Powers of transfer matrices via eigenfunctions

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Abstract

The parameters of the transfer matrix describing a first-order optical system that is a cascade of k identical subsystems defined by the transfer matrix M, are determined from considering the subsystem's eigenfunctions. A condition for the cascade to be cyclic is derived. Particular examples of cyclic first-order optical systems are presented. Structure and properties of eigenfunctions of cyclic transforms are considered. A method of optical signal encryption by using cyclic first-order systems is proposed.

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1 Introduction

Cascades of first-order optical systems [1] including thin lenses, spherical mirrors, graded index media, etc. have recently attracted much attention in the field of optical signal processing. In particular they are used in phase space tomography [2], where the complex amplitude (in the case of coherent light) or the correlation function (in the case of partially coherent light) is reconstructed from the intensity distributions measured at the output planes of the cascade subsystems. The application of first-order optical system cascades for the characterization of the complex structure of optical fields has been proposed in Refs. [3, 4, 5]. In general, the optical field propagating through a cascade of first-order systems is, in a certain way, similar to the wavelet transform, and this promises to be important for optical signal processing. Note also that a laser cavity can be represented as a cascade of first-order optical systems, as well.

A cascade is usually constructed from a number of identical first-order optical subsystems. Each of them is described in the paraxial approximation of the scalar diffraction theory through the canonical integral transform, also known as the generalized Fresnel transform (GFT) [6, 7, 8, 9, 10] . Thus, the evolution of the complex field amplitude $f(x)$ during propagation through a first-order optical system, is the GFT of the input field amplitude $f_i(x)$

$$f_o(u) = R^M \left[f_i(x) \right](u) = \int_{-\infty}^{\infty} f_i(x) K_M(x,u) dx, \tag{1}$$

with the kernel

$$K_M(x,u) = \begin{cases} \left(1/\sqrt{iB}\right) \exp\left(i\pi \left(Ax^2 + Du^2 - 2xu\right)/B\right) & B \neq 0 \\ \sqrt{A} \exp\left(i\pi Cu^2/A\right) \delta(x - Au) & B = 0 \end{cases} \tag{2}$$

parametrized by a real 2×2 matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3}$$

with the determinant $AD - BC$ equal to 1. The parameters A, B, C, D depend on the concrete first-order system and the wavelength. For the sake of simplicity we will consider the one-dimensional case.

As an example, we mention that the GFT parametrized by a matrix with $A = D = \cos\theta$ and $B = -C = \sin\theta$ corresponds, except for a factor $\exp(i\theta/2)$, to the fractional Fourier transform (FT) [11, 12, 13, 14].

Due to the cascading property of the canonical integral transform

$$R^{M_2} R^{M_1} = R^{M_3} \tag{4}$$

where $M_3 = M_2 \times M_1$, the complex field amplitude at the output plane of the cascade of k identical first-order systems, each of which described by the same transfer matrix M, can be represented as the GFT for a matrix M^k of the input complex field amplitude. The connection between the parameters of the transfer matrix M and its power M^k was considered in Ref. [15]. Based on matrix calculus it was shown that

$$\begin{aligned} M^k &= \xi_k M - \mu_k I, \\ \mu_k &= \lambda_1 \lambda_2 \xi_{k-1}, \\ \xi_k &= \lambda_1^{k-1} + \lambda_1^{k-2}\lambda_2 + ... + \lambda_1 \lambda_2^{k-2} + \lambda_2^{k-1}, \end{aligned} \tag{5}$$

where λ_1 and λ_2 are the eigenvalues of the matrix M.

In this paper we derive an alternative method of determining the parameters of the cascade transfer matrix based on the analysis of the self-imaging phenomenon in first-order optical systems. This approach allows us to formulate a simple condition for a cascade of canonical integral transforms to be cyclic and to classify first-order optical systems in accordance with this definition. We show that there is a wide group of various wavefronts which are self-reproducible under propagation through a cyclic first-order optical system; meanwhile, the form of the self-reproducible wavefronts for noncyclic systems is strictly defined.

We investigate the structure and the properties of the eigenfunctions for the cyclic GFTs and propose a method for their generation. We show that any complex field amplitude can be decomposed into the finite set of orthogonal eigenfunctions for a cyclic GFT. The signal decomposition into this set of self-GFT functions is then used for the optical encryption by using cyclic first-order systems.

2 Eigenfunctions for the generalized Fresnel transform

The self-imaging phenomenon of coherent fields in a first-order optical system is described in the framework of the eigenfunctions of the GFT. An input complex field amplitude $f_i(x)$ is an eigenfunction $f_M(x)$ of the canonical operator R^M corresponding to the given optical system if

$$R^M \left[f_M(x) \right](u) = a f_M(u), \tag{6}$$

where $a = \exp(i2\pi\varphi)$ is the (generally complex) eigenvalue [6]. From Parseval's relation for the canonical transform of the field amplitude with finite energy, $\int |f(x)|^2 \, dx < \infty$, we have $|a| = 1$, and therefore φ is real. Note that for infinite-energy wavefronts φ can be complex. The structure and the properties of the eigenfunctions of the particular cases of the GFT corresponding to the Fourier transform and to the fractional Fourier transform were investigated in Refs. [16, 17, 18, 19, 20, 21, 22].

It is easy to see from Eqs. (4) and (6) that an eigenfunction $f_M(x)$ for the canonical integral operator R^M with eigenvalue a, is also an eigenfunction with eigenvalue a^k for the GFT parametrized by the matrix M^k, where k is an integer.

From the linearity of the GFT and from the definition (6) it follows that a sum of eigenfunctions for a given GFT operator R^M with identical eigenvalues a, is also an eigenfunction for R^M with the same eigenvalue a.

The structure of the eigenfunctions for the GFT (the so-called self-GFT functions) has been considered in Ref. [6]. It was shown there that the functions

$$\Phi_n(x) = \left(\sqrt{\pi}2^n\lambda n!\right)^{-\frac{1}{2}} \exp\left(-\tfrac{1}{2}(1+i\beta)(x/\lambda)^2\right) H_n\left(x/\lambda\right) \tag{7}$$

are eigenfunctions for the GFT parametrized by the matrix (3) with eigenvalue $a = \exp(-i(n+\frac{1}{2})\theta)$, where $H_n(u)$ are the Hermite polynomials and where the parameters θ, λ, and β are defined from the parameters of the transfer matrix by

$$\begin{aligned} \theta &= \arccos\left(\tfrac{1}{2}(A+D)\right) \\ \lambda^2 &= 2B\left(4-(A+D)^2\right)^{-\frac{1}{2}} \\ \beta &= (A-D)\left(4-(A+D)^2\right)^{-\frac{1}{2}}. \end{aligned} \tag{8}$$

This implies the relationship

$$R^M\left[\Phi_n(x)\right](u) = \exp\left(-i(n+\tfrac{1}{2})\theta\right)\Phi_n(u) \tag{9}$$

for the GFT parametrized by a matrix M with

$$\begin{aligned} A &= \cos\theta + \beta\sin\theta \\ B &= \lambda^2\sin\theta \\ C &= -\left((\beta^2+1)/\lambda^2\right)\sin\theta \\ D &= \cos\theta - \beta\sin\theta. \end{aligned} \tag{10}$$

Note that $\beta = 0$ only if $A = D = \cos\theta$. In that case we have $B = \lambda^2\sin\theta$ and $C = -(1/\lambda^2)\sin\theta$, which represents the scaled fractional Fourier transform with eigenfunctions

$$\Phi_n(x) = \left(\sqrt{\pi}2^n\lambda n!\right)^{-\frac{1}{2}} \exp\left(-\tfrac{1}{2}(x/\lambda)^2\right) H_n(x/\lambda). \tag{11}$$

Unfortunately, in limiting cases like the Fresnel transform ($\lambda^2 \to \infty$ and $\theta \to 0$) and the scaling transform ($\lambda^4 \to 0$ and $\beta^2+1 \to 0$), the application of the relationships (7) and (8) for the construction of the eigenfunctions is problematic. We therefore confine ourselves to systems described by a matrix M for which $|A+D| \neq 2$ and $B, C \neq 0$. Moreover, we remark that, since $\cos\theta = \frac{1}{2}(A+D)$, the eigenvalues $a = \exp(-i(n+\frac{1}{2})\theta)$ depend on the parameters A and D. If $|A+D| > 2$, the parameter θ becomes complex: $\theta = \Re(\theta) + i\Im(\theta)$, with $\Re(\theta) = \pi k$.

The set of functions $\{\Phi_n(x)\}$ forms a complete orthonormal set. Hence, a self-GFT function with eigenvalue a can be represented as a superposition of certain $\Phi_n(x)$ modes with the same eigenvalue a. In order to have the same a, the indices $\{n\}$ should satisfy the relationship

$$2\pi(N+\varphi) = -(n+\tfrac{1}{2})\theta, \tag{12}$$

where φ is a constant defining the eigenvalue $\exp(i2\pi\varphi)$ of this eigenfunction and where N is an integer.

It has been proved in Ref. [6] that for the optical system described by the GFT parametrized by a matrix with parameters A and D such that $\theta/2\pi = \arccos(\frac{1}{2}(A+D))/2\pi$ is complex or irrational, the functions $\Phi_n(x)$ are the only solutions of Eq. (6). Let us consider, as an example, the eigenfunctions for the GFT parametrized by a matrix with parameters $A = D = \cosh\alpha$ and $B = C = \sinh\alpha$. Since $A = D$, it follows from Eqs. (10) that $\beta = 0$ and $B = -\lambda^4 C$, which yields

$\lambda^2 = i$ and $\theta = i\alpha$. The set of orthonormal eigenfunctions (7) with eigenvalues $a = \exp((n+\frac{1}{2})\alpha)$ for this system can now be written as

$$\Phi_n(x) = \left(\sqrt{\pi}2^n \exp(i\pi/4)n!\right)^{-\frac{1}{2}} \exp\left(\tfrac{1}{2}ix^2\right) H_n\left(x\exp(-i\pi/4)\right). \tag{13}$$

We conclude that the chirp function $\Phi_0(x) = \left(\sqrt{\pi}\exp(i\pi/4)\right)^{-\frac{1}{2}} \exp\left(\frac{1}{2}ix^2\right)$ is self-reproducible under propagation through this system. Note that the eigenvalues of the different modes $\Phi_n(x)$ and $\Phi_m(x)$ for the same value α are different. This means that a superposition of these modes is not an eigenfunction of the corresponding GFT.

If the parameters of the transfer matrix are such that $\theta/2\pi = \arccos(\frac{1}{2}(A+D))/2\pi$ is rational, then θ can be represented as $\theta = 2\pi m/k$ with m and k integers, and there are several sets of indices $\{n\}$ that satisfy Eq. (12). The structure and the properties of the eigenfunctions for the GFT characterized by rational $\theta/2\pi$ will be considered in Section 4.

3 Powers of transfer matrices and cyclic cascades

As we have learned before, an eigenfunction $\Phi_n(x)$ for the operator R^M defined by Eqs. (7) and (10) with eigenvalue a is also an eigenfunction for the operator R^{M^k} with eigenvalue $a^k = \exp(-i(n+\frac{1}{2})k\theta)$, where k is an integer. Therefore, the parameters of the k-th power M^k of the matrix M have to satisfy equations which are similar to Eqs. (10):

$$\begin{aligned} A^{(k)} &= \cos k\theta + \beta \sin k\theta \\ B^{(k)} &= \lambda^2 \sin k\theta \\ C^{(k)} &= -((\beta^2+1)/\lambda^2)\sin k\theta \\ D^{(k)} &= \cos k\theta - \beta \sin k\theta. \end{aligned} \tag{14}$$

From Eqs. (8), (10), and (14), we conclude that the parameters of the matrix M^k can alternatively be represented in terms of the parameters of the matrix M:

$$\begin{aligned} A^{(k)} &= \cos k\theta + \tfrac{1}{2}(A-D)\sin k\theta/\sin\theta \\ B^{(k)} &= B\sin k\theta/\sin\theta \\ C^{(k)} &= C\sin k\theta/\sin\theta \\ D^{(k)} &= \cos k\theta - \tfrac{1}{2}(A-D)\sin k\theta/\sin\theta \end{aligned} \tag{15}$$

with $\cos\theta = \frac{1}{2}(A+D)$. Equations (15) allow an easy determination of the resulting matrix M^k of the cascade of k identical first-order systems. For the fractional Fourier transform system, for instance, determined by $\beta = 0$ and $\lambda^2 = 1$, and hence by $A = D = \cos\theta$ and $B = -C = \sin\theta$, we immediately have $A^{(k)} = D^{(k)} = \cos k\theta$, and $B^{(k)} = -C^{(k)} = \sin k\theta$.

From Eqs. (15) it is easy to see that if

$$\theta = 2\pi m/k, \tag{16}$$

with k and m integers, we have

$$M^k = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{17}$$

This implies that the cascade of k optical systems described by a matrix M with parameters A and D such that

$$\tfrac{1}{2}(A+D) = \cos(2\pi m/k), \tag{18}$$

produces the identity transform, and hence $M = I^{1/k}$. We call these systems cyclic of order k. Note that, without loss of generality, we may choose $0 \le m < k$.

In the special case that $m = 1$ in Eqs. (16) and (18), we denote the corresponding k-th order cyclic transfer matrix by M_k; its parameters follow from Eqs. (10) with $\theta = 2\pi/k$:

$$\begin{aligned} A_k &= \cos(2\pi/k) + \beta \sin(2\pi/k) \\ B_k &= \lambda^2 \sin(2\pi/k) \\ C_k &= -\left((\beta^2+1)/\lambda^2\right) \sin(2\pi/k) \\ D_k &= \cos(2\pi/k) - \beta \sin(2\pi/k). \end{aligned} \tag{19}$$

From Eqs. (15) we conclude that the general k-th order cyclic matrix M with $\theta = 2\pi m/k$, can be expressed as the m-th power M_k^m of M_k; M_k can thus be considered as the m-th root $M^{1/m}$ of M. Moreover, the j-th power of M is equal to the l-th power of M_k, where the integers j and l are related to each other by $mj = l + Nk$,

$$M^j = M_k^{mj} = M_k^{l+Nk} = M_k^l = M^{l/m}, \tag{20}$$

and where we have used the property that M_k^k is the identity matrix. We conclude that cascade properties of any cyclic transform of order k defined by a transfer matrix M can be described by the matrix $M_k = M^{1/m}$.

As a first example of a cyclic system, we mention the fractional Fourier transform system for an angle $\theta = 2\pi m/k$. Note that the cascade of a finite number of identical fractional FT systems cannot produce the identity transformation if $\theta/2\pi$ is irrational.

It is well-known that a cascade of $k = 4$ Fourier transforming systems produces the identity transformation. We now determine an entire class of canonical systems exhibiting this property. From Eq. (16) it follows that such systems have the property $\theta = \pi m/2$. Using Eqs. (10), we conclude that for even m one gets the inverse or the identity transformation, $A = D = \pm 1$ and $B = C = 0$, while for odd m the transfer matrix can be expressed as

$$M = \begin{pmatrix} A & B \\ -(A^2+1)/B & -A \end{pmatrix}. \tag{21}$$

It is easy to see that the case $A = 0$ and $B = 1$ corresponds to the Fourier transforming system. In general the inverse matrix, the identity matrix, and the matrix (21) are the 4-th root of the identity matrix: $M^4 = I$. The kernel of the GFT parametrized by matrix (21) is written as

$$K_{I^{1/4}}(x,u) = \left(1/\sqrt{iB}\right) \exp\left(i\pi\left(A(x^2-u^2) - 2xu\right)/B\right). \tag{22}$$

Note that the square of the matrix (21) corresponds to the inverse matrix.

Let us consider an optical setup that performs such a transform. A typical optical system consisting of a thin lens with focal length f (normalized to the wavelength) can be described by the transfer matrix

$$M = \begin{pmatrix} 1 - z_2/f & z_1 + z_2 - z_1 z_2/f \\ -1/f & 1 - z_1/f \end{pmatrix},$$

where z_1 and z_2 are the (normalized) distances from the lens to the input and the output plane, respectively. Such an optical systems performs a GFT parametrized by a $I^{1/4}$ matrix (21) if $A + D = 0$, which is equivalent to $2f = z_1 + z_2$. We then have

$$I^{1/4} = \begin{pmatrix} (z_1 - z_2)/2f & (z_1^2 + z_2^2)/2f \\ -1/f & -(z_1 - z_2)/2f \end{pmatrix}.$$

In particular for $z_1 = z_2 = f$ one has a Fourier transforming system. Another choice is $z_1 = 0$ and $z_2 = 2f$, which yields the matrix

$$I^{1/4} = \begin{pmatrix} -1 & 2f \\ -1/f & 1 \end{pmatrix}.$$

Let us now construct the class of transforms described by the matrix which is the k-th root of the Fourier transfer matrix. In this case we have

$$M^k = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{23}$$

and it follows from Eqs. (15) that $A = D = \cos\theta$ and $B = -C = (-1)^m \sin\theta = \sin((-1)^m\theta)$, with $k\theta = (2m+1)\pi/2$. Note that the similar result was obtained in [15]. In general the transformation described by the k-th root of the Fourier transfer matrix corresponds to the fractional FT at the angle $\theta = (-1)^m(2m+1)\pi/2k$. Due to the periodicity property of the fractional FT $R^{\theta+2\pi n} = R^\theta$, the cascades of j $(j = 1, ..., 4k)$ identical systems for different m produce the same transformations of the input fields corresponding to the fractional FT at the angles $\{\theta = \pi j/2k \,|\, j = 1, ..., 4k\}$ but taken in a different order. Speaking about angle evolution of the fractional FT, it is more common to consider consequently increasing angles in the region $[0, 2\pi[$. This corresponds to $m = 0$.

4 Structure and properties of eigenfunctions for cyclic canonical transforms

As we have seen above, cascade properties of any cyclic transform of order k defined by a transfer matrix M with $\frac{1}{2}(A+D) = \cos(2\pi m/k)$, can be described by the matrix $M_k = M^{1/m}$. Taking into account that an eigenfunction for the operator R^M is also an eigenfunction for the operator R^{M^j} and in particular, if $mj = 1 + Nk$, an eigenfunction of the operator $R^{M^{1/m}}$, we can conclude that the GFTs parametrized by the matrices M and $M^{1/m} = M_k$ have the same eigenfunctions. Let us find the eigenfunctions for the GFT related with the matrix M_k, whose parameters follow from Eqs. (19).

A self-GFT function with eigenvalue $a = \exp(i2\pi\varphi)$ can be represented as a superposition of the modes $\Phi_n(x)$ whose indices $\{n\}$ satisfy the relationship (12)

$$N + \varphi = -(n + \tfrac{1}{2})/k, \tag{24}$$

where φ is a constant defining the eigenvalue of this self-GFT function and where N is an integer. It is easy to see that we have k different sets of modes for which relation (24) holds

$$n = L + kl,$$

where $L = 0, 1, \ldots, k-1$ and l is an integer. Then a self-GFT function for the matrix (19) with eigenvalue $a = \exp\left(-i2\pi(L + \frac{1}{2})/k\right)$ is defined as

$$f_k^L(u) = \sum_{l=0}^{\infty} g_{L+kl}\Phi_{L+kl}(u), \tag{25}$$

where g_{L+kl} are complex constants. This function is also an eigenfunction with eigenvalue $a = \exp\left(-i2\pi(L + \frac{1}{2})m/k\right)$ for the general k-th order cyclic GFT parametrized by the matrix M_k^m for any integer m.

Let us briefly mention the main properties of the self-GFT functions for cyclic operators which are similar to the properties of the self-fractional FT functions considered in [22]. As well as in the case of the self-fractional FT functions, the self-GFT functions for the same operator with different eigenvalues (i.e., different indices L) are orthogonal to each other, because they are expanded into disjoint series of the orthogonal modes Φ_n.

Since the Φ_n modes form a complete orthogonal set, any function $g(u)$ can be represented as their superposition

$$g(u) = \sum_{n=0}^{\infty} g_n \Phi_n(u). \tag{26}$$

Subdividing the series into partial ones

$$g(u) = \sum_{L=0}^{k-1} \left(\sum_{l=0}^{\infty} g_{L+kl} \Phi_{L+kl}(u) \right) = \sum_{L=0}^{k-1} f_k^L(u) \tag{27}$$

we conclude that any function $g(u)$ can be represented as a linear superposition of k orthogonal self-GFT functions $f_k^L(u)$ of a given cyclic operator of order k. Note that there are many of cyclic operators of order k, which differ from each other by the parameters λ and β. The decomposition of the optical image into the set of self-GFT functions can be useful for the analysis of complex images, their processing through first-order optical systems, in optical testing, and in the development of filtering devices.

On the other hand, a self-GFT function for the operator R^M described by the transfer matrix (19), can be constructed from any generator function $g(x)$ through the following procedure

$$f_k^L(u) = \frac{1}{k} \sum_{l=0}^{k-1} \exp\left(\frac{i2\pi(L+\frac{1}{2})l}{k} \right) R^{M^l}\left[g(x)\right](u). \tag{28}$$

An optical configuration for the synthesis of a self-GFT function is similar to the one that was proposed for the synthesis of self-Fourier functions [23].

Note that self-imaging of the complex field amplitude, being a self-GFT function for the corresponding cyclic cascade of first-order optical systems, is similar to the Talbot effect where periodic wavefronts of a certain period are self-reproducible under propagation through a cascade of identical Fresnel systems.

5 Optical encryption by using cyclic first-order optical systems

The application of optical systems for data security and encryption is a perspective field in optical engineering [24]. The main advantages of optical encryption are the possibility of parallel processing of two-dimensional optical signals and the possibility of hiding information in several wave parameters like amplitude, phase, wavelength, polarization, etc. In this paper we consider the application of cyclic first-order optical systems for signal encryption.

The method of optical encryption proposed here is based on the signal decomposition into the set of self-GFT functions. As it follows from Eq. (27), any signal $g(u)$ from L^2 can be represented as a sum of k self-GFT functions $f_k^L(u)$. In the limiting case $k \to \infty$ we have the generalized Hermite-Gauss expansion (26).

The encryption procedure of a signal $g(u)$ consists of (i) its decomposition into the set of k orthogonal self-GFT functions $f_k^L(u)$ $(L = 0, ..., k-1)$, (ii) multiplication of each self-GFT function by a secret factor $a_L \neq 0$, and (iii) the composition of an encrypted signal $G(u) = \sum_{L=0}^{k-1} a_L f_k^L(u)$ from the weighted self-GFT functions $a_L f_k^L(u)$. The complete procedure is represented in the following scheme:

$$g(u) \Longrightarrow \left\{ \begin{array}{lc} f_k^0(u) & \to \\ ... & \to \\ f_k^L(u) & \to \\ ... & \to \\ f_k^{k-1}(u) & \to \end{array} \left[\begin{array}{l} \otimes a_0 \\ ... \\ \otimes a_L \\ ... \\ \otimes a_{k-1} \end{array} \right] \begin{array}{ll} \to & a_0 f_k^0(u) \\ \to & ... \\ \to & a_L f_k^L(u) \\ \to & ... \\ \to & a_{k-1} f_k^{k-1}(u) \end{array} \right\} \Longrightarrow G(u) \tag{29}$$

For convenience, we suppose that the signal $g(u)$ itself is not a self-GFT function for the given first-order optical system.

The procedure does not change the set of self-GFT functions that compose the initial signal, but produces only an alteration of their contributions in the composition. The information is hidden by choosing an order k of self-GFT functions and manipulating with the secret key-factors a_L, which, in general, may be complex. Therefore, for a given cyclic system of order k one gets $2k$ keys. The optical set-up introducing the key-factor multiplication consists of a number of constant amplitude-phase screens, different for every self-GFT function. The encrypted signal can then be transmitted via a common, unprotected way.

The procedure of decrypting is similar to the encryption procedure as can be seen from the following scheme

$$G(u) \Longrightarrow \left\{ \begin{array}{lc} a_0\, f_k^0(u) & \rightarrow \\ \dots & \rightarrow \\ a_L\, f_k^L(u) & \rightarrow \\ \dots & \rightarrow \\ a_{k-1}\, f_k^{k-1}(u) & \rightarrow \end{array} \left[\begin{array}{c} \otimes a_0^{-1} \\ \dots \\ \otimes a_L^{-1} \\ \dots \\ \otimes a_k^{-1} \end{array} \right] \begin{array}{ll} \rightarrow & f_k^0(u) \\ \rightarrow & \dots \\ \rightarrow & f_k^L(u) \\ \rightarrow & \dots \\ \rightarrow & f_k^{k-1}(u) \end{array} \right\} \Longrightarrow g(u) \qquad (30)$$

and is realized using the same optical equipment by only changing the amplitude-phase screens.

Let us demonstrate the method by the simplest example of signal encryption, i.e., its decomposition in even and odd functions ($k = 2$), which are the self-fractional Fourier functions. We suppose that our signal is neither even nor odd; for even or odd signals, a larger-order encryption procedure k should be applied. The encrypted signal can be written as

$$G(u) = (g(u)(a_0 + a_1) + g(-u)(a_0 - a_1))/2,$$

where $a_0 \neq a_1$. In accordance with this procedure a real signal $g(u) = u/(u + b)$ transforms to the signal $G(u) = u(ba_1 - ua_0)/(b^2 - u^2)$, which is, in general, complex (if at least one of the factors a_0 or a_1 is complex).

We finally note that the generalization of the encryption procedure to the two-dimensional case increases information security due to the extension of the number of keys and the possibility of applying different orders k_x and k_y for the coordinates x and y.

6 Conclusions

Starting from the analysis of the eigenfunctions of the GFT parametrized by a matrix M, we have derived the expressions for the k-th power M^k of the transfer matrix describing the cascade of k GFTs. It has been shown that matrices for which the parameters A and D are such that $\arccos(\frac{1}{2}(A + D)) = 2\pi m/k$, describe cyclic transformations of order k, which means that a cascade of k such transformations produces the identity transform. It has been found that certain powers of the cyclic transform correspond to its mth root.

We have derived the general expression for the eigenfunctions for cyclic GFT and have discussed their main properties. In particular it has been shown that any function (complex field amplitude) can be represented as a linear superposition of k orthogonal eigenfunctions of a cyclic GFT of order k. A generation procedure of the GFT eigenfunctions has been proposed.

A method of optical encryption by using cyclic first-order systems has been discussed. It is based on a signal decomposition into the set of self-GFT functions and changing their coefficients by multiplying them by the secret codes, which in general are complex numbers. The composition of these weighted self-GFT functions is an encrypted signal and can be transmitted via a common, unprotected way. The decrypting procedure is similar to the encrypting one. Both of them can be realized by using quadratic refractive index optical systems like lenses, mirrors, optical fibers, etc.

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