Gabor's signal expansion and the Gabor transform on a non-separable time-frequency lattice

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Abstract

Gabor's signal expansion and the Gabor transform are formulated on a general, non-separable time-frequency lattice instead of on the traditional rectangular lattice. The representation of the general lattice is based on the rectangular lattice via a shear operation, which corresponds to a description of the general lattice by means of a lattice generator matrix that has the Hermite normal form. The shear operation on the lattice is associated with simple operations on the signal, on the synthesis and the analysis window, and on Gabor's expansion coefficients; these operations consist of multiplications by quadratic phase terms. Following this procedure, the well-known biorthogonality condition for the window functions in the rectangular sampling geometry, can be directly translated to the general case. In the same way, a modified Zak transform can be defined for the non-separable case, with the help of which Gabor's signal expansion and the Gabor transform can be brought into product forms that are identical to the ones that are well known for the rectangular sampling geometry.

1 Introduction

Recently a new sampling lattice – the quincunx lattice – has been introduced [1] as a sampling geometry in the Gabor scheme, which geometry is different from the traditional rectangular sampling geometry [2]. While the basic operations in Gabor's signal expansion and the Gabor transform – time shifting and modulation – are independent operations in the case of a rectangular time-frequency lattice, these operations are no longer independent in the case of non-separable time-frequency lattices, of which the quincunx lattice forms a particular example. In order to have a higher packing density in the time-frequency plane, the non-separable lattice can be adapted to the shape of the windowed Fourier transformed analysis window. The windowed Fourier

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transform of a Gaussian function, for instance, yields circular contour lines; in this case, a quincunx (or hexagonal) lattice yields the highest packing density. Some numerical comparisons between a rectangular lattice, a quincunx lattice and a mismatched lattice for speech signals can be found in [3]. In this paper we show how results that hold for rectangular sampling (see, for instance, [4–6]) can be transformed to the general, non-separable case.

After a short introduction of Gabor's signal expansion and the Gabor transform on the well-known rectangular lattice, we define time shifting and modulation in the Gabor scheme on a general, non-separable lattice. The general lattice is described by a lattice generator matrix, and we pay special attention to the case in which the lattice generator matrix takes the Hermite normal form [7].

We continue with demonstrating how a general, non-separable lattice can be derived from a rectangular lattice by means of a simple shear operation, once the lattice generator matrix is represented in its Hermite normal form. Moreover, we show how this shear operation on the rectangular lattice corresponds to multiplication of the signal, the window functions, and the expansion coefficients by quadratic phase terms. We thus arrive at a simple procedure to transform results that are well known for the rectangular sampling geometry, to the general, non-separable case. As an example we show how the biorthogonality condition for the window functions reads in the case of a non-separable time-frequency lattice.

As a second example we show how the well-known product forms [4,5] for Gabor's signal expansion and the Gabor transform in terms of the Fourier transform of the expansion coefficients and the Zak transforms of the signal and the window functions, can be formulated in the non-separable case.

2 Gabor's signal expansion on a rectangular lattice

We start with the usual Gabor expansion [2,4,6] on a rectangular time-frequency lattice, in which case a signal $\varphi(t)$ can be expressed as a linear combination of properly shifted and modulated versions $g_{mk}(t)$ of a synthesis window $g(t)$:

$$\varphi(t) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_{mk} g_{mk}(t), \tag{1}$$

with

$$g_{mk}(t) = g(t - m\alpha T)e^{jk\beta\Omega t}. \tag{2}$$

The time step αT and the frequency step $\beta\Omega$ satisfy the relationships $\Omega T = 2\pi$ and $\alpha\beta \leq 1$; note that the factor $1/\alpha\beta$ represents the degree of oversampling, and that in his original paper [2] Gabor considered the case of critical sampling, i.e. $\alpha\beta = 1$. The expansion coefficients a_{mk} follow from sampling the windowed Fourier transform with analysis window $w(t)$,

$$\int_{-\infty}^{\infty} \varphi(t) w^*(t-\tau) e^{-j\omega t} dt,$$

on the rectangular lattice ($\tau = m\alpha T, \omega = k\beta\Omega$):

$$a_{mk} = \int_{-\infty}^{\infty} \varphi(t) w_{mk}^*(t) dt = \langle \varphi, w_{mk} \rangle, \tag{3}$$

where the asterisk (*) denotes complex conjugation. The latter relationship is known as the Gabor transform.

The synthesis window $g(t)$ and the analysis window $w(t)$ are related to each other in such a way that their shifted and modulated versions constitute two sets of functions that are biorthogonal:

$$\sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} g(t_1 - m\alpha T) w^*(t_2 - m\alpha T) e^{jk\beta\Omega(t_1 - t_2)} = \delta(t_1 - t_2). \tag{4}$$

The biorthogonality relation (4) leads immediately to the equivalent but simpler expression

$$\frac{T}{\beta} \sum_{m=-\infty}^{\infty} g(t - m\alpha T) w^* \left(t - \left[m + \frac{k}{\alpha\beta} \right] \alpha T \right) = \delta_k, \tag{5}$$

where δ_k is the Kronecker delta. In the case of critical sampling, i.e., $\alpha\beta = 1$, Eq. (5) reduces to

$$\alpha T \sum_{m=-\infty}^{\infty} g(t - m\alpha T) w^*(t - [m+k]\alpha T) = \delta_k.$$

3 Gabor's signal expansion on a non-separable lattice

The rectangular (or separable) lattice that we considered in the previous section can be obtained by integer combinations of two orthogonal vectors

$$\boldsymbol{v}_0 = [\alpha T,\ 0]^T \quad \text{and} \quad \boldsymbol{v}_1 = [0,\ \beta\Omega]^T .$$

We thus express the lattice Λ in the form

$$\Lambda = \{ n_0 \boldsymbol{v}_0 + n_1 \boldsymbol{v}_1 | n_0, n_1 \in \mathbb{Z} \}. \tag{6}$$

We now consider Gabor's signal expansion on a time-frequency lattice that is no longer separable. We call a time-frequency lattice non-separable, if the time-shifts and the modulations in the shifted and modulated windows $g_{mk}(t)$ and $w_{mk}(t)$ are not independent operations anymore. Such a lattice is obtained by integer combinations of two linearly independent, but no longer orthogonal vectors, which we express in the forms

$$D\boldsymbol{v}_0 = [a\alpha TD,\ c\beta\Omega]^T \quad \text{and} \quad D\boldsymbol{v}_1 = [b\alpha TD,\ d\beta\Omega]^T, \tag{7}$$

with a, b, c and d integers, $\alpha, \beta \in \mathbb{R}^+$, $D = ad - bc$, and $\Omega T = 2\pi$ again. The first component in the vectors $\boldsymbol{v}_0$ and $\boldsymbol{v}_1$ corresponds to a time-shift $a\alpha T$ and $b\alpha T$, respectively, while the second component corresponds to a modulation by a frequency $c\beta\Omega/D$ and $d\beta\Omega/D$, respectively.

Each point $\boldsymbol{\lambda} \in \Lambda$ in the time-frequency plane can be obtained by a matrix-vector product [cf. Eqs. (6) and (7)]

$$\forall_{\boldsymbol{\lambda}\in\Lambda}\exists_{\boldsymbol{n}\in\mathbb{Z}^2}\ \boldsymbol{\lambda} = \boldsymbol{U}\boldsymbol{L}\boldsymbol{n}, \quad \text{with} \quad \boldsymbol{U} = \frac{1}{D}\begin{bmatrix} \alpha TD & 0 \\ 0 & \beta\Omega \end{bmatrix} \quad \text{and} \quad \boldsymbol{L} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The column vectors $\boldsymbol{v}_0$ and $\boldsymbol{v}_1$ are the columns of the lattice generator matrix $\boldsymbol{UL}$. Note, moreover, that D is equal to the determinant of the matrix $\boldsymbol{L}$. We assume that the integers a and b have no common divisors, and that the same holds for the integers c and d; hence $\gcd(a, b) = 1$ and $\gcd(c, d) = 1$. A possible common divisor can be unified with α and β. The area of a cell (a parallelogram) in the time-frequency plane is equal to the determinant of the lattice generator matrix $\boldsymbol{UL}$, which determinant is equal to $\alpha\beta\Omega T = \alpha\beta 2\pi$. It is well known that the set of shifted and modulated versions of the window is not complete in the case that $\alpha\beta > 1$. The equality $\alpha\beta = 1$ corresponds to critical sampling, and $\alpha\beta < 1$ corresponds to oversampling. Note that we only consider those lattice generator matrices that can be decomposed into the two matrices $\boldsymbol{U}$ and $\boldsymbol{L}$. The corresponding lattices have samples on the time- and frequency-axes and are therefore suitable for a discrete-time approach, as well.

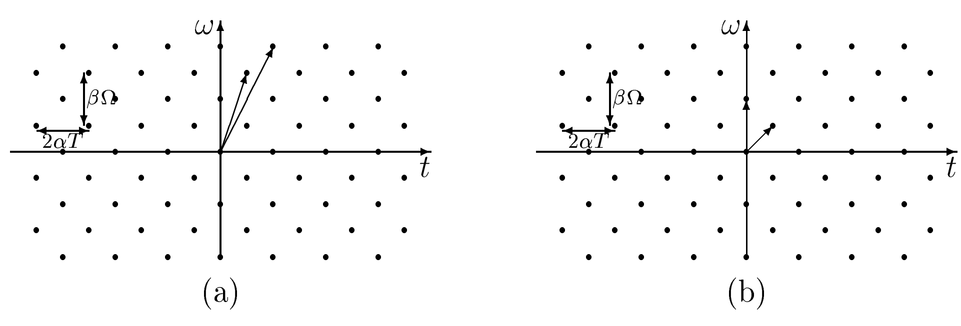

Fig. 1. The quincunx lattice and two possible combinations of generator vectors.

It is clear that for a given matrix $\boldsymbol{U}$, there are a lot of matrices $\boldsymbol{L}$ that generate the same lattice Λ. As an example, we have plotted in Figs. 1a and 1b two possible combinations of lattice generators in the case of a quincunx lattice.

One form, the Hermite normal form [7], see Fig. 1b, is very interesting; it reads

$$\boldsymbol{L}' = \begin{bmatrix} 1 & 0 \\ -r & D \end{bmatrix},$$

where the integer $-r$ equals $h_0 c + h_1 d$, with the integers h_0 and h_1 such that $h_0 a + h_1 b = 1$. Note that these integers h_0 and h_1 exist, since $\gcd(a, b) = 1$, and that they can be obtained by the Euclidean algorithm (see, for instance, [8]). The columns of the matrix $\boldsymbol{U}\boldsymbol{L}'$ are equal to $[\alpha T,\ -r\beta\Omega/D]^T$ and $[0,\ \beta\Omega]^T$, respectively. Consequently, the shifted and modulated versions $g_{mk}(t)$ of the synthesis window $g(t)$ on the lattice Λ take the form

$$g_{mk}(t) = g(t - m\alpha T)e^{-jmr\beta\Omega t/D}e^{jk\beta\Omega t}. \tag{8}$$

The shifted and modulated versions $w_{mk}(t)$ of the analysis window $w(t)$ are defined similarly. With this modified definition (8) of the set of shifted and modulated window functions, the original expressions for Gabor's signal expansion (1) and the Gabor transform (3) remain valid in the non-separable case. Note that in the case of a rectangular lattice, i.e., $D = 1$ and $r = 0$, Eq. (8) indeed reduces to Eq. (2).

4 From the separable to the non-separable case

A lot of algorithms to calculate the Gabor transform and Gabor's signal expansion for the rectangular case in an efficient way have been developed in the last two decades. Therefore it would be very interesting if these algorithms could be reused for the non-separable case. We will show that this is possible by rewriting Eq. (8) as follows

$$\begin{aligned} g_{mk}(t) = & g(t - m\alpha T)e^{jr\beta\Omega(t - m\alpha T)^2/2\alpha TD} \\ & \times e^{-jr\beta\Omega t^2/2\alpha TD}e^{-j\pi\alpha\beta r m^2/D}e^{jk\beta\Omega t}. \end{aligned}$$

The same is done for the shifted and modulated versions $w_{mk}(t)$ of the analysis window $w(t)$. Substitution of this into the Gabor expansion yields

$$\varphi'(t) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \langle \varphi', w'_{mk} \rangle g'_{mk}(t) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a'_{mk} g'_{mk}(t),$$

where

$$\begin{aligned}\varphi'(t) &= \varphi(t)e^{jr\beta\Omega t^2/2\alpha TD},\\ g'(t) &= g(t)e^{jr\beta\Omega t^2/2\alpha TD},\\ w'(t) &= w(t)e^{jr\beta\Omega t^2/2\alpha TD}\end{aligned} \tag{9}$$

and

$$a'_{mk} = a_{mk}e^{-j\pi\alpha\beta rm^2/D}. \tag{10}$$

The shifted and modulated versions of the synthesis window $g'(t)$ and the analysis window $w'(t)$ are now on a rectangular lattice [cf. Eq. (2)], i.e.,

$$g'_{mk}(t) = g'(t - m\alpha T)e^{jk\beta\Omega t} \quad \text{and} \quad w'_{mk}(t) = w'(t - m\alpha T)e^{jk\beta\Omega t}.$$

Consequently, algorithms for the rectangular lattice can be reused, but the signal $\varphi(t)$, the windows $g(t)$ and $w(t)$ have to be pre-multiplied by a quadratic phase term. After calculating the Gabor coefficients a'_{mk}, these Gabor coefficients a'_{mk} have to be post-multiplied by a quadratic phase term to obtain the Gabor coefficients a_{mk} for the non-separable lattice. A similar procedure holds for the Gabor expansion. In this case, after reconstructing the signal $\varphi'(t)$ by using an efficient algorithm for a rectangular lattice, the signal $\varphi'(t)$ has to be post-multiplied by a quadratic phase term to obtain the original signal $\varphi(t)$.

As an example, we take the Gabor expansion on a quincunx lattice. Here we have $r = -1$ and $D = 2$. This quincunx lattice is depicted in Fig. 1b. The lattice generator matrix $\boldsymbol{UL}'$ has the form

$$\begin{bmatrix}\alpha T & 0\\ \frac{1}{2}\beta\Omega & \beta\Omega\end{bmatrix}.$$

The columns of the matrix $\boldsymbol{UL}'$ are the generator vectors of the quincunx lattice. These vectors are depicted in Fig. 1b, as well. Substituting $r = -1$ and $D = 2$ into Eq. (9) yields

$$\begin{aligned}\varphi'(t) &= \varphi(t)e^{-j\beta\Omega t^2/4\alpha T},\\ g'(t) &= g(t)e^{-j\beta\Omega t^2/4\alpha T},\\ w'(t) &= w(t)e^{-j\beta\Omega t^2/4\alpha T}\end{aligned}$$

and substituting $r = -1$ and $D = 2$ into Eq. (10) yields

$$a'_{mk} = a_{mk}e^{j\frac{1}{2}\pi\alpha\beta m^2}.$$

Following the procedure outlined above and substituting the ‘primed’ window functions into the biorthogonality relation [cf. Eq. (5)], it is not difficult to show that, in the case of a quincunx sampling geometry, the biorthogonality relation takes the form

$$\frac{T}{\beta}\sum_{m=-\infty}^{\infty} g(t-m\alpha T)w^*\left(t-\left[m+\frac{k}{\alpha\beta}\right]\alpha T\right)(-1)^{mk}=\delta_k,$$

while in the general non-separable case this relation reads

$$\frac{T}{\beta}\sum_{m=-\infty}^{\infty} g(t-m\alpha T)w^*\left(t-\left[m+\frac{k}{\alpha\beta}\right]\alpha T\right)e^{-j(2\pi r/D)mk}=\delta_k.$$

5 The Fourier transform and the Zak transform in the case of a non-separable lattice

It is well known (see, for instance, [4,6]) that in the case of critical sampling, $\alpha\beta=1$, Gabor's signal expansion (1) and the Gabor transform (3) can be transformed into product form. We therefore need the Fourier transform $\bar{a}(t/T,\omega/\Omega)$ of the two-dimensional array of Gabor coefficients a_{mk}, defined by

$$\bar{a}(x,y)=\sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} a_{mk}e^{-j2\pi(my-kx)}, \tag{11}$$

and the Zak transforms $\tilde{\varphi}(x\alpha T,y\Omega/\alpha;\alpha T)$, $\tilde{g}(x\alpha t,y\Omega/\alpha;\alpha T)$, and $\tilde{w}(x\alpha T,y\Omega/\alpha;\alpha T)$ of the signal $\varphi(t)$ and the window functions $g(t)$ and $w(t)$, respectively, where the Zak transform $\tilde{f}(t,\omega;\tau)$ of a function $f(t)$ is defined as (see, for instance, [4,6,9,10])

$$\tilde{f}(t,\omega;\tau)=\sum_{n=-\infty}^{\infty} f(t+n\tau)e^{-jn\tau\omega}. \tag{12}$$

Upon substituting from the Fourier transform (11) and the Zak transforms [cf. Eq. (12)] into Eqs. (1) and (3), it is not too difficult to show that Gabor's signal expansion (1) can be transformed into the product form

$$\tilde{\varphi}\left(x\alpha T,y\frac{\Omega}{\alpha};\alpha T\right)=\bar{a}(x,y)\tilde{g}\left(x\alpha T,y\frac{\Omega}{\alpha};\alpha T\right),$$

while the Gabor transform (3) can be transformed into the product form

$$\bar{a}(x,y)=\alpha T\tilde{\varphi}\left(x\alpha T,y\frac{\Omega}{\alpha};\alpha T\right)\tilde{w}^*\left(x\alpha T,y\frac{\Omega}{\alpha};\alpha T\right).$$

In the case of oversampling by a rational factor, $\alpha\beta=q/p\leq 1$, with p and q positive integers, $p\geq q\geq 1$, use of the Fourier transform (11) and the Zak transform (12) leads to the sum-of-products forms [5]

$$\tilde{\varphi}\left((x+s)\frac{\alpha pT}{q}, y\frac{\Omega}{\alpha}; \alpha pT\right) =$$
$$\frac{1}{p}\sum_{r=0}^{p-1} \bar{a}\left(x, y+\frac{r}{p}\right) \tilde{g}\left((x+s)\frac{\alpha pT}{q}, \left[y+\frac{r}{p}\right]\frac{\Omega}{\alpha}; \alpha T\right), \qquad (13)$$

and

$$\bar{a}\left(x, y+\frac{r}{p}\right) =$$
$$\frac{\alpha pT}{q}\sum_{s=0}^{q-1} \tilde{\varphi}\left((x+s)\frac{\alpha pT}{q}, y\frac{\Omega}{\alpha}; \alpha pT\right) \tilde{w}^*\left((x+s)\frac{\alpha pT}{q}, \left[y+\frac{r}{p}\right]\frac{\Omega}{\alpha}; \alpha T\right). \qquad (14)$$

We now combine the p functions

$$a_r(x,y) = \bar{a}\left(x, y+\frac{r}{p}\right) \quad (r = 0, 1, \ldots, p-1)$$

into the p-dimensional column vector of functions

$$\boldsymbol{a} = \left[a_0(x,y), a_1(x,y), \ldots, a_{p-1}(x,y)\right]^T$$

and, likewise, the q functions

$$\varphi_s(x,y) = \tilde{\varphi}\left((x+s)\frac{\alpha pT}{q}, y\frac{\Omega}{\alpha}; \alpha pT\right) \quad (s = 0, 1, \ldots, q-1)$$

into the q-dimensional column vector of functions

$$\boldsymbol{\phi} = \left[\varphi_0(x,y), \varphi_1(x,y), \ldots, \varphi_{q-1}(x,y)\right]^T.$$

Moreover, we combine the $q \times p$ functions

$$g_{sr} = \tilde{g}\left((x+s)\frac{\alpha pT}{q}, \left[y+\frac{r}{p}\right]\frac{\Omega}{\alpha}; \alpha T\right)$$

and

$$w_{sr} = \tilde{w}\left((x+s)\frac{\alpha pT}{q}, \left[y+\frac{r}{p}\right]\frac{\Omega}{\alpha}; \alpha T\right),$$

where $r = 0, 1, \ldots, p-1$ and $s = 0, 1, \ldots, q-1$, into the $(q \times p)$-dimensional matrices of functions

$$\boldsymbol{G} = \begin{bmatrix} g_{00}(x,y) & g_{01}(x,y) & \dots & g_{0,p-1}(x,y) \\ g_{10}(x,y) & g_{11}(x,y) & \dots & g_{1,p-1}(x,y) \\ . & . & \dots & . \\ . & . & \dots & . \\ g_{q-1,0}(x,y) & g_{q-1,1}(x,y) & \dots & g_{q-1,p-1}(x,y) \end{bmatrix}$$

and

$$\boldsymbol{W} = \begin{bmatrix} w_{00}(x,y) & w_{01}(x,y) & \dots & w_{0,p-1}(x,y) \\ w_{10}(x,y) & w_{11}(x,y) & \dots & w_{1,p-1}(x,y) \\ . & . & \dots & . \\ . & . & \dots & . \\ w_{q-1,0}(x,y) & w_{q-1,1}(x,y) & \dots & w_{q-1,p-1}(x,y) \end{bmatrix},$$

respectively. With the help of these vectors and matrices, the sum-of-products forms (13) and (14) can be expressed in the elegant matrix-vector products

$$\boldsymbol{\phi} = \frac{1}{p}\boldsymbol{G}\boldsymbol{a} \tag{23}$$

and

$$\boldsymbol{a} = \frac{\alpha p T}{q}\boldsymbol{W}^*\boldsymbol{\phi}, \tag{24}$$

respectively, where, as usual, the asterisk (*) denotes conjugation *and* transposition.

The matrix-vector products (23) and (24) hold in the case of the rationally oversampled, separable-lattice geometry. In the non-separable case they hold as well, but then for the 'primed' vectors $\boldsymbol{a}'$ and $\boldsymbol{\phi}'$ and the 'primed' matrices $\boldsymbol{G}'$ and $\boldsymbol{W}'$, derived from the 'primed' functions (9) and (10). If we substitute from these 'primed' functions into the Fourier transform (11) and the Zak transform (12), the 'primed' Fourier transform takes the form

$$\begin{aligned} \bar{a}'(x,y) &= \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} a'_{mk} e^{-j2\pi(my-kx)} \\ &= \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} a_{mk} e^{-j\alpha\beta(\pi r/D)m^2} e^{-j2\pi(my-kx)}, \end{aligned}$$

while the 'primed' Zak transform takes the form

$$
\begin{aligned}
&\tilde{f}'(t,\omega;\tau) = \sum_{n=-\infty}^{\infty} f'(t+n\tau)e^{-jn\tau\omega} \\
&= \sum_{n=-\infty}^{\infty} f(t+n\tau)e^{jr\beta\Omega(t+n\tau)^2/2\alpha TD}e^{-jn\tau\omega} \\
&= e^{j(\beta/\alpha)(\pi r/D)(t/T)^2} \\
&\times \sum_{n=-\infty}^{\infty} f(t+n\tau)e^{j(\beta/\alpha)(\pi r/D)(n\tau/T)^2}e^{-jn\tau(\omega - r\beta\Omega t/\alpha TD)}.
\end{aligned}
$$

6 Conclusions

We have shown how the Gabor scheme for the rectangular (or separable) lattice can be extended to the general, non-separable lattice in a structured way; this was achieved by describing the non-separable lattice by means of a lattice generator matrix. We have written the generator matrix in the Hermite normal form to obtain a shear representation on the shifted and modulated windows, which shear representation then led to a modification of the rectangular Gabor scheme and resulted in the Gabor scheme on a non-separable lattice. Efficient algorithms that are well known for the rectangular case, could thus easily be applied to the non-separable case: we have shown that the shear operation merely involves pre- and post-multiplications by quadratic phase terms.

As examples we have formulated the biorthogonality condition for the analysis and the synthesis window in the non-separable sampling geometry explicitly, and we have formulated product forms for Gabor's signal expansion and the Gabor transform in terms of the (modified) Fourier transform of Gabor's expansion coefficients and the (modified) Zak transforms of the signal and the window functions.

Finally we note that, in general, other transformations can be used to transform the results in the rectangular Gabor scheme to the non-separable case. In particular we want to mention the fractional Fourier transform [11] in this context [12]. The fractional Fourier transform is a generalization of the classical Fourier transform and can be seen as a rotation by an angle θ in the time-frequency plane. The non-separable lattice is now obtained by a rotation instead of a shear.

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