Mode mapping in paraxial lossless optics

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Under the Collins transformation, the orthonormal set of Hermite-Gaussian modes maps into an orthonormal set of beams with a Gaussian envelope. Among them are the Laguerre-Gaussian beams and the recently introduced Hermite-Laguerre-Gaussian beams. Compact expressions for the complex field amplitudes of these modes are derived. The obtained results are useful for the description of light propagation through first-order optical systems, for the solution of the phase retrieval problem by non-interferometric techniques, and for the design of mode converters and information processing systems.

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Paraxial optical systems are actively used for information processing, for phase retrieval, for transformation and characterization of optical beams. In particular, generation of vortex beams attracts a lot of interest of the scientific community, due to their possible applications in micro-object manipulation, communication, etc. New families of Gaussian beams,[1,2] closely related to the Hermite-Gaussian (HG) and Laguerre-Gaussian (LG) ones, and new schemes for laser mode converters, such as fractional Fourier transform (FT) $\pi/2$ converters,[3] have recently been proposed. Here we define a generalized class of Gaussian beams, $ABCD$-HG modes, obtained from the HG or LG ones after propagation through a first-order optical system described by the Collins integral.[4] This class includes as particular cases the Gaussian beams considered in Refs. [1, 2]. The introduction of the novel class of orthonormal modes simplifies the description of light propagation through first-order optical systems and the design of optical schemes for mode converters and information processing.

A lossless paraxial system (or first-order optical **ABCD**-system with real matrices **ABCD**) is described by its symplectic ray transformation matrix[5] $\mathbf{T}$ that relates the position $\mathbf{r}_i = (x_i, y_i)^t$ and direction $\mathbf{q}_i = (u_i, v_i)^t$ of an incoming ray to the position $\mathbf{r}_o = (x_o, y_o)^t$ and direction $\mathbf{q}_o = (u_o, v_o)^t$ of the outgoing ray:

$$\begin{pmatrix} \mathbf{r}_o \\ \mathbf{q}_o \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \mathbf{r}_i \\ \mathbf{q}_i \end{pmatrix} = \mathbf{T} \begin{pmatrix} \mathbf{r}_i \\ \mathbf{q}_i \end{pmatrix}; \qquad (1)$$

as usual, the superscript ^{t} denotes transposition. The symplecticity of the **ABCD**-matrix yields the relations

$$\begin{aligned} &\mathbf{A}\mathbf{B}^t = \mathbf{B}\mathbf{A}^t, \quad \mathbf{C}\mathbf{D}^t = \mathbf{D}\mathbf{C}^t, \quad \mathbf{A}\mathbf{D}^t - \mathbf{B}\mathbf{C}^t = \mathbf{I}, \\ &\mathbf{A}^t\mathbf{C} = \mathbf{C}^t\mathbf{A}, \quad \mathbf{B}^t\mathbf{D} = \mathbf{D}^t\mathbf{B}, \quad \mathbf{A}^t\mathbf{D} - \mathbf{C}^t\mathbf{B} = \mathbf{I}. \end{aligned} \qquad (2)$$

Any symplectic matrix is associated to two integral transforms which differ by a $\pm$ sign, see Ref. [6], Section 9. Under the assumption that $\mathbf{B}$ is a non-singular matrix, the action of a first-order optical system on the complex field amplitude $f_i(\mathbf{r}_i)$ at its input can be described by the Collins integral,[4]

$$f_o(\mathbf{r}_o) = \frac{\exp(i\phi)}{\sqrt{\det i\mathbf{B}}} \iint f_i(\mathbf{r}_i) \exp\left[i\pi\left(\mathbf{r}_i^t\mathbf{B}^{-1}\mathbf{A}\mathbf{r}_i - 2\mathbf{r}_i^t\mathbf{B}^{-1}\mathbf{r}_o + \mathbf{r}_o^t\mathbf{D}\mathbf{B}^{-1}\mathbf{r}_o\right)\right] \mathrm{d}\mathbf{r}_i, \quad (3)$$

where the square root $\sqrt{\det i\mathbf{B}}$ is chosen such that its argument is in the region $[\pi/2; \pi]$. The constant phase factor $\exp(i\phi)$, with which we can cope, if necessary, with the optical path length and the metaplectic sign problem, is rather irrelevant in the scope of this paper and will be dropped in the remainder of it.

There are two sets of orthonormal modes that are widely used in optics: HG and LG ones. An arbitrary square integrable two-dimensional function, representing for example the complex field amplitude $f(\mathbf{r})$, can be written as a linear superposition of HG or LG modes. The orthonormal HG mode is given by

$$\mathcal{H}_{n,m}(\mathbf{r}; w_x, w_y) = \mathcal{H}_n(x; w_x)\, \mathcal{H}_m(y; w_y), \qquad (4)$$

with

$$\mathcal{H}_n(x; w) = 2^{1/4}\,(2^n n!\, w)^{-1/2}\, H_n\left(\sqrt{2\pi}\, x/w\right) \times \exp\left(-\pi\, x^2/w^2\right), \quad (5)$$

where $H_n(\cdot)$ denotes the Hermite polynomials.[7] The orthonormal form for the LG mode[8] is

$$\mathcal{L}_{n,m}(\mathbf{r}; w) = 2^{1/2} w^{-1} \left[(\min\{n,m\})!/(\max\{n,m\})!\right]^{1/2} \times \left(\sqrt{2\pi}\, r/w\right)^{|n-m|} \exp[i(n-m)\varphi]\, L_{\min\{n,m\}}^{(|n-m|)}\left(2\pi\, r^2/w^2\right) \times \exp\left[-\pi\, r^2/w^2\right], \quad (6)$$

where $x = r\cos\varphi$ and $y = r\sin\varphi$ and where $L_n^{(\alpha)}(\cdot)$ denotes the generalized Laguerre polynomial.[7]

There are several first-order optical systems that are special with respect to the HG and LG modes. One of them is the separable fractional FT system, described by its ray transformation matrix[6] $\mathbf{T}_{fFT}$

$$\left(\begin{array}{cccc} \cos\gamma_x & 0 & w_x^2 \sin\gamma_x & 0 \\ 0 & \cos\gamma_y & 0 & w_y^2 \sin\gamma_y \\ -w_x^{-2}\sin\gamma_x & 0 & \cos\gamma_x & 0 \\ 0 & -w_y^{-2}\sin\gamma_y & 0 & \cos\gamma_y \end{array} \right), \tag{7}$$

for which the HG and LG modes are eigenfunctions of the corresponding Collins transformation. Note that for the (rotationally symmetric) LG modes, the fractional Fourier transformer should be isotropic: $w_x = w_y = w$ and $\gamma_x = \gamma_y = \gamma$.

Other first-order optical systems – mode converters – transform, for instance, HG modes into LG ones and vice versa. To treat systems that convert HG modes $\mathcal{H}_{n,m}(\mathbf{r}; w_x, w_y)$ into others, we introduce normalized and dimensionless ray transformation submatrices $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ as

$$\left(\begin{array}{cc} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{array} \right) = \left(\begin{array}{cc} \mathbf{W}_o^{-1}\mathbf{A}\mathbf{W}_i & \mathbf{W}_o^{-1}\mathbf{B}\mathbf{W}_i^{-1} \\ \mathbf{W}_o\mathbf{C}\mathbf{W}_i & \mathbf{W}_o\mathbf{D}\mathbf{W}_i^{-1} \end{array} \right) \tag{8}$$

where

$$\mathbf{W}_i = \left(\begin{array}{cc} w_x & 0 \\ 0 & w_y \end{array} \right) \tag{9}$$

with w_x and w_y the widths of the HG mode at the system's input, and with $\mathbf{W}_o$ a similar diagonal matrix, appropriately chosen, at its output. Several HG-to-LG mode converters[1, 3, 8, 9] have been proposed; we mention

$$\mathbf{a}_1 = \mathbf{d}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{b}_1 = -\mathbf{c}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\mathbf{a}_2 = \mathbf{d}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{b}_2 = -\mathbf{c}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Case 1 is described in Ref. [1], Eq. (6) with $\alpha = \pi/4$, while case 2 is described in Ref. [8], Eq. (14) with $\phi = \pi/4$; the two systems are related by a Fourier transformation. The LG beams at the output of these systems differ from one another only by a constant phase factor. The inverses of these operations perform a LG-to-HG transformation.

Our goal is to find the expression for the beams at the output of an **ABCD**-system starting from the orthonormal HG modes. Note that one could also start from the set of LG modes and obtain the same class of beams, based on the additivity of first-order systems and the knowledge of the LG-HG converter matrix. We will use the following notation: for a HG input beam $\mathcal{H}_{n,m}(\mathbf{r}; w_x, w_y)$ [see Eqs. (4) and (5)] we get the $ABCD$-HG beam $\mathcal{H}^{ABCD}_{n,m}(\mathbf{r}; w_x, w_y)$ at the output of an **ABCD**-system, after applying the Collins integral (3).

Let us now derive the expression for the complex field amplitude of the beam $\mathcal{H}^{ABCD}_{n,m}(\mathbf{r}; w_x, w_y)$. Applying Collins integral to the generating function of the HG modes, we can find the generating function of the $ABCD$-HG beams,

$$\begin{aligned} &\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \mathcal{H}^{ABCD}_{n,m}(\mathbf{r}; w_x, w_y) \left(\frac{2^{n+m}}{n!m!}\right)^{1/2} s_x^n s_y^m \\ &= 2^{1/2}/\sqrt{\det \mathbf{W}_o \det(\mathbf{a}+i\mathbf{b})} \\ &\times \exp\left[-\mathbf{s}^t(\mathbf{a}+i\mathbf{b})^{-1}(\mathbf{a}-i\mathbf{b})\mathbf{s} + 2\sqrt{2\pi}\,\mathbf{s}^t(\mathbf{a}+i\mathbf{b})^{-1}\boldsymbol{\rho}\right] \\ &\times \exp\left[-\pi\boldsymbol{\rho}^t(\mathbf{d}-i\mathbf{c})(\mathbf{a}+i\mathbf{b})^{-1}\boldsymbol{\rho}\right], \end{aligned} \tag{10}$$

where $\mathbf{s} = (s_x, s_y)^t$ and where we have introduced normalized and dimensionless space variables $\boldsymbol{\rho} = (\xi, \eta)^t = \mathbf{W}_o^{-1}\mathbf{r}$. Subsequently we find the derivative and recurrence relations of this generalized class of Gaussian beams,[9] by differentiating the generating function (10) with respect to $\boldsymbol{\rho}$ and $\mathbf{s}$, respectively:

$$\begin{aligned} &\left[\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}\right]^t \mathcal{H}^{ABCD}_{n,m}(\mathbf{r}) \\ &= 2\sqrt{\pi}\left(\mathbf{a}^t + i\mathbf{b}^t\right)^{-1}\left[\sqrt{n}\,\mathcal{H}^{ABCD}_{n-1,m}(\mathbf{r}), \sqrt{m}\,\mathcal{H}^{ABCD}_{n,m-1}(\mathbf{r})\right]^t \\ &\quad - 2\pi\,\mathcal{H}^{ABCD}_{n,m}(\mathbf{r})\,(\mathbf{d}-i\mathbf{c})\,(\mathbf{a}+i\mathbf{b})^{-1}\,[\xi,\eta]^t, \end{aligned} \tag{11}$$

$$\begin{aligned} &2\sqrt{\pi}\,[\xi,\eta]^t\,\mathcal{H}^{ABCD}_{n,m}(\mathbf{r}) \\ &= (\mathbf{a}+i\mathbf{b})\left[\sqrt{n+1}\,\mathcal{H}^{ABCD}_{n+1,m}(\mathbf{r}), \sqrt{m+1}\,\mathcal{H}^{ABCD}_{n,m+1}(\mathbf{r})\right]^t \\ &\quad + (\mathbf{a}-i\mathbf{b})\left[\sqrt{n}\,\mathcal{H}^{ABCD}_{n-1,m}(\mathbf{r}), \sqrt{m}\,\mathcal{H}^{ABCD}_{n,m-1}(\mathbf{r})\right]^t, \end{aligned} \tag{12}$$

The combination of Eqs. (11) and (12) leads to the following relationship for the $ABCD$-HG beams

$$\begin{array}{rcl} 2\sqrt{\pi(n+1)}\,\mathcal{H}_{n+1,m} & = & \mathcal{P}\,\mathcal{H}_{n,m} \\ 2\sqrt{\pi(m+1)}\,\mathcal{H}_{n,m+1} & = & \mathcal{Q}\,\mathcal{H}_{n,m} \end{array} \tag{13}$$

with the operators

$$\begin{array}{rcl} \mathcal{P} & = & 2\pi(U_{11}\xi + U_{12}\eta) - Z_{11}\dfrac{\partial}{\partial\xi} - Z_{12}\dfrac{\partial}{\partial\eta} \\ \mathcal{Q} & = & 2\pi(U_{21}\xi + U_{22}\eta) - Z_{21}\dfrac{\partial}{\partial\xi} - Z_{22}\dfrac{\partial}{\partial\eta} \end{array} \tag{14}$$

and the matrices

$$\mathbf{U} = (\mathbf{a}-i\mathbf{b})^t[(\mathbf{d}-i\mathbf{c})(\mathbf{a}+i\mathbf{b})^{-1}]^* \tag{15}$$

$$\mathbf{Z} = (\mathbf{a}-i\mathbf{b})^t. \tag{16}$$

Note that the operators $\mathcal{P}$ and $\mathcal{Q}$ commute since $\mathbf{Z}\mathbf{U}^t = \mathbf{U}\mathbf{Z}^t$, and that we are thus led to an alternative definition of the $ABCD$-HG beams:

$$\mathcal{H}^{ABCD}_{n,m}(\mathbf{r}) = \frac{\mathcal{P}^n\,\mathcal{Q}^m\,\mathcal{H}^{ABCD}_{0,0}(\mathbf{r})}{2^{n+m}\sqrt{\pi^{n+m}n!m!}}. \tag{17}$$

Based on the formula[9] $\iint \exp\left(-\pi\,\mathbf{r}_i^t\mathbf{P}\mathbf{r}_i - i2\pi\,\mathbf{r}_i^t\mathbf{q}\right)\mathrm{d}\mathbf{r}_i = \exp\left(-\pi\mathbf{q}^t\mathbf{P}^{-1}\mathbf{q}\right)/\sqrt{\det\mathbf{P}}$ for the calculation of Collins integral for the HG fundamental mode

$\mathcal{H}_{0,0}(\mathbf{r}; w_x, w_y)$, an explicit form for the fundamental ($n = m = 0$) $ABCD$-HG mode is found:

$$\mathcal{H}_{0,0}^{ABCD}(\mathbf{r}) = 2^{1/2}/\sqrt{\det \mathbf{W}_o \det(\mathbf{a}+i\mathbf{b})} \times \exp\left[-\pi\,\boldsymbol{\rho}^t\,(\mathbf{d}-i\mathbf{c})\,(\mathbf{a}+i\mathbf{b})^{-1}\,\boldsymbol{\rho}\right]. \quad (18)$$

One can easily confirm the orthonormality relationship between $\mathcal{H}_{n,m}^{ABCD}(\mathbf{r}; w_x, w_y)$ and $\mathcal{H}_{l,k}^{ABCD}(\mathbf{r}; w_x, w_y)$,

$$\iint \mathcal{H}_{n,m}^{ABCD}(\mathbf{r}; w_x, w_y)[\mathcal{H}_{l,k}^{ABCD}(\mathbf{r}; w_x, w_y)]^* \mathrm{d}\mathbf{r} = \delta_{nl}\delta_{mk}, \quad (19)$$

which is a direct consequence of the orthonormality relationship between the HG beams $\mathcal{H}_{n,m}(\mathbf{r}; w_x, w_y)$ and $\mathcal{H}_{l,k}(\mathbf{r}; w_x, w_y)$ at the input of the first-order optical system and the unitarity of this system.

Note that if the ray transformation matrix $\mathbf{T}$ commutes with the ray transformation matrix $\mathbf{T}_{FT}$ of a FT system,[6] cf. (7) with $\gamma_x = \gamma_y = \pi/2$ and $w_x = w_y$, we have $\mathbf{d} - i\mathbf{c} = \mathbf{a} + i\mathbf{b}$ and therefore $\mathbf{U} = \mathbf{Z}$. In this case the operators reduce to

$$\begin{aligned} \mathcal{P} &= -Z_{11}\,(\partial/\partial\xi - 2\pi\xi) - Z_{12}\,(\partial/\partial\eta - 2\pi\eta) \\ \mathcal{Q} &= -Z_{21}\,(\partial/\partial\xi - 2\pi\xi) - Z_{22}\,(\partial/\partial\eta - 2\pi\eta) \end{aligned} \quad (20)$$

and the $ABCD$-HG beams reduce to the two-dimensional HG modes considered in Ref. [2]. Since

$$\left(\frac{d}{dt} - 2\pi t\right)^n \exp(-\pi t^2) = \exp(\pi t^2)\left(\frac{d}{dt}\right)^n \exp(-2\pi t^2),$$

the beams may then as well be expressed in the form

$$\mathcal{H}_{n,m}^{ABCD}(\mathbf{r}) = \frac{2^{1/2}(-1)^{n+m}\exp[\pi(\xi^2+\eta^2)]}{2^{n+m}\sqrt{\pi^{n+m}n!m!\det\mathbf{W}_o\det(\mathbf{a}+i\mathbf{b})}} \times \left(Z_{11}\frac{\partial}{\partial\xi} + Z_{12}\frac{\partial}{\partial\eta}\right)^n \left(Z_{21}\frac{\partial}{\partial\xi} + Z_{22}\frac{\partial}{\partial\eta}\right)^m \times \exp[-2\pi(\xi^2+\eta^2)]. \quad (21)$$

The recently introduced Hermite-Laguerre-Gaussian (HLG) modes[1] are a particular case of the two dimensional HG modes and are therefore $ABCD$-HG modes. They can be obtained from HG modes by the Collins transformation parameterized by the matrices

$$\begin{aligned} \mathbf{a} = \mathbf{d} &= \frac{1}{\sqrt{2}}\begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}, \\ \mathbf{b} = -\mathbf{c} &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \quad (22)$$

Simple expressions of these modes, which in the particular case $\alpha = \pi/4$ reduce to LG modes, can be obtained from Eq. (21). (Note that in Ref. [1], the output field amplitude $\mathcal{G}_{n,m}(\mathbf{r}|\alpha)$ is defined with an additional rotation of the coordinate system through the angle α.)

Since HG modes are eigenfunctions of the separable fractional FT (7) with unimodular eigenvalues, any system described by the matrix $\mathbf{T}_{ABCD}\mathbf{T}_{fFT}$ with $\mathcal{H}_{n,m}(\mathbf{r}; w_x, w_y)$ at its input, produces at its output the beam $\mathcal{H}_{n,m}^{ABCD}(\mathbf{r}; w_x, w_y)$ with an additional constant phase shift. This fact enlarges the number of optical configurations used to produce certain kinds of beams, for example, HG-to-LG converters. Moreover, since the two systems characterized by $\mathbf{T}_{ABCD}$ and $\mathbf{T}_{ABCD}\mathbf{T}_{fFT}$ produce – up to the constant phase shift – the same output $\mathcal{H}_{n,m}^{ABCD}(\mathbf{r}; w_x, w_y)$ from a HG mode at their inputs, this implies that the $ABCD$-HG mode is an eigenfunction of the Collins transform corresponding to the matrix $\mathbf{T} = \mathbf{T}_{ABCD}\mathbf{T}_{fFT}\mathbf{T}_{ABCD}^{-1}$.

We can also use the orthonormality of the $ABCD$-HG modes to describe the evolution of an arbitrary complex field during its propagation through a first-order optical system. Since the HG modes form a complete set, any function $f_i(\mathbf{r})$ can be decomposed as

$$f_i(\mathbf{r}) = \sum_m \sum_n q_{nm}\mathcal{H}_{n,m}(\mathbf{r}; w_x, w_y), \quad (23)$$

$$q_{nm} = \iint [\mathcal{H}_{n,m}(\mathbf{r}; w_x, w_y)]^* f_i(\mathbf{r})\,\mathrm{d}\mathbf{r}. \quad (24)$$

Every HG mode $\mathcal{H}_{n,m}(\mathbf{r}; w_x, w_y)$ maps into the $\mathcal{H}_{n,m}^{ABCD}(\mathbf{r}; w_x, w_y)$ mode at the output of the **ABCD**-system. Then we can write the expression for the complex field amplitude in the output plane in the form

$$f_o(\mathbf{r}) = \sum_m \sum_n q_{nm}\mathcal{H}_{n,m}^{ABCD}(\mathbf{r}; w_x, w_y). \quad (25)$$

This is a generalization of the expression used for the propagation through a fractional FT system, to the case of a general **ABCD**-system. The mapping of orthonormal modes can be used for phase recovery from intensity distribution, for analysis of imaging systems, and for beam designing.

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