Gabor's signal expansion and the Zak transform, with oversampling by an integer factor

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Abstract

Gabor's expansion of a signal into a discrete set of shifted and modulated versions of an elementary signal is reviewed and its relation to sampling of the sliding-window spectrum is shown. It is indicated how Gabor's expansion coefficients can be found as samples of the sliding-window spectrum, where the window function – which still has to be determined – is related to the elementary signal. Gabor's critical sampling as well as the case of oversampling by an integer factor are considered.

The Zak transform is introduced and its intimate relationship to Gabor's signal expansion is demonstrated. It is shown how the Zak transform can be helpful in determining Gabor's expansion coefficients and how it can be used in finding window functions that correspond to a given elementary signal.

An arrangement is described which is able to generate Gabor's expansion coefficients of a rastered, one-dimensional signal by coherent-optical means.

1 Introduction

It is sometimes convenient to describe a time signal $\varphi(t)$, say, not in the time domain, but in the frequency domain by means of its *frequency spectrum*. The frequency spectrum shows us the *global* distribution of the energy of the signal as a function of frequency. However, one is often more interested in the momentary or *local* distribution of the energy as a function of frequency.

The need for a *local frequency spectrum* arises in several disciplines. It arises in music, for instance, where a signal is usually described not by a time function nor by the Fourier transform of that function, but by its *musical score*; indeed, when a composer writes a score, he prescribes the frequencies of the tones that should be present at a certain moment. It arises in optics: geometrical optics is usually treated in terms of *rays*, and the signal is described by giving the directions (cf. frequencies) of the rays (cf. tones) that should be present at a certain position (cf. time moment). It arises also in mechanics, where the position and the momentum of a particle are given simultaneously, leading to a description of mechanical phenomena in a *phase space*.

A strong candidate for a local frequency spectrum is *Gabor's signal expansion*. In this paper we will consider Gabor's signal expansion and its companion, the *Gabor transform*, and we will show under which condition these two signal representations form a transform pair. Extensive use will be made of the Zak transform, which offers a possibility to transform Gabor's signal expansion and the Gabor transform in mathematically more attractive product forms.

2 Gabor's signal expansion and Gabor transform

In 1946 Gabor [1] suggested the expansion of a signal $\varphi(t)$ into a discrete set of properly shifted (over discrete distances mT) and modulated (with discrete frequencies $k\Omega$ and $\Omega = 2\pi/T$) versions of a Gaussian elementary signal $g(t) = 2^{1/4}\exp[-\pi(t/T)^2]$. Although Gabor restricted himself to an elementary signal that had a Gaussian shape, his signal expansion holds for rather arbitrarily shaped elementary signals $g(t)$. In general, Gabor's signal expansion takes the form

$$\varphi(t) = \sum_m \sum_k a_{mk} g(t - m\alpha T) e^{jk\beta\Omega t}, \tag{1}$$

where the time shift αT and the frequency shift $\beta\Omega$ satisfy the relationships $\Omega T = 2\pi$ and $\alpha\beta \leq 1$, and where m and k may take all integer values. (Unless otherwise stated, all summations and integrations in this paper extend from $-\infty$ to $+\infty$.) In his original paper, Gabor restricted himself to a *critical* sampling of the time-frequency domain, i.e., $\alpha\beta = 1$, in which case the expansion coefficients a_{mk} can be interpreted as independent data, i.e., degrees of freedom of the signal.

For $\alpha\beta > 1$, the set of shifted and modulated versions of the elementary signal is *not complete* and thus cannot represent any arbitrary signal, while for $\alpha\beta < 1$, the set is *overcomplete*, which implies that Gabor's expansion coefficients become dependent and can no longer be identified as degrees of freedom. In the special case $\alpha\beta = 1$, it has been shown [2] how a *window function* $w(t)$ can be found such that the expansion coefficients can be determined via the so-called *Gabor transform*

$$a_{mk} = \int \varphi(t) w^*(t - m\alpha T) e^{-jk\beta\Omega t} dt. \tag{2}$$

It is the aim of this paper to show how a window function can be found when the parameters α and β satisfy the relation $\alpha\beta = 1/p < 1$, where p is a positive *integer*.

That a window function can be found in the case of *oversampling* (i.e., $\alpha\beta < 1$) is not surprising. To see this let us consider the *continuous* analogues of Gabor's signal expansion and the Gabor transform. The Gabor transform (2) can be considered as a sampled version [2] of the *sliding-window spectrum* or *windowed Fourier transform* $s(t, \omega)$ of the signal $\varphi(t)$, defined as

$$s(t, \omega) = \int \varphi(\tau) w^*(\tau - t) e^{-j\omega\tau} d\tau, \tag{3}$$

where the sampling appears on the time-frequency lattice ($t = m\alpha T, \omega = k\beta\Omega$). Gabor's expansion coefficients follow from the sliding-window spectrum through the relation $a_{mk} = s(m\alpha T, k\beta\Omega)$. It is well known that the signal $\varphi(t)$ can be reconstructed from its sliding-window spectrum $s(t, \omega)$ in many different ways, one of them reading

$$\varphi(\tau) \int |w(t)|^2 dt = \frac{1}{2\pi} \iint s(t, \omega) w(\tau - t) e^{j\omega\tau} dt d\omega. \tag{4}$$

It is not difficult to see that the latter signal representation is a continuous analogue of Gabor's signal expansion (1), and that it can be derived from this discrete expansion by letting the time step αT and the frequency step $\beta\Omega$ tend to zero. In fact, the signal representation (4) is identical to Gabor's signal expansion (1) with an *infinitely dense* sampling lattice. We conclude that in this limiting case, the window function $w(t)$ may be chosen proportional to the elementary signal $g(t)$. A transition from Gabor's signal expansion to its continuous analogue can be achieved by letting $\alpha\beta = 1/p \downarrow 0$.

3 Fourier transform and Zak transform

We will make extensive use of the Fourier transform of a two-dimensional array and of the Zak transform [3] of a one-dimensional function. The *Fourier transform* $\bar{a}(t, \omega; T)$ of an array a_{mk} is defined according to

$$\bar{a}(t, \omega; T) = \sum_m \sum_k a_{mk} e^{-j(m\omega T - k\Omega t)}, \tag{5}$$

with T and Ω arbitrary normalization constants that satisfy the relation $\Omega T = 2\pi$; the Fourier transform of an array will throughout be denoted by the same symbol as the array itself, but marked by a bar on top of it. We remark that the Fourier transform is periodic, both in the frequency variable ω (with period $\Omega = 2\pi/T$) and in the time variable t (with period T):

$$\bar{a}\left(t + mT, \omega + k\frac{2\pi}{T}; T\right) = \bar{a}(t, \omega; T) \tag{6}$$

for all integer values of m and k. In considering a Fourier transform we can therefore restrict ourselves to the *fundamental Fourier interval* $(-\frac{1}{2}T < t \leq \frac{1}{2}T, -\frac{1}{2}\Omega < \omega \leq \frac{1}{2}\Omega)$.

The *Zak transform* $\tilde{\varphi}(t,\omega;\tau)$ of a signal $\varphi(t)$ is defined as a one-dimensional Fourier transformation of the sequence $\varphi(t+n\tau)$ (with n taking on all integer values and t being a mere parameter), hence

$$\tilde{\varphi}(t,\omega;\tau) = \sum_n \varphi(t+n\tau)e^{-jn\omega\tau}; \tag{7}$$

the Zak transform of a signal will throughout be denoted by the same symbol as the signal itself, but marked by a tilde on top of it. We remark that the Zak transform is periodic in the frequency variable ω (with period $2\pi/\tau$) and quasi-periodic in the time variable t (with quasi-period τ):

$$\tilde{\varphi}\left(t+m\tau,\omega+k\frac{2\pi}{\tau};\tau\right) = \tilde{\varphi}(t,\omega;\tau)e^{jm\omega\tau} \tag{8}$$

for all integer values of m and k. Note that the Zak transform provides a means to represent an arbitrarily long one-dimensional time function (or one-dimensional frequency function) by a two-dimensional time-frequency function on a rectangle with *finite area* 2π. In considering a Zak transform we can therefore restrict ourselves to the *fundamental Zak interval* $(-\frac{1}{2}\tau < t \leq \frac{1}{2}\tau, -\pi/\tau < \omega \leq \pi/\tau)$.

We want to make an interesting observation to which we will return later on in this paper. Suppose that, for small τ for instance, we can approximate a function $g(t)$ by the piecewise constant function

$$g(t) = \sum_n g_n \mathrm{rect}\left(\frac{t-n\tau}{\tau}\right), \tag{9}$$

where $\mathrm{rect}(x) = 1$ for $-\frac{1}{2} < x \leq \frac{1}{2}$ and $\mathrm{rect}(x) = 0$ outside that interval. In the time interval $-\frac{1}{2}\tau < t \leq \frac{1}{2}\tau$, the Zak transform $\tilde{g}(t,\omega;\tau)$ then takes the form

$$\tilde{g}(t,\omega;\tau) = \sum_n g_n e^{-jn\omega\tau} = \bar{g}(\omega\tau); \tag{10}$$

note that this Zak transform does not depend on the time variable t, and that the one-dimensional Fourier transform $\bar{g}(\omega\tau)$ of the sequence g_n arises.

With the help of the Fourier transform (5) and the Zak transform (7), and with $1/\alpha\beta = p$ being a positive integer, the Gabor transform (2) and Gabor's signal expansion (1) can be transformed into the *product forms* [4]

$$\bar{a}(t,\omega;T) = \alpha pT\tilde{\varphi}\left(\alpha pt,\frac{\omega}{\alpha};\alpha pT\right)\tilde{w}^*\left(\alpha pt,\frac{\omega}{\alpha};\alpha T\right) \tag{11}$$

and

$$\tilde{\varphi}\left(\alpha pt,\frac{\omega}{\alpha};\alpha pT\right) = \frac{1}{p}\sum_{r=<p>}\bar{a}\left(t,\omega+r\frac{\Omega}{p};T\right)\tilde{g}\left(\alpha pt,\frac{\omega+r\Omega/p}{\alpha};\alpha T\right), \tag{12}$$

respectively, where the expression $r = <p>$ is used throughout as a short-hand notation for an interval of p successive integers ($r = 0,1,2,...,p-1$, for instance; due to the periodicity of the Fourier transform and the Zak transform, however, any sequence of p successive integers can be chosen).

When we consider in the fundamental Fourier interval $(-\frac{1}{2}T < t \leq \frac{1}{2}T, -\frac{1}{2}\Omega < \omega \leq \frac{1}{2}\Omega)$ the three functions $\bar{a}(t,\omega;T)$, $\tilde{\varphi}(\alpha pt,\omega/\alpha;\alpha pT)$, and $\tilde{w}(\alpha pt,\omega/\alpha;\alpha T)$ that arise in Eq. (11), and the three functions $\tilde{\varphi}(\alpha pt,\omega/\alpha;\alpha pT)$, $\bar{a}(t,\omega;T)$, and $\tilde{g}(\alpha pt,\omega/\alpha;\alpha T)$ that arise in Eq. (12), we note that whereas the Fourier transform $\bar{a}$ appears only *once*, the Zak transforms $\tilde{\varphi}$, $\tilde{g}$, and $\tilde{w}$ appear *p-fold*: $\tilde{\varphi}(\alpha pt,\omega/\alpha;\alpha pT)$ as p identical stripes with height Ω/p and width T, and $\tilde{g}(\alpha pt,\omega/\alpha;\alpha T)$ and $\tilde{w}(\alpha pt;\omega/\alpha;\alpha T)$ as p stripes with width T/p and height Ω, which stripes are identical to each other apart from the factor $\exp[j\omega T]$ [cf. the periodicity property (8) of the Zak transform].

4 Transform pair

Using the product forms (11) and (12), it is not difficult to show that Gabor's signal expansion (1) and the Gabor transform (2) form a *transform pair*, if the Zak transform $\tilde{g}$ of the elementary signal $g(t)$ and the

Zak transform $\tilde{w}$ of the window function $w(t)$ satisfy the condition

$$\sum_{r=<p>} \tilde{w}^* \left(\alpha pt, \frac{\omega + r\Omega/p}{\alpha}; \alpha T\right) \tilde{g} \left(\alpha pt, \frac{\omega + r\Omega/p}{\alpha}; \alpha T\right) = \frac{1}{\alpha T}. \tag{13}$$

One way to meet this condition is by letting $\tilde{g}$ and $\tilde{w}$, in the frequency interval $-\frac{1}{2}\Omega/p < \omega \leq \frac{1}{2}\Omega/p$, satisfy the p relations

$$\tilde{w}^* \left(\alpha pt, \frac{\omega + r\Omega/p}{\alpha}; \alpha T\right) \tilde{g} \left(\alpha pt, \frac{\omega + r\Omega/p}{\alpha}; \alpha T\right) = \frac{c_r}{\alpha T} \tag{14}$$

for $r = < p >$, with the additional condition

$$\sum_{r=<p>} c_r = 1 \tag{15}$$

for the p constants c_r. Thus, in any interval of width Ω/p,

$$(r - \tfrac{1}{2})\frac{\Omega}{p} < \omega \leq (r + \tfrac{1}{2})\frac{\Omega}{p}, \tag{16}$$

the Zak transform $\tilde{w}(\alpha pt, \omega/\alpha; \alpha T)$ is chosen inverse proportional to the Zak transform $\tilde{g}^*(\alpha pt, \omega/\alpha; \alpha T)$. Note that for $p = 1$ the set of p equations (14) reduces to *one* equation, and that the window function $w(t)$ is completely determined by the elementary signal $g(t)$ through its Zak transform $\tilde{w}(\alpha t, \omega/\alpha; \alpha T) = 1/\alpha T \tilde{g}^*(\alpha t, \omega/\alpha; \alpha T)$. Determining the window function might be difficult, however, if the Zak transform of the elementary signal has zeros, which usually is the case [5]. As an example we have depicted in Figs. 1 and 2 the Zak transforms $\tilde{g}(\alpha t, \omega/\alpha; \alpha T)$ that correspond to the Gaussian elementary signal $g(t) = 2^{1/4} \exp[-\pi(t/T)^2]$ for $\alpha = 1$ (Gabor's original choice) and $\alpha = \frac{1}{3}$, respectively; we remark that the Zak transform of this Gaussian signal has zeros for $(t = \frac{1}{2}T + mT, \omega = \frac{1}{2}\Omega + k\Omega)$. And note that for $\alpha = \frac{1}{3}$ (and also for smaller values of α, of course) the Zak transform becomes almost independent of t, as we observed before.

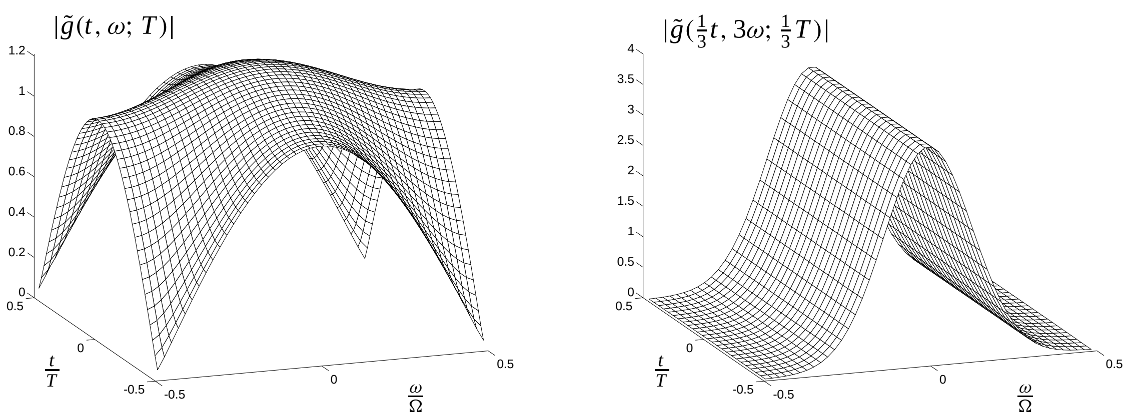

Figure 1: The Zak transform $\tilde{g}(t, \omega; T)$ in the case of a Gaussian elementary signal $g(t) = 2^{1/4} \exp[-\pi(t/T)^2]$.

Figure 2: The Zak transform $\tilde{g}(\frac{1}{3}t, 3\omega; \frac{1}{3}T)$ in the case of a Gaussian elementary signal $g(t) = 2^{1/4} \exp[-\pi(t/T)^2]$.

For $p > 1$, the window function that corresponds to a given elementary signal is *not unique*. This is in accordance with the fact that in the case of oversampling the set of shifted and modulated versions of the elementary signal is overcomplete, and that Gabor's expansion coefficients are dependent and can no longer be considered as degrees of freedom, as we have mentioned before. The additional freedom in constructing a window function can be used to avoid the problems caused by the occurrence of zeros in the

Zak transform of the elementary signal: we simply choose in Eq. (14) the constants c_r equal to zero in those intervals where $\tilde{g}(\alpha pt, [\omega + r\Omega/p]/\alpha; \alpha T)$ has zeros. In this way we can construct a window function if there is at least one interval in which the Zak transform $\tilde{g}$ does not vanish.

We might choose the non-vanishing constants c_r according to

$$c_r = C\alpha T \frac{p}{T}\frac{p}{\Omega} \int_{-\frac{1}{2}T/p}^{\frac{1}{2}T/p} \int_{-\frac{1}{2}\Omega/p}^{\frac{1}{2}\Omega/p} \left| \tilde{g}\left(\alpha pt, \frac{\omega + r\Omega/p}{\alpha}; \alpha T\right)\right|^2 dt d\omega, \tag{17}$$

where C is a real constant that still has to be determined in accordance with condition (15). It can be shown [4] that for this choice of the constants c_r, a window function $w(t)$ is constructed that resembles the elementary signal $g(t)$. Indeed, in the limiting case $p \to \infty$, this window function becomes exactly proportional to the elementary signal, $w(t) \to Cg(t)$, with $C = p \int |w(t)|^2 dt$, which is in accordance with the continuous analogues (3) and (4) of the Gabor transform (2) and Gabor's signal expansion (1). As an example we have depicted in Fig. 3 the Zak transform $\tilde{w}(\frac{1}{3}t, 3\omega; \frac{1}{3}T)$ of the window function $w(t)$ that corresponds to the Gaussian elementary signal for $\alpha = \frac{1}{3}$ and $\beta = \frac{1}{15}$, and thus $p = 45$. Note that the Zak transforms $\tilde{g}(\frac{1}{3}t, 3\omega; \frac{1}{3}T)$ (see Fig. 2) and $\tilde{w}(\frac{1}{3}t, 3\omega; \frac{1}{3}T)$ (see Fig. 3) are roughly the same. The corresponding window function $w(t)$ has been depicted in Fig. 4.

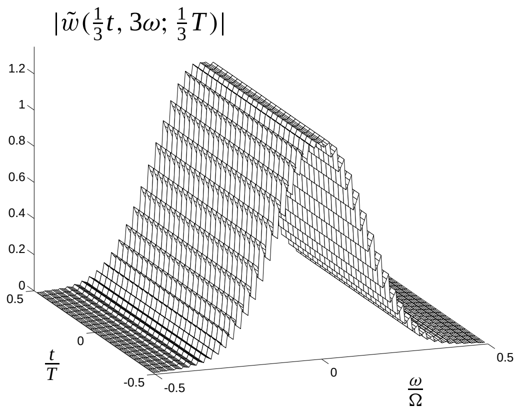

Figure 3: The Zak transform $\tilde{w}(\frac{1}{3}t, 3\omega; \frac{1}{3}T)$ of the window function $w(t)$ that corresponds to the Gaussian elementary signal $g(t) = 2^{1/4}\exp[-\pi(t/T)^2]$ for $\alpha = \frac{1}{3}$ and $\beta = \frac{1}{15}$.

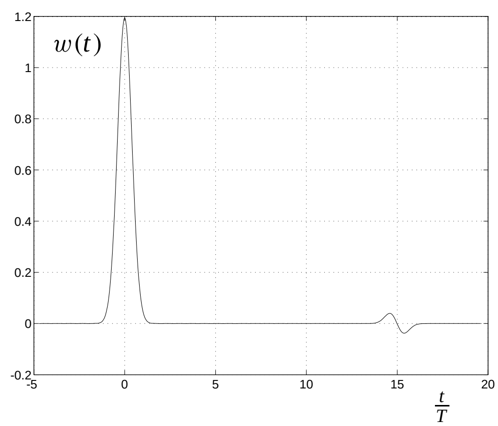

Figure 4: The window function $w(t)$ that corresponds to the Gaussian elementary signal $g(t) = 2^{1/4}\exp[-\pi(t/T)^2]$ for $\alpha = \frac{1}{3}$ and $\beta = \frac{1}{15}$.

If we consider the product form (11) of the Gabor transform, we observe an easy possibility to determine Gabor's expansion coefficients:

- we determine the Zak transform $\tilde{\varphi}(\alpha pt, \omega/\alpha; \alpha pT)$ of the signal $\varphi(t)$ and the Zak transform $\tilde{w}(\alpha pt, \omega/\alpha; \alpha T)$ of the window function $w(t)$;
- we then find the Fourier transform $\bar{a}(t, \omega; T)$ by means of the product rule (11);
- we finally determine Gabor's expansion coefficients a_{mk} from $\bar{a}(t, \omega; T)$ via an inverse Fourier transformation.

5 Coherent-optical setup

The above-mentioned way of generating Gabor's expansion coefficients can be implemented by means of a coherent-optical setup. We therefore present the one-dimensional signal $\varphi(t)$ on an optical transparency in a *raster format*, where – like on a video monitor – the horizontal direction is the 'fast' direction and the vertical direction the 'slow' one, say, and illuminate this transparency by a plane wave of monochromatic laser light.

If we identify the horizontal width of the raster with T, the required Zak transform $\tilde{\varphi}(\alpha pt, \omega/\alpha; \alpha pT)$ can then be generated by an astigmatic coherent-optical system that performs a one-dimensional optical Fourier transformation in the vertical direction and an ideal imaging in the horizontal direction; such an astigmatic system can be realized, for instance, using a combination of a spherical and a cylindrical lens. The two-dimensional output signal of the astigmatic system can thus be identified with $\tilde{\varphi}(\alpha pt, \omega/\alpha; \alpha pT)$, where the time variable t corresponds to the horizontal coordinate and the frequency variable ω to the vertical one. The output signal $\tilde{\varphi}(\alpha pt, \omega/\alpha; \alpha pT)$ is then multiplied by the Zak transform $\alpha pT\tilde{w}^*(\alpha pt, \omega/\alpha; \alpha T)$ to yield the Fourier transform $\bar{a}(t, \omega; T)$, by placing an appropriate transparency in the output plane of the astigmatic system; note that we can restrict ourselves to one period of the Fourier transform, using a rectangular aperture. With the help of a spherical lens, we finally generate the two-dimensional optical Fourier transform of the product $\alpha pT\tilde{\varphi}(\alpha pt, \omega/\alpha; \alpha pT)\tilde{w}^*(\alpha pt, \omega/\alpha; \alpha T) = \bar{a}(t, \omega; T)$, and the Gabor coefficients a_{mk} appear on a lattice in the back focal plane of the spherical lens. An important feature of this optical arrangement is that the two-dimensional nature of an optical processing system, with its parallel processing capabilities and its large space-bandwidth product, is fully utilized. Due to the possibility of avoiding the problems that arise from the occurrence of zeros in the Zak transform of the elementary signal, the required optical transparency can indeed be fabricated.

6 Conclusion

We have introduced Gabor's expansion of a signal into a discrete set of shifted and modulated versions of an elementary signal. We have also described the inverse operation – the Gabor transform – with which Gabor's expansion coefficients can be determined, and we have shown the relationship between the Gabor transform and sampling of the sliding-window spectrum. It is well known that in the case of Gabor's original, *critical* sampling, and using the Fourier transform and the Zak transform, the Gabor transform and Gabor's signal expansion can be transformed into product forms.

Determination of the expansion coefficients via the product forms may be difficult, however, because of the occurrence of zeros in the Zak transform. One way of avoiding the problems that arise from these zeros is to sample the time-frequency domain on a denser lattice. In this paper we have shown that in the case of *oversampling* by an integer factor, the Gabor transform can again be transformed into a product form; furthermore, Gabor's signal expansion can then be transformed into a sum-of-products form. Using these product forms, it was possible to show that the Gabor transform and Gabor's signal expansion form a transform pair.

The process of oversampling introduces dependence between the Gabor coefficients. In controlling this dependence, we were able to avoid the problems that arise from the occurrence of zeros in the Zak transform. The additional freedom caused by oversampling, allowed us to construct the window function in such a way that it is mathematically well-behaved. Moreover, it was shown that for a very large oversampling and with a proper choice of the design parameters, the window function can become proportional to the elementary signal; this result is in accordance with the continuous analogue of Gabor's signal expansion.

Finally, a coherent-optical arrangement was described which is able to generate Gabor's expansion coefficients of a rastered, one-dimensional signal via the Zak transform.

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