# Power filtering of $n$-th order in the fractional Fourier domain

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**Abstract.** The main properties of the power filtering operation in the fractional Fourier domain and its relationship to the differentiation operation are considered. The application of linear power filtering for solving the phase retrieval problem from only intensity distributions is proposed. The optical configuration for experimental realization of the method is discussed.

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The usefulness of the Fourier transform (FT) is often related to the simplification of the differentiation operation, which permits to solve a number of differential equations. Thus, power filtering in the Fourier domain is related to the signal derivative as

$$R^{-\pi/2}\left[F_{\pi/2}(u)u^n\right](x) = (-i)^n \frac{d^n f(x)}{dx^n}, \tag{1}$$

where $F_{\pi/2}(u)$ is the FT of $f(x)$ and $R^{-\pi/2}[\cdot]$ denotes the inverse Fourier operation, cf. Eq. (2). Note also that the first and the second derivatives – or power filtering for $n = 1, 2$ – are widely used for edge characterization.

The introduction of the fractional FT to quantum mechanics, paraxial optics, and signal/image processing [1, 2, 3, 4] has permitted to simplify the description of the corresponding systems and to develop new methods for signal analysis [5]. Thus, the fractional correlation and convolution operations related to filtering in the fractional Fourier domain were extensively studied during the past years [6, 7, 8]. Nevertheless, the question of power filtering in the fractional Fourier domain and its relationship to the differentiation operation does not seem to have been considered in the literature and is the subject of this paper.

The fractional FT is a generalization of the ordinary FT. Its kernel depends on a parameter that can be interpreted as a rotation angle in phase space. The fractional FT of a function $f(x)$ for the angle $\alpha$ is defined as [4]

$$R^{\alpha}\left[f(x)\right](u) = F_{\alpha}(u) = \int_{-\infty}^{\infty} f(x)K(\alpha, x, u)dx, \tag{2}$$

where the kernel is given by

$$K(\alpha, x, u) = \frac{\exp(i\alpha/2)}{\sqrt{i2\pi \sin\alpha}} \exp\left[i\frac{(x^2+u^2)\cos\alpha - 2xu}{2\sin\alpha}\right]. \tag{3}$$

The kernel of the fractional FT is periodic in $\alpha$. For $\alpha = 0$ the fractional FT corresponds to the identity operation, $F_0(u) = f(u)$, and for $\alpha = \pi/2$ and $\alpha = 3\pi/2$ it reduces to the FT and the inverse FT, respectively; moreover $F_\pi(u) = f(-u)$. The fractional FT is additive with respect to $\alpha$, $R^{\alpha+\beta} = R^\alpha R^\beta$, and – as can easily be seen from Eq. (3) – the kernel $K(-\alpha, x, u)$ of the inverse fractional FT $R^{-\alpha}$ is equal to $K^*(\alpha, x, u)$.

The kernel of the fractional FT is, except for a phase shift $\alpha/2$, a propagator of the nonstationary Schrödinger equation for a harmonic oscillator (here dimensionless variables are used):

$$\left[\frac{\partial}{\partial\alpha} - \frac{i}{2}\frac{\partial^2}{\partial x^2} + \frac{i}{2}x^2\right]\Psi(x,\alpha) = 0.$$

The same equation describes in the paraxial approximation of the scalar diffraction theory the complex field amplitude $\Psi(r, z)$ during its propagation through a medium with a quadratic index of refraction $n^2 = n_0^2(1 - g^2r^2)$ [3, 5]. In this case we have $\alpha = gz$ and $x = \sqrt{kg}r$, where $r$ and $z$ are the transversal and longitudinal coordinates, respectively, and $k$ is the wave number. The fractional FT of the complex field amplitude as well as the ordinary FT can also be performed by using a thin lens with focal distance $\mathbf{f}$, with the input and output planes located at distances $d = 2\mathbf{f}\sin^2\alpha/2$ before and after the lens, respectively [9].

Based on the additivity property of the fractional FT, Eq. (1) can easily be generalized as

$$R^{-\pi/2}\left[F_{\alpha+\pi/2}(u)u^n\right](x) = (-i)^n\frac{d^nF_\alpha(x)}{dx^n}. \tag{4}$$

Then the ordinary $n$-power filtering operation performed in any $\beta = \alpha + \pi/2$ fractional Fourier domain is related to the $n$th derivative of the $\alpha$-fractional FT. Moreover, from the additivity property it also follows that $R^\beta\left[F_\alpha(u)g(u)\right](x) = R^{\beta+\alpha}\left[R^{-\alpha}\left[F_\alpha(u)g(u)\right](y)\right](x)$ with, in our case, $g(u) = u^n$.

In this paper we restrict ourselves to considering the operation $R^{-\alpha}\left[F_\alpha(u)u^n\right](x)$, because it corresponds to a logical generalization of the $n$-th order power filtering [see Eq. (1)] in the fractional FT domain. Note that the operation $R^\alpha[f(u)u^n](x)$ for different integers $n$ was considered in [1, 2, 4]. Since

$$\int K(-\alpha, x, u)K(\alpha, \xi, u)u^n du = (-i\sin\alpha)^n\exp[-i(x^2-\xi^2)\cot\alpha/2]\ \delta^{(n)}(x-\xi),$$

we have

$$\begin{aligned}
R^{-\alpha}\left[F_\alpha(u)u^n\right](x) &= \int_{-\infty}^{\infty}K(-\alpha,x,u)F_\alpha(u)u^n du \\
&= \int_{-\infty}^{\infty}f(\xi)d\xi\int_{-\infty}^{\infty}K(-\alpha,x,u)K(\alpha,\xi,u)u^n du \\
&= (-i\sin\alpha)^n\exp(-ix^2\cot\alpha/2)\ \frac{d^n}{dx^n}f(x)\exp(ix^2\cot\alpha/2) \\
&= (-i\sin\alpha)^n\exp(-ix^2\cot\alpha/2)\sum_{k=0}^{n}\begin{pmatrix} n \\ k\end{pmatrix}\frac{d^{n-k}\exp(ix^2\cot\alpha/2)}{dx^{n-k}}\frac{d^kf(x)}{dx^k} \\
&= (i\sin\alpha)^n\sum_{k=0}^{n}\begin{pmatrix} n \\ k\end{pmatrix}(-1)^k\left(\sqrt{\cot\alpha/2i}\right)^{n-k}H_{n-k}\left(x\sqrt{\cot\alpha/2i}\right)\frac{d^kf(x)}{dx^k},
\end{aligned} \tag{5}$$

where $H_n(z) = (-1)^n \exp(z^2) d^n \exp(-z^2)/dz^n$ are the Hermite polynomials [10]. This equation is a generalization of Eq. (1) in the fractional Fourier domain. Alternatively, based on the multiplication rule derived in [1, 2], $R^\alpha[f(u)u^n](x) = (x\cos\alpha + i\sin\alpha\,\mathfrak{D})^n R^\alpha[f(u)](x)$, the $n$-th order power filtering operation can be written in the form

$$R^{-\alpha}[F_\alpha(u)u^n](x) = (x\cos\alpha - i\sin\alpha\,\mathfrak{D})^n f(x), \tag{6}$$

where $\mathfrak{D}$ denotes the differential operator $d/dx$. It is easy to see from Eqs. (5) and (6) that

$$R^{-\alpha}\left[F_\alpha(u)u^n\right](x) = (-1)^n R^{-\alpha-\pi}\left[F_{\alpha+\pi}(u)u^n\right](x). \tag{7}$$

Moreover, for real $f(x)$ we obtain that $\left\{R^{-\alpha}\left[F_\alpha(u)u^n\right](x)\right\}^* = R^\alpha\left[F_{-\alpha}(u)u^n\right](x)$, and therefore $\left|R^{-\alpha}\left[F_\alpha(u)u^n\right](x)\right|^2 = \left|R^\alpha\left[F_{-\alpha}(u)u^n\right](x)\right|^2$.

We start with the case $n = 1$, which is related to the first derivative of the signal $f(x)$. Filtering in the fractional Fourier domain with mask $u$ yields the following result

$$\begin{aligned} R^{\mp\alpha}\left[F_{\pm\alpha}(u)u\right](x) &= \cos\alpha\; xf(x) \mp i\sin\alpha\;\frac{df(x)}{dx} \\ &= \cos\alpha\; R^0\left[F_0(u)u\right](x) \pm \sin\alpha\; R^{-\pi/2}\left[F_{\pi/2}(u)u\right](x), \end{aligned} \tag{8}$$

which can be considered as a weighted sum of corresponding filtering results in the position and Fourier domains.

Taking Eq. (8) for two different angles $\alpha$ and $\beta$, we can find the first derivative $df/dx = iR^{-\pi/2}\left[F_{\pi/2}(u)u\right](x)$ and the product $xf(x) = R^0\left[F_0(u)u\right](x)$ as:

$$\begin{aligned} -i\frac{df(x)}{dx} &= \frac{1}{\sin(\alpha-\beta)}\left\{\cos\beta\; R^{-\alpha}\left[F_\alpha(u)u\right](x) - \cos\alpha\; R^{-\beta}\left[F_\beta(u)u\right](x)\right\}, \\ xf(x) &= \frac{-1}{\sin(\alpha-\beta)}\left\{\sin\beta\; R^{-\alpha}\left[F_\alpha(u)u\right](x) - \sin\alpha\; R^{-\beta}\left[F_\beta(u)u\right](x)\right\}. \end{aligned} \tag{9}$$

In the particular case $\beta = \alpha + \pi/2$, Eqs. (9) can be written in the form of a matrix-vector product as

$$\begin{pmatrix} R^{-\pi/2}\left[F_{\pi/2}(u)u\right](x) \\ R^0\left[F_0(u)u\right](x) \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} R^{-\alpha-\pi/2}\left[F_{\alpha+\pi/2}(u)u\right](x) \\ R^{-\alpha}\left[F_\alpha(u)u\right](x) \end{pmatrix}. \tag{10}$$

The two expressions in the right-hand side of Eq. (10) are invariant with respect to $\alpha$, and the same holds for the sum of the intensities in two Fourier-conjugated domains:

$$\left|R^{-\alpha-\pi/2}\left[F_{\alpha+\pi/2}(u)u\right](x)\right|^2 + \left|R^{-\alpha}\left[F_\alpha(u)u\right](x)\right|^2 = |xf(x)|^2 + \left|\frac{df(x)}{dx}\right|^2. \tag{11}$$

For the particular case $\beta = -\alpha$, or, equivalently, for the case $\beta = \pi - \alpha$ and using Eq. (7), Eqs. (9) yield

$$\begin{aligned} -i\frac{df(x)}{dx} &= \frac{1}{2\sin\alpha}\left\{R^{-\alpha}\left[F_\alpha(u)u\right](x) - R^{\alpha}\left[F_{-\alpha}(u)u\right](x)\right\}, \\ xf(x) &= \frac{1}{2\cos\alpha}\left\{R^{-\alpha}\left[F_\alpha(u)u\right](x) + R^{\alpha}\left[F_{-\alpha}(u)u\right](x)\right\}. \end{aligned} \tag{12}$$

Note that the expressions in the right-hand side of Eqs. (12) are also invariant with respect to $\alpha$.

From Eq. (8) we conclude that the sum of the squared moduli of the filtered signal – first-order power filtered in the $+\alpha$ and the $-\alpha$ fractional Fourier domain – is related to the squared moduli of the signal derivative and the signal intensity as

$$\frac{1}{2}\left\{\left|R^{-\alpha}\left[F_{\alpha}(u)u\right](x)\right|^2+\left|R^{\alpha}\left[F_{-\alpha}(u)u\right](x)\right|^2\right\}=\cos^2\alpha\,|xf(x)|^2+\sin^2\alpha\left|\frac{df(x)}{dx}\right|^2 \quad (13)$$

and that their difference is connected to the amplitude $|f(x)|$ and the phase $\varphi(x) = \arg f(x)$ as

$$\frac{1}{2\sin 2\alpha}\left\{\left|R^{-\alpha}\left[F_{\alpha}(u)u\right](x)\right|^2-\left|R^{\alpha}\left[F_{-\alpha}(u)u\right](x)\right|^2\right\}=x\,|f(x)|^2\,\frac{d\varphi(x)}{dx}. \quad (14)$$

We stress again the $\alpha$-invariance of the expression in Eq. (14 ). Equation (14) can be applied for solving the phase retrieval problem, at least for $\alpha \neq n\pi/2$. Indeed, the phase derivative $d\varphi/dx$, and therefore the phase $\varphi(x)$ up to a constant term, can be reconstructed from the knowledge of the intensity $|f(x)|^2$ and the intensity distributions at the output of two fractional FT filters with mask $u$:

$$\frac{d\varphi(x)}{dx}=\frac{1}{2\sin 2\alpha\;x\,|f(x)|^2}\left\{\left|R^{-\alpha}\left[F_{\alpha}(u)u\right](x)\right|^2-\left|R^{\alpha}\left[F_{-\alpha}(u)u\right](x)\right|^2\right\}. \quad (15)$$

In order to illustrate the efficiency of this method, numerical simulations have been carried out for the signal with Gaussian amplitude and sinusoidal phase $f(x) = \exp[-6x^2]\exp[i\,0.025\sin(40\pi x)]$, where $x \in [-0.5; 0.5]$. In Fig. 1a we have depicted the original phase derivative, $\pi\cos 40\pi x$, whereas the reconstructed derivatives – determined using Eq. (15) – are shown in Figs. 1b through 1f for several values of the fractional angle $\alpha$. The fractional FTs have been calculated following the procedure proposed in [11]. We remark that the reconstruction is good for those values of $x$ for which $x|f(x)|^2$ is not too small. Note moreover that in the central region $x \in [-0.1; 0.1]$ the quality of reconstruction is better for small angles, whereas for $|x| \in [0.3; 0.5]$ the better results are observed for $\alpha \geq \pi/4$. We have limited the number of sensor points to 128, in order to make the simulations closer to the experiments. In general, increasing the number of sensor points improves the reconstruction quality.

As it is relatively simple to perform a fractional FT in optics, the proposed method can be used for the phase retrieval of optical fields from intensity distribution data. The hybrid opto-electronic processor realizing the procedure (15) is shown in Fig. 2. The optical beam is first divided by a beam splitter into two parts, which then propagate through optical systems similar to the one represented in Fig. 3. The intensity distributions at the output of these systems are registered by CCD cameras and proceed to a computer where the additional algebraic operations are made. The optical set-up (see Fig. 3) proposed for the experimental measurements of the two intensity distributions $\left|R^{-\alpha}\left[F_{\alpha}(u)u\right](x)\right|^2$ and $\left|R^{\alpha}\left[F_{-\alpha}(u)u\right](x)\right|^2$ consists of two thin lenses with focal distances $\mathbf{f}_1$ and $\mathbf{f}_2$, and a filter with transfer function $g(u) = u$; negative values of $g(u)$ (for $u < 0$) are realized by placing a $\pi$ phase step over the negative half of the $u$ plane. The configuration parameters (distances $d_1$ and $d_2$, and focal distances $\mathbf{f}_1$ and $\mathbf{f}_2$ of the two lenses) for each of the two optical systems in Fig. 2 are chosen in such a way that $\beta = \alpha$ for one beam, and $\beta = 2\pi - \alpha$ for the other. The optical set-up in Fig. 3 can be considered as a cascade of two subsystems, one from the input plane to the filtering plane, and one from the filtering plane to the output plane, with the amplitude plate in the filtering plane. The first subsystem performs a fractional FT for the angle $\beta$, if the relation $d_1 = 2\,\mathbf{f}_1\sin^2\beta/2$ is satisfied [9]; likewise, the second subsystem performs an inverse fractional FT, if the

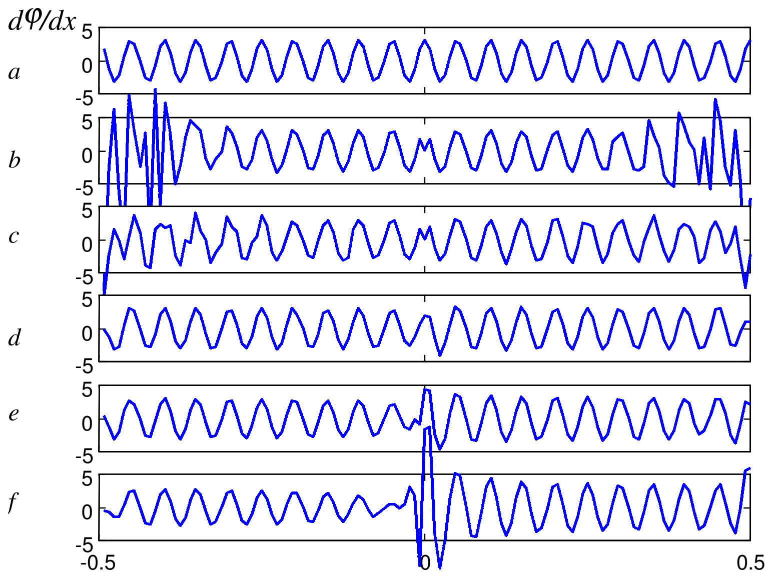


**Figure 1.** Original and reconstructed phase derivative of the signal $f(x) = \exp[-6x^2 + i\,0.025\sin(40\pi x)]$ for different values of the fractional angle $\alpha$: (a) $\alpha = 0$, original phase derivative; (b) $\alpha = 0.1\,\pi/2$; (c) $\alpha = 0.25\,\pi/2$; (d) $\alpha = 0.5\,\pi/2$; (e) $\alpha = 0.75\,\pi/2$; (f) $\alpha = 0.9\,\pi/2$.

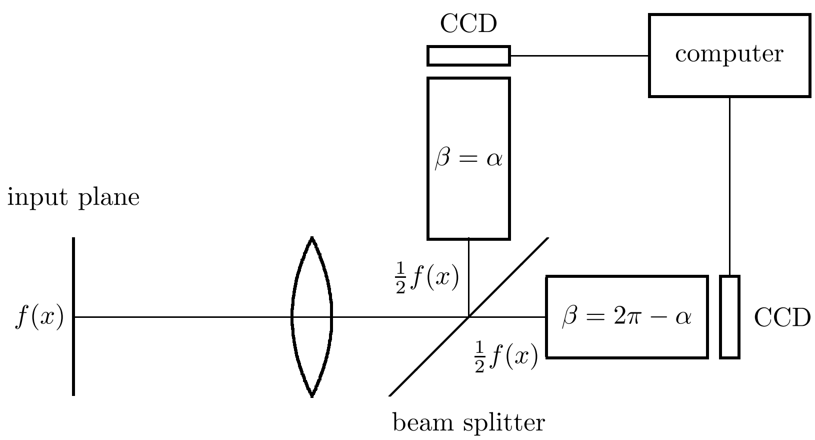


**Figure 2.** The optical field $f(x)$ in the input plane is imaged through a beam splitter onto two optical, fractional power-filtering systems (see Fig. 3) with fractional angles $\beta = \alpha$ and $\beta = 2\pi - \alpha$, whose output fields are captured by CCD cameras for further electronic processing.

relation $d_2 = 2\,\mathbf{f}_2 \sin^2(\pi - \beta/2)$ holds. Thus, the optical field $f(x)/2$ is first fractional Fourier transformed to yield $F_\beta(u)/2 = R^\beta\,[f(x)/2]\,(u)$ just before the filtering mask. After passing through the filter, the optical field reads $F_\beta(u)u/2$. Finally, the inverse fractional FT system yields in its output plane the optical field $R^{-\beta}\,[F_\beta(u)u/2]\,(x)$, whose intensity distribution is registered by a CCD camera. Linear filtering in the

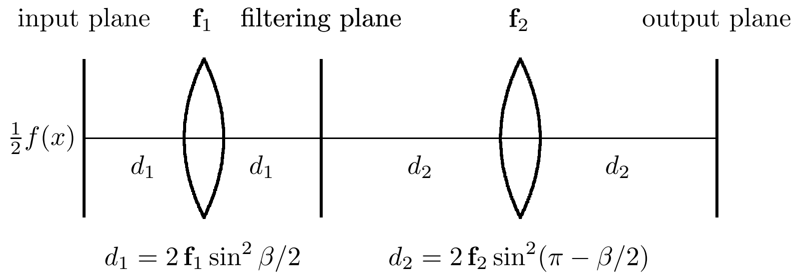


**Figure 3.** A cascade of a fractional FT system and an inverse fractional FT system, each consisting of one thin lens with focal length $\mathbf{f}_{1,2}$, preceded and followed by two identical distances $d_{1,2}$ of free space. The relations between the distances, the focal lengths, and the fractional angle $\beta$ read $d_1 = 2\mathbf{f}_1 \sin^2\beta/2$ and $d_2 = 2\mathbf{f}_2 \sin^2(\pi-\beta/2)$.

fractional Fourier domain can thus be used for phase retrieval of optical fields. Note that the proposed method is non-interferometric and non-iterative.

Let us now consider Eq. (5) for the case $n = 2$, which is related to the second derivative of the signal $f(x)$. Filtering in the fractional Fourier domain with mask $u^2$ yields [cf. Eq. (8)]

$$R^{\mp\alpha}\left[F_{\pm\alpha}(u)u^2\right](x) = \cos^2\alpha\; x^2 f(x) - \sin^2\alpha\;\frac{d^2f(x)}{dx^2} \mp i\cos\alpha\sin\alpha\;\left[f(x) + 2x\frac{df(x)}{dx}\right] \quad (16)$$

and hence

$$\tfrac{1}{2}\left\{R^{-\alpha}\left[F_\alpha(u)u^2\right](x) + R^{\alpha}\left[F_{-\alpha}(u)u^2\right](x)\right\} = \cos^2\alpha\; x^2 f(x) - \sin^2\alpha\;\frac{d^2f(x)}{dx^2}. \quad (17)$$

If we take the latter relationship for two different angles $\alpha$ and $\beta$, we can find the second derivative $d^2f/dx^2$ as [cf. Eq. (9)]

$$\frac{d^2f(x)}{dx^2} = \frac{1}{2(\cos^2\beta-\cos^2\alpha)}\left\{\cos^2\alpha\left(R^{-\beta}\left[F_\beta(u)u^2\right](x) + R^{\beta}\left[F_{-\beta}(u)u^2\right](x)\right) - \cos^2\beta\left(R^{-\alpha}\left[F_\alpha(u)u^2\right](x) + R^{\alpha}\left[F_{-\alpha}(u)u^2\right](x)\right)\right\}. \quad (18)$$

If we combine Eq. (16) for the angle $\alpha$ with the similar expression for the angle $\alpha+\pi/2$, we get the relationship

$$R^{-\alpha-\pi/2}\left[F_{\alpha+\pi/2}(u)u^2\right](x) + R^{-\alpha}\left[F_\alpha(u)u^2\right](x) = x^2 f(x) - \frac{d^2f(x)}{dx^2}, \quad (19)$$

from which we conclude that the sum $R^{-\alpha-\pi/2}\left[F_{\alpha+\pi/2}(u)u^2\right](x) + R^{-\alpha}\left[F_\alpha(u)u^2\right](x)$ is invariant with respect to $\alpha$.

We have derived the general expression for the power filtering of $n$-th order in the fractional Fourier domain, which stresses its relation to the differentiation operation. The main properties and invariants of linear and quadratic fractional filtering have been found. In particular it has been shown that the signal derivative and the corresponding power filtering in the Fourier domain, can be represented as a linear combination of the related fractional power filtering operations. The application of linear filtering in the fractional Fourier domain for phase retrieval from only intensity distributions has been proposed. Its efficiency has been demonstrated by numerical simulations. A simple optical configuration for the experimental realization of the method has been discussed.

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