New Focus on Fourier Optics Techniques

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Abstract
We present a short overview on the application of fractional cyclic and linear canonical transformations to optical signal processing and dedicate some of the discussions to the particular features found in the fractional Fourier transform domain.

1. Introduction

Almost forty years ago, Anthony Van der Lugt initiated, with his pioneering work on optical filtering and optical signal processing, an epoque of expansion of Fourier Optics. Essentially he introduced for the first time a new configuration for optical data processing, the so called optical correlator and later named Van der Lugt correlator, [1]. The optical correlation operation is based upon the capacity of a convergent lens to perform an operation proportional to the Fourier Transform, in two dimensions, of a particular object, so that this response is located at the real back focal plane of the lens. By now from various decades analogical optical processors, joint transforms correlators, correlators with space and time integration, adapted filters, etc. are extensively used, [2]. Other tools for signal processing such as wavelets and bilinear distributions are optically implemented as well, [3].

Recently, the Fourier Optics area has expanded with new contributions related to non conventional transformations, the so called fractional ones, [4-7]. For example, it has been proposed the applications of fractional Fourier transforms (FFT), [5], for spatially variant filtering, characters recognition, encryption, water marks, implementation of neural networks, etc. On the other hand, the fractional Hilbert transform can be applied to edge enhancement, [8]. The optical configurations performing such operations have been designed and so the related optical data are experimentally recorded.

The linear canonical transformations are other type of tools having great interest for optical processing. We notice that some fractional transformations, as the mentioned FFT, belong to that kind as well. The Fresnel transform is another example of linear canonical transform.

Our current general interest is a deep study on the mathematical properties of non conventional transformations and the possible applications to optical information processing.

2. Fractional cyclic transformations

In recent papers, [6-7], we have proposed a general method for the fractionalization of a cyclic transformation (such as Fourier, Hilbert, Hankel, Hartley, Sine, etc.). We define a cyclic transformation as an operator that produces the identity transformation after being applied an integer number N of times. For example, the Fourier and Hilbert transforms are cyclic with a period N=4. The Hankel, Hartley, Cosine and Sine transforms have a period N=2. The fractional transform related to a particular transformation depends on a parameter whose value equals one when producing the original transform and equals Nn (n is an integer) when producing an identity transformation. The additivity property holds with respect to that parameter. Moreover, common properties of the cyclic fractional transformations can be formulated. Most part of cyclic transformations, as it is the case for the Fourier, Hartley and Hankel transforms, have associated an infinite number of fractional transformations. The usefulness of a specific fractional transform is related with its optical feasibility as well as with its application in signal/image processing. It has been demonstrated that some fractional transformations such as Fourier, Hankel, Hartley, Hilbert, can be implemented in optical systems of the first order [6-8]. We have considered some specific tasks in which the fractional FT is applicable and then is named Optical Fractional Fourier Transform (OFFT).

3. Optical Fractional Fourier Transform

The fractional FT is a generalization of the ordinary FT[1-6]. Its kernel depends on the parameter α which can be interpreted as a rotation angle in phase space. The fractional FT of a function $f(x)$ for angle α is defined as

$$F_\alpha(u) = R^\alpha[f(x)](u) = \int K_\alpha(x,u) f(x) dx \tag{1}$$

where the kernel is given by

$$K_\alpha(x,u) = \frac{\exp(i\alpha/2)}{\sqrt{i\sin\alpha}} \exp\left(i\pi \frac{(x^2+u^2)\cos\alpha - 2xu}{\sin\alpha} \right) \tag{2}$$

Thus for $\alpha=0$ it corresponds to the identity transform, for $\alpha=\pi/2$ and $\alpha=3\pi/2$ it reduces to the FT and inverse FT, correspondingly. Moreover $F_\pi(u)=f(-u)$. The fractional FT is continuous, periodic $R^{\alpha+2\pi n}=R^\alpha$, and additive $R^{\alpha+\beta} = R^\alpha R^\beta$ with respect to the parameter α. The inverse fractional FT is the fractional FT for angle $-\alpha$. It is easy to see from Eq. (1-2) that $K_\alpha^*(x,u)= K_{-\alpha}(x,u)$.

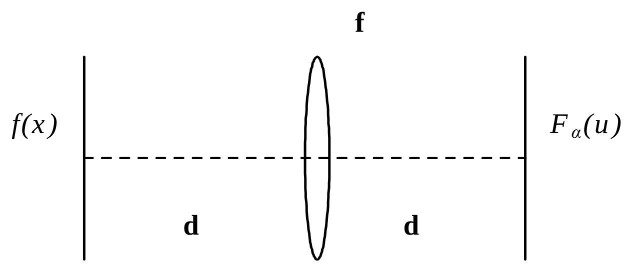

Figure 1

Optical set up for the fractional Fourier transform performance: $d=2f\sin^2(\alpha/2)$.

The fractional FT describes, in the paraxial approximation of the scalar diffraction theory, the evolution of the complex field amplitude during its propagation through quadratic refractive index medium (such as lenses, spherical mirrors etc.). The fractional FT of a complex wave amplitude *f(x)* can be performed by using different optical set ups proposed for example by A. Lohmann. One of them is represented in Fig.1. It consists on an image forming system with a thin lens having focal distance f. Depending on the distance *d* and f values we obtain at the output plane the fractional FT of the input field for the different angles α.
The main properties of the fractional FT are collected in the Table 1.

Table 1
Properties of the fractional Fourier transform

Shift theorem	$R^{\alpha}[f(x-y)](u)=R^{\alpha}[f(x)](u-y\cos\alpha)\exp(i\pi\sin\alpha(y^2\cos\alpha-2uy))$
Scaling theorem ($\tan\beta=c^2\tan\alpha$)	$R^{\alpha}[f(ct)](u)=\sqrt{\frac{\cos\beta}{\cos\alpha}}\exp\left(i\frac{\alpha-\beta}{2}\right)\exp\left(i\pi u^2\cot\alpha\left(1-\frac{\cos^2\beta}{\cos^2\alpha}\right)\right)\times R^{\beta}[f(t)]\left(\frac{u\sin\beta}{c\sin\alpha}\right)$
Parseval´s equality	$\int f(x)g^*(x)dx=\int F_{\alpha}(u)G_{\alpha}^*(u)du$
Wigner distribution rotation	$W_{F_{\alpha}}(x,u)=W_f(x\cos\alpha-u\sin\alpha,x\sin\alpha+u\cos\alpha)$

Note that the first two theorems lead to the fact that the fractional convolution (correlation) is shift and scale variant. From the Parseval's equality for the fractional FT it follows the energy conservation law: $\int |f(x)|^2\, dx=\int |F_{\alpha}(u)|^2\, du$. The last property stresses the interpretation of the parameter α as a rotation angle at the phase plane. Thus, the fractional FT produces the rotation of the Wigner distribution (WD) [4,5,7]. The Wigner distribution is a powerful tool, applied to the signal analysis and signal characterization (wave fields) not solely in optics but in astronomy, quantum mechanics, telecommunications, image treatment, etc. Moreover, the square modulii of the OFFT correspond to the WD projection associated with intensity distributions or probability and enabling direct measurements in optics and quantum mechanics.

In various areas of science, as it is in the case of optics, the intensity measurements are the only ones experimentally realizable. The recovering of the phase of a complex signal, from intensity data, is a very crucial problem in modern science and in optical computing in particular. The new alternative procedures to the classic interferometric techniques, based on waves propagation through some particular optical systems, increase the possibilities for phase recovering.

The rotation of the WD under the OFFT is the base of the so called space-phase tomography [9] allowing the entire reconstruction of the WD from intensity measurements, and, consequently, the complex amplitude of the field in the case of fully coherent fields or the correlation function in the case of partially coherent fields.

Another method for phase recovering of a fully coherent optical field (and in the case of one dimensional signals), is based on the measurements of two WD close projections, and has been proposed recently by some of us, [10].

It has been also proposed by us a filtering operation in the fractional Fourier domain, enabling phase recovering from intensity data measurement of the filtered signals, [11]. Thus the derivative of the phase can be reconstructed from the knowledge of the intensity $|f(x)|^2$ and the intensity distributions at the output of two fractional FT filters with mask u:

$$\frac{d\varphi(x)}{dx} = \frac{1}{x|f(x)|^2 2\sin 2\alpha}\left\{\left|R^{-\alpha}\left[F_{\alpha}(u)u\right](x)\right|^2 - \left|R^{\alpha}\left[F_{-\alpha}(u)u\right](x)\right|^2\right\}.$$

As a generalization, an optical field is characterized not by its WD, that is a four variables function, but by its moments. In order to estimate all the global moments, it is possible to calculate the minimum number of projections of the WD, up to an order n, [12]. In general, the optical determination of the WD or its global and local moments, from intensity data, opens new perspectives in the optical information processing.

The optical filtering in the fractional domains, differs from the classic Fourier domain, and can be applied to space-variant image recognition. The former results obtained by us show that, in most of the studied cases, the phase of the fractional Fourier transform contains more information on the image structure than the amplitude itself, [13]. This result is leading to specific design and application of phase filters and correlation in the fractional domains.

4. Applications to fractal analysis

After the introduction by Mandelbrot of the concept of fractal geometry, the study of the interaction of light with fractal structures and the discovering of fractal properties of some electromagnetic fields, have produced the development of fractal electrodynamics and fractal optics. It has been proposed the application of fractional and canonical transformations to the study and analysis of fractal fields, [14-16].

Our experimental results and numerical simulations have shown the powerful of the canonical transformations for the analysis of fractal structures, [17]. In particular, by studying the evolution of the Fresnel diffraction by fractal structure, one can construct the fractal tree reveling the hierarchical structure of the fractal, estimating the scale parameter, the fractal dimension, etc.

It has been demonstrated [17] on the example of the Cantor bars grating of level 5 partially shown in Fig. 2 a. The grating is formed by the iterative procedure of dividing a bar into three equal segments and removing its middle third.

Moving the CCD camera along the optical axis a sequence of 40 Fresnel diffraction patterns was obtained. Selecting the same small region Δy from every pattern and sticking them consecutively in growing order of z, we build up the image (see Fig. 2 b) representing a part of fractal tree, which reveals the hierarchical structure of Cantor set. The scale of gray colors indicates the intensity levels. One can observe that during propagation in free space the field associated with the fractal of level n transforms into the field associated with the structure of

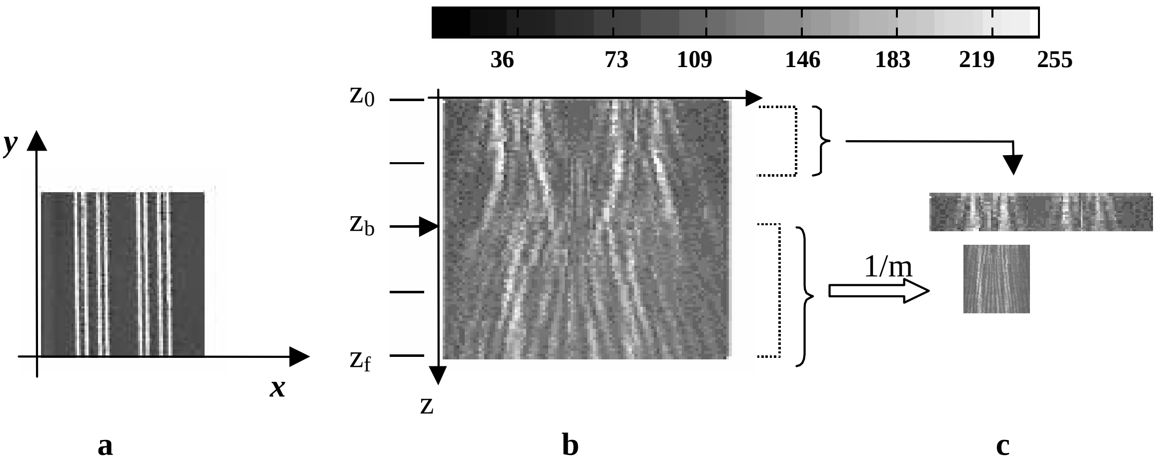

Figure 2

(a) Part of the triadic Cantor set of level 5, registered by CCD camera. (b) The hierarchical tree of the Cantor set, obtained from the observation of the intensity evolution of the diffractive patterns along the optical patterns *z*. (c) Demonstration of the self-affinity of the diffraction patterns.

of lower level *n-1*. The bifurcation point, where this transformation occurs, is indicated by z_b in Fig. 2 b. From the analysis of Fig. 2 b we can conclude that another bifurcation occurs at the distance z_0 from the object plane. It is the first registered diffraction pattern due to specific peculiarity of our experimental set up. The experimental observation of the evolution of the Fresnel diffraction patterns verifies the self-affine behavior of the diffraction patterns with respect to longitudinal *z* and transversal coordinates. Thus, the ratio between the distances, where the similar intensity distributions are observed, $m^2=z_b/z_0=7.5$ that corresponds to the scaling factor $m\approx 3$. Moreover after rescaling the coordinate *x* of the fractal tree for distances $z\in[z_b,z_f]$ by factor $1/m$ we obtain the image (see, Fig 2 c) which coincides with every of two parts of the Cantor tree for distances $z\in[z_0,z_1]$.

Since there exists a large class of natural scenes and images such as mammographies, radiographies, various textures, behaving as deterministic fractals, and, since it has been also discovered that some optical fields have intrinsically similar structure, it is now one of our main objectives to develop new optical methods for their analysis and characterization.

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