozaktas, and A. W. Lohmann, “Self Fourier functions and fractional Fourier transforms,” *Opt. Commun.*, vol. 105, no. 1–2, pp. 36–38, 1994.
[106] T. Alieva, “On the self-fractional Fourier functions,” *J. Phys. A, Math. Gen.*, vol. 29, no. 15, pp. L377–L379, 1996.
[107] T. Alieva and A. M. Barbé, “Self-fractional Fourier images,” *J. Mod. Opt.*, vol. 46, no. 1, pp. 83–99, 1999.
[108] T. Alieva and A. M. Barbé, “Self-fractional Fourier functions and selection of modes,” *J. Phys. A, Math. Gen.*, vol. 30, no. 8, pp. L211–L215, 1997.
[109] T. Alieva and M. J. Bastiaans, “Mode analysis in optics through fractional transforms,” *Opt. Lett.*, vol. 24, no. 17, pp. 1206–1208, 1999.
[110] P. Pellat-Finet, “Fresnel diffraction and the fractional-order Fourier transform,” *Opt. Lett.*, vol. 19, no. 18, pp. 1388–1390, 1994.
[111] P. Pellat-Finet and G. Bonnet, “Fractional order Fourier transform and Fourier optics,” *Opt. Commun.*, vol. 111, no. 1–2, pp. 141–154, 1994.
[112] H. M. Ozaktas and D. Mendlovic, “Fractional Fourier trans-

form as a tool for analyzing beam propagation and spherical mirror resonators," *Opt. Lett.*, vol. 19, no. 21, pp. 1678–1680, 1994.

[113] L. M. Bernardo and O. D. D. Soares, "Fractional Fourier transforms and optical systems," *Opt. Commun.*, vol. 110, no. 5–6, pp. 517–522, 1994.

[114] Y. B. Karasik, "Expression of the kernel of a fractional Fourier transform in elementary functions," *Opt. Lett.*, vol. 19, no. 11, pp. 769–770, 1994.

[115] A. Sahin, H. M. Ozaktas, and D. Mendlovic, "Optical implementation of the two-dimensional fractional Fourier transform with different orders in the two dimensions," *Opt. Commun.*, vol. 120, no. 3–4, pp. 134–138, 1995.

[116] A. W. Lohmann and B. H. Soffer, "Relationships between the Radon-Wigner and fractional Fourier transforms," *J. Opt. Soc. Am. A*, vol. 11, no. 6, pp. 1798–1811, 1994.

[117] A. Ya. Bekshaev, M. S. Soskin, and M. V. Vasnetsov, "Optical vortex symmetry breakdown and decomposition of the orbital angular momentum of light beams," *J. Opt. Soc. Am. A*, vol. 20, no. 8, pp. 1635–1643, 2003.

[118] S. I. Jin and S. Y. Lee, "Joint transform correlator with fractional Fourier transform," *Opt. Commun.*, vol. 207, no. 1–6, pp. 161–168, 2002.

[119] G. Gonon, O. Richoux, and C. Depollier, "Acoustic wave propagation in a 1-D lattice: analysis of nonlinear effects by the fractional Fourier transform method," *Signal Process.*, vol. 83, no. 11, pp. 2469–2480, 2003.

[120] J. L. Du, Z. Cui, Y. X. Zhang, C. L. Du, J. Yang, and Y. K. Guo, "Enhancement of photolithography resolution by fractional Fourier domain filtering," *Microelectron. Eng.*, vol. 67–68, pp. 31–38, 2003.

[121] B. Javidi, "Securing information with optical technologies," *Phys. Today*, vol. 50, no. 3, pp. 27–32, 1997.

[122] A. I. Zayed, "A class of fractional integral transforms: A generalization of the fractional Fourier transform," *IEEE Trans. Signal Process.*, vol. 50, no. 3, pp. 619–627, 2002.

[123] D. S. Yeung, Q. W. Ran, E. C. C. Tsang, and K. L. Teo, "Complete way to fractionalize Fourier transform," *Opt. Commun.*, vol. 230, no. 1–3, pp. 55–57, 2004.

[124] T. Alieva and M. J. Bastiaans, "Powers of transfer matrices determined by means of eigenfunctions," *J. Opt. Soc. Am. A*, vol. 16, no. 10, pp. 2413–2418, 1999.

[125] C. C. Shih, "Fractionalization of Fourier transform," *Opt. Commun.*, vol. 118, no. 5–6, pp. 495–498, 1995.

[126] B. W. Dickinson and K. Steiglitz, "Eigenvectors and functions of the discrete Fourier transform," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 30, pp. 25–31, 1982.

[127] D. A. Linden, "A discussion of sampling theorems," *Proc. IRE*, vol. 47, no. 7, pp. 1219–1226, 1959.

[128] L. Yu, Y. Y. Lu, X. M. Zeng, M. C. Huang, M. Z. Chen, W. D. Huang, and Z.Z. Zhu, "Deriving the integral representation of a fractional Hankel transform from a fractional Fourier transform," *Opt. Lett.*, vol. 23, no. 15, pp. 1158–1160, 1998.

[129] L. Yu, W. D. Huang, M. C. Huang, Z. Z. Zhu, X. M. Zeng, and W. Ji, "The Laguerre Gaussian series representation of two-dimensional fractional Fourier transform," *J. Phys. A, Math. Gen.*, vol. 31, no. 46, pp. 9353–9357, 1998.

[130] S. T. Liu, J. X. Jiang, Y. Zhang, and J. D. Zhang, "Generalized fractional Fourier transforms," *J. Phys. A, Math. Gen.*, vol. 30, no. 3, pp. 973–981, 1997.

[131] T. Alieva and M. J. Bastiaans, "Fractional cosine and sine transforms," *Asian J. Phys.*, vol. 12, no. 1, pp. 25–32, 2003.

[132] D. M. Zhao, H. Mao, H. J. Liu, X. F. Wei, F. Jing, and Q. H. Zhu, "Effect of spherically aberrated lens on the fractional Fourier transformation systems," *Opt. Commun.*, vol. 227, no. 4–6, pp. 213–220, 2003.

[133] I. Moreno, J. A. Davis, and K. Crabtree, "Fractional Fourier transform optical system with programmable diffractive lenses," *Appl. Opt.*, vol. 42, no. 32, pp. 6544–6548, 2003.

[134] D. Dragoman and M. Dragoman, "Reconfigurable electro-optical waveguide for optical processing," *Appl. Opt.*, vol. 42, no. 32, pp. 6439–6444, 2003.

[135] A. Klappenecker and M. Rotteler, "Engineering functional quantum algorithms," *Phys. Rev. A*, vol. 67, no. 1, art. no. 010302, 2003.

[136] Y. J. Cai and Q. A. Lin, "Properties of a flattened Gaussian beam in the fractional Fourier transform plane," *J. Opt. A, Pure Appl. Opt.*, vol. 5, no. 3, pp. 272–275, 2003.

[137] D. M. Zhao, H. D. Mao, M. X. Shen, H. J. Liu, F. Jing, Q. H. Zhu, and X. F. Wei, "Propagation of flattened Gaussian beams in apertured fractional Fourier transforming systems," *J. Opt. A, Pure Appl. Opt.*, vol. 6, no. 1, pp. 148–154, 2004.

[138] Y. J. Cai, and Q. Lin, "Fractional Fourier transform for elliptical Gaussian beams," *Opt. Commun.*, vol. 217, no. 1–6, pp. 7–13, 2003.

[139] Y. J. Cai, D. Ge, and Q. Lin, "Fractional Fourier transform for partially coherent and partially polarized Gaussian-Schell model beams," *J. Opt. A, Pure Appl. Opt.*, vol. 5, no. 5, pp. 453–459, 2003.

[140] Y. J. Cai and Q. Lin, "Transformation and spectrum properties of partially coherent beams in the fractional Fourier transform plane," *J. Opt. Soc. Am. A*, vol. 20, no. 8, pp. 1528–1536, 2003.

[141] Q. Lin and Y. J. Cai, "Fractional Fourier transform for partially coherent Gaussian-Schell model beams," *Opt. Lett.*, vol. 27, no. 19, pp. 1672–1674, 2002.

[142] A. A. Malyutin, "Simple scheme for the astigmatic transformation of laser modes," *Quantum Electron.*, vol. 33, no. 11, pp. 1015–1018, 2003.

[143] Q. F. Tan, Y. B. Yan, G. F. Jin, and D. Y. Xu, "True beam smoothing in the fractional Fourier transform domain," *J. Mod. Opt.*, vol. 50, no. 14, pp. 2147–2153, 2003.

[144] A. E. Cetin, H. Ozaktas, and H. M. Ozaktas, "Resolution enhancement of low resolution wavefields with POCS algorithm," *Electron. Lett.*, vol. 39, no. 25, pp. 1808–1810, 2003.

[145] J. T. Sheridan, B. Hennelly, and D. Kelly, "Motion detection, the Wigner distribution function, and the optical fractional Fourier transform," *Opt. Lett.*, vol. 28, no. 11, pp. 884–886, 2003.

[146] H. Sun, G. S. Liu, H. Gu, and W. M. Su, "Application of the fractional Fourier transform to moving target detection in airborne SAR," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 38, no. 4, pp. 1416–1424, 2002.

[147] D. Sazbon, Z. Zalevsky, E. Rivlin, and D. Mendlovic, "Using Fourier/Mellin-based correlators and their fractional versions in navigational tasks," *Pattern Recognit.*, vol. 35, no. 12, pp. 2993–2999, 2002.

[148] I. S. Yetik and A. Nehorai, "Beamforming using the frac-

tional Fourier transform," *IEEE Trans. Signal Process.*, vol. 51, no. 6, pp. 1663–1668, 2003.
[149] J. A. Armstrong, "Fresnel holograms: Their imaging properties and aberrations," *IBM J. Res. Dev.*, vol. 9, no. 3, pp. 171–178, 1965.
[150] M. Liebling, T. Blu, and M. Unser, "Fresnelets: New multiresolution wavelet bases for digital holography," *IEEE Trans. Image Process.*, vol. 12, no. 1, pp. 29–43, 2003.
[151] R. Tudela, E. Martin-Badosa, I. Labastida, S. Vallmitjana, I. Juvells, A. Carnicer, "Full complex Fresnel holograms displayed on liquid crystal devices," *J. Opt. A, Pure Appl. Opt.*, vol. 5, no. 5, pp. S189–S194, 2003.
[152] Y. S. Zeng, Y. K. Guo, F. H. Gao, and J. H. Zhu, "Principle and application of multiple fractional Fourier transform holography," *Opt. Commun.*, vol. 215, no. 1–3, pp. 53–59, 2003.
[153] S. C. Pei and J. J. Ding, "Fractional cosine, sine, and Hartley transforms," *IEEE Trans. Signal Process.*, vol. 50, no. 7, pp. 1661–1680, 2002.
[154] C. C. Tseng, "Eigenvalues and eigenvectors of generalized DFT, generalized DHT, DCT-IV and DST-IV matrices," *IEEE Trans. Signal Process.*, vol. 50, no. 4, pp. 866–877, 2002.
[155] H. Y. Fan and H. L. Lu, "Eigenmodes of fractional Hankel transform derived by the entangled-state method," *Opt. Lett.*, vol. 28, no. 9, pp. 680–682, 2003.
[156] A. Torre, "Linear and radial canonical transforms of fractional order," in *J. Comput. Appl. Math.*, vol. 153, no. 1–2, pp. 477–486, 2003.