6.3 GABOR'S SIGNAL EXPANSION FOR A NON-ORTHOGONAL SAMPLING GEOMETRY[0]

6.3.1 Historical Perspective

In 1946 [5], Gabor suggested the representation of a time signal in a combined time-frequency domain; in particular he proposed to represent the signal as a superposition of shifted and modulated versions of a so-called elementary signal or synthesis window $g(t)$. Moreover, as a synthesis window $g(t)$ he chose a Gaussian signal, because such a signal has a good localization, both in the time domain and in the frequency domain. The other choice that Gabor made, was that his signal expansion was formulated on a rectangular lattice in the time-frequency domain, $(mT, k\Omega)$, and that the sampling distances T and Ω satisfied the relation $\Omega T = 2\pi$.

The coefficients in Gabor's signal expansion can be determined by using an analysis window $w(t)$. In the case of critical sampling, i.e., $\Omega T = 2\pi$, the analysis window $w(t)$ follows uniquely from the given synthesis window $g(t)$. However, such a unique analysis window appears to have some mathematically very unattractive properties. For this reason, the expansion should be formulated on a denser lattice, $\Omega T < 2\pi$. This makes the analysis window no longer unique and thus allows for finding an analysis window that is optimal in some way. We can, for instance, look for the analysis window that resembles best the synthesis window; a better resemblance can then be reached for a higher degree of oversampling.

A better resemblance can also be reached if we adapt the structure of the lattice to the form of the window as it is represented in the time-frequency domain. For the Gaussian window, for instance, its time-frequency representation has circular contour lines, and it is well known that circles are better packed on a hexagonal lattice than on a rectangular lattice. Gabor's signal expansion on such a hexagonal, non-orthogonal lattice then leads to a better resemblance between the window functions $g(t)$ and $w(t)$ than the expansion on a rectangular, orthogonal lattice does.

6.3.2 Gabor's Signal Expansion on a Rectangular Lattice

We start with the usual Gabor expansion [1]-[5] on a rectangular time-frequency lattice, in which case a signal $\varphi(t)$ can be expressed as a linear combination of properly shifted and modulated versions $g_{mk}(t) = g(t - mT)\exp(jk\Omega t)$ of a synthesis window $g(t)$:

$$\varphi(t) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_{mk}\, g_{mk}(t). \tag{6.3.1}$$

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The time step T and the frequency step Ω satisfy the relationship $\Omega T \leq 2\pi$; note that the factor $2\pi/\Omega T$ represents the degree of oversampling, and that in his original paper [5] Gabor considered the case of critical sampling, i.e. $\Omega T = 2\pi$. The expansion coefficients a_{mk} follow from sampling the windowed Fourier transform with analysis window $w(t)$, $\int_{-\infty}^{\infty} \varphi(t)\, w^*(t-\tau) \exp(-j\omega t)\, dt$, on the rectangular lattice $(\tau = mT, \omega = k\Omega)$:

$$a_{mk} = \int_{-\infty}^{\infty} \varphi(t)\, w_{mk}^*(t)\; dt. \tag{6.3.2}$$

This relationship is known as the Gabor transform.

The synthesis window $g(t)$ and the analysis window $w(t)$ are related to each other in such a way that their shifted and modulated versions constitute two sets of functions that are biorthogonal:

$$\sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} g_{mk}(t_1)\, w_{mk}^*(t_2) = \delta(t_1 - t_2). \tag{6.3.3}$$

If the biorthogonality condition (6.3.3) is satisfied, the Gabor transform (6.3.2) and Gabor's signal expansion (6.3.1) form a transform pair in the following sense: if we start with an arbitrary signal $\varphi(t)$ and determine its expansion coefficients a_{mk} via the Gabor transform (6.3.2), the signal can be reconstructed via the Gabor expansion (6.3.1).

The biorthogonality relation (6.3.3) leads immediately to the equivalent but simpler expression

$$\frac{2\pi}{\Omega} \sum_{m=-\infty}^{\infty} g(t - mT)\, w^* \left(t - \left[mT + n\frac{2\pi}{\Omega}\right]\right) = \delta_n, \tag{6.3.4}$$

where δ_n is the Kronecker delta. In the case of critical sampling, i.e., $\Omega T = 2\pi$, the biorthogonality relation (6.3.4) reduces to

$$T \sum_{m=-\infty}^{\infty} g(t - mT)\, w^*(t - [m+n]T) = \delta_n \tag{6.3.5}$$

and the analysis window $w(t)$ follows uniquely from a given synthesis window $g(t)$, or vice versa. An elegant way to find the analyis window if the synthesis window is given, is presented in the next section.

6.3.3 Fourier Transform and Zak Transform

It is well known (see, for instance, [1]-[4]) that in the case of critical sampling, $\Omega T = 2\pi$, Gabor's signal expansion (6.3.1) and the Gabor transform (6.3.2) can be transformed into product form. We therefore need the Fourier transform

$\bar{a}(t/T, \omega/\Omega)$ of the two-dimensional array of Gabor coefficients a_{mk}, defined by

$$\bar{a}(x,y) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_{mk} \exp[-j2\pi(my - kx)], \tag{6.3.6}$$

and the Zak transforms $\tilde{\varphi}(xT, 2\pi y/T; T)$, $\tilde{g}(xT, 2\pi y/T; T)$, and $\tilde{w}(xT, 2\pi y/T; T)$ of the signal $\varphi(t)$ and the window functions $g(t)$ and $w(t)$, respectively, where the Zak transform $\tilde{f}(t, \omega; \tau)$ of a function $f(t)$ is defined as (see, for instance, [1], [4])

$$\tilde{f}(t, \omega; \tau) = \sum_{n=-\infty}^{\infty} f(t + n\tau) \exp(-jn\tau\omega). \tag{6.3.7}$$

Note that the Fourier transform $\bar{a}(x,y)$ is periodic in x and y with period 1, and that the Zak transform $\tilde{f}(t, \omega; \tau)$ is periodic in ω with period $2\pi/\tau$ and quasi-periodic in t with period τ: $\tilde{f}(t + m\tau, \omega + 2\pi k/\tau; \tau) = \tilde{f}(t, \omega; \tau) \exp(jm\omega\tau)$.

Upon substituting from the Fourier transform (6.3.6) and the Zak transforms [cf. Eq. (6.3.7)] into Eqs. (6.3.1) and (6.3.2), it is not too difficult to show that Gabor's signal expansion (6.3.1) can be transformed into the product form

$$\tilde{\varphi}\left(xT, y\frac{2\pi}{T}; T\right) = \bar{a}(x,y)\, \tilde{g}\left(xT, y\frac{2\pi}{T}; T\right), \tag{6.3.8}$$

while the Gabor transform (6.3.2) can be transformed into the product form

$$\bar{a}(x,y) = T\ \tilde{\varphi}\left(xT, y\frac{2\pi}{T}; T\right)\ \tilde{w}^*\left(xT, y\frac{2\pi}{T}; T\right). \tag{6.3.9}$$

In particular the product form (6.3.9) is useful for determining Gabor's expansion coefficients. Since a Zak transform is merely a Fourier transform [cf. Eq. (6.3.7)], the expansion coefficients can be determined by Fourier transformations and multiplications; and if things are formulated for discrete-time signals, we can use the *fast* Fourier transform to formulate a fast algorithm for the Gabor transform [2], [3].

The relationship between the Zak transforms of the analysis window $w(t)$ and the synthesis window $g(t)$ then follows from substituting from Eq. (6.3.9) into Eq. (6.3.8) and reads

$$T\ \tilde{g}\left(xT, y\frac{2\pi}{T}; T\right)\ \tilde{w}^*\left(xT, y\frac{2\pi}{T}; T\right) = 1. \tag{6.3.10}$$

From the latter relationship we conclude that (the Zak transform of) the analysis window $w(t)$ follows uniquely from (the Zak transform of) the given synthesis window $g(t)$. In general, however, the unique analysis window $w(t)$ has some very unattractive mathematical properties. We are therefore urged to consider Gabor's signal expansion on a denser lattice, in which case the analysis window is no longer unique. This enables us to choose an analysis window that is better suited to our purpose of determining Gabor's expansion coefficients.

6.3.4 Rational Oversampling

In the case of oversampling by a rational factor, $2\pi/\Omega T = p/q \geq 1$, with p and q relatively prime, positive integers, $p > q \geq 1$, Gabor's expansion (6.3.1) and the Gabor transform (6.3.2) can be transformed into the sum-of-products forms [2], [3], cf. Eqs. (6.3.8) and (6.3.9),

$$\varphi_s(x,y) \;=\; \frac{1}{p}\sum_{r=0}^{p-1} g_{sr}(x,y)\, a_r(x,y) \quad (s=0,1,\;\ldots\;,q-1) \tag{6.3.11}$$

$$a_r(x,y) \;=\; \frac{pT}{q}\sum_{s=0}^{q-1} w_{sr}^*(x,y)\, \varphi_s(x,y) \quad (r=0,1,\;\ldots\;,p-1), \tag{6.3.12}$$

respectively, where we have introduced the shorthand notations

$$\begin{aligned} a_r(x,y) &= \bar{a}(x, y + r/p) \\ \varphi_s(x,y) &= \tilde{\varphi}((x+s)pT/q, 2\pi y/T; pT) \\ g_{sr}(x,y) &= \tilde{g}((x+s)pT/q, 2\pi(y+r/p)/T; T) \\ w_{sr}(x,y) &= \tilde{w}((x+s)pT/q, 2\pi(y+r/p)/T; T), \end{aligned}$$

with $0 \leq x < 1$ and $s = 0,1,\;\ldots\;,q-1$ (and hence $0 \leq (x+s)/q < 1$), and $0 \leq y < 1/p$ and $r = 0,1,\;\ldots\;,p-1$ (and hence $0 \leq y + r/p < 1$). The relationship between the Zak transforms of the analysis window $w(t)$ and the synthesis window $g(t)$ then follows from substituting from Eq. (6.3.12) into Eq. (6.3.11) and reads [cf. Eq. (6.3.10)]

$$\frac{T}{q}\sum_{r=0}^{p-1} g_{s_1 r}(x,y)\, w_{s_2 r}^*(x,y) = \delta_{s_1 - s_2}, \tag{6.3.13}$$

with $s_1, s_2 = 0,1,\;\ldots\;,q-1$. The latter relationship represents a set of q^2 equations for pq unknowns, which set of equations is underdetermined since $p > q$, and we conclude that the analysis window does not follow uniquely from the synthesis window.

After combining the p functions $a_r(x,y)$ into a p-dimensional column vector $\boldsymbol{a}(x,y)$, the q functions $\varphi_s(x,y)$ into a q-dimensional column vector $\boldsymbol{\phi}(x,y)$, and the $q \times p$ functions $g_{sr}(x,y)$ and $w_{sr}(x,y)$ into the $q \times p$-dimensional matrices $\boldsymbol{G}(x,y)$ and $\boldsymbol{W}(x,y)$, respectively, the sum of products forms can be expressed as matrix-vector and matrix-matrix multiplications:

$$\boldsymbol{\phi}(x,y) \;=\; \frac{1}{p}\;\boldsymbol{G}(x,y)\,\boldsymbol{a}(x,y) \tag{6.3.14}$$

$$\boldsymbol{a}(x,y) \;=\; \frac{pT}{q}\;\boldsymbol{W}^*(x,y)\,\boldsymbol{\phi}(x,y) \tag{6.3.15}$$

$$\boldsymbol{I}_q \;=\; \frac{T}{q}\;\boldsymbol{G}(x,y)\;\boldsymbol{W}^*(x,y), \tag{6.3.16}$$

where $\boldsymbol{I}_q$ denotes the $q \times q$-dimensional identity matrix and where, as usual, the asterisk in connection with vectors and matrices denotes complex conjugation *and* transposition.

The latter relationship again represents q^2 equations for pq unknowns, and the $p \times q$ matrix $\boldsymbol{W}^*(x,y)$ cannot be found by a simple inversion of the $q \times p$ matrix $\boldsymbol{G}(x,y)$. An 'optimum' solution that is often used, is based on the generalized inverse and reads $\boldsymbol{W}^*_{opt}(x,y) = (q/T)\, \boldsymbol{G}^*(x,y)\, [\boldsymbol{G}(x,y)\, \boldsymbol{G}^*(x,y)]^{-1}$. This solution for $\boldsymbol{W}(x,y)$ is optimum in the sense that (i) it yields the analysis window $w(t)$ with the lowest L^2 norm, (ii) it yields the Gabor coefficients a_{mk} with the lowest L^2 norm, and (iii) it yields the analysis window that – in an L^2 sense, again – best resembles the synthesis window.

The 'optimum' solution gets better if the degree of oversampling p/q becomes higher. However, there is another way of finding a better solution, based on the structure of the lattice. If the lattice structure is adapted to the form of the window function as it is represented in the time-frequency domain, the 'optimum' solution will be better, even for a lower degree of oversampling. We will therefore consider the case of a non-orthogonal sampling geometry, but we will do that in such a way that we can relate this non-orthogonal sampling to orthogonal sampling. In that case we will still be able to use product forms of Gabor's expansion and the Gabor transform, and benefit from all the techniques that have been developed for them.

6.3.5 Non-orthogonal Sampling

The rectangular (or orthogonal) lattice that we considered in the previous sections, where sampling occurred on the lattice points $(\tau = mT, \omega = k\Omega)$, can be obtained by integer combinations of two orthogonal vectors $[T,0]^t$ and $[0,\Omega]^t$, see Fig. 6.1(a), which vectors constitute the lattice generator matrix

$$\begin{bmatrix} T & 0 \\ 0 & \Omega \end{bmatrix}.$$

We now consider a time-frequency lattice that is no longer orthogonal. Such a lattice is obtained by integer combinations of two linearly independent, but no longer orthogonal vectors, which we express in the forms $[aT, c\Omega]^t$ and $[bT, d\Omega]^t$, with a, b, c and d integers, and which constitute the lattice generator matrix

$$\begin{bmatrix} aT & bT \\ c\Omega & d\Omega \end{bmatrix} = \begin{bmatrix} T & 0 \\ 0 & \Omega \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Without loss of generality, we may assume that the integers a and b have no common divisors, and that the same holds for the integers c and d; possible common divisors can be absorbed in T and Ω. Note that we only consider lattices that have samples on the time and frequency axes and that are therefore suitable for a discrete-time approach, as well.

The area of a cell (a parallelogram) in the time-frequency plane, spanned by the two vectors $[aT, c\Omega]^t$ and $[bT, d\Omega]^t$, is equal to the determinant of the lattice

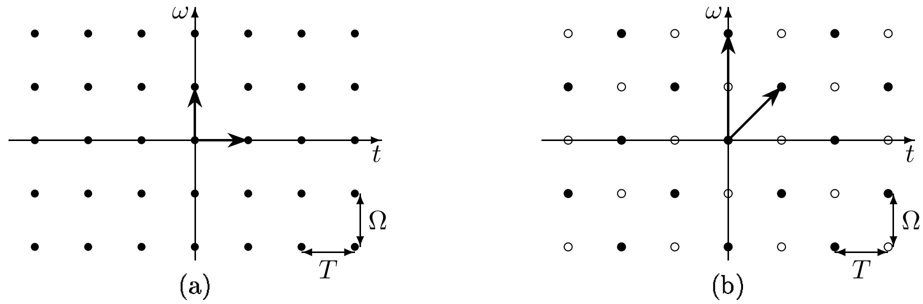

Figure 6.1: (a) A rectangular lattice with lattice vectors $[T, 0]^t$ and $[0, \Omega]^t$, and thus $R = 0$ and $D = 1$; and (b) a hexagonal lattice with lattice vectors $[T, \Omega]^t$ and $[0, 2\Omega]^t$, and thus $R = 1$ and $D = 2$.

generator matrix, which determinant is equal to $\Omega T D$, with $D = |ad - bc|$. To be usable as a proper Gabor sampling lattice, this area should satisfy the condition $D \leq 2\pi/\Omega T$.

There are a lot of lattice generator matrices that generate the same lattice. We will use the one that is based on the Hermite normal form, unique for any lattice,

$$\begin{bmatrix} T & 0 \\ R\Omega & D\Omega \end{bmatrix} = \begin{bmatrix} T & 0 \\ 0 & \Omega \end{bmatrix} \begin{bmatrix} 1 & 0 \\ R & D \end{bmatrix},$$

where R and D are relatively prime integers and $0 \leq |R| < D$. Sampling then occurs on the lattice points $(\tau = mT, \omega = [mR + nD]\Omega)$, and it is evident that these points of the non-orthogonal lattice form a subset of the points $(\tau = mT, \omega = k\Omega)$ of the orthogonal lattice. To be more specific: the non-orthogonal lattice is formed by those points of the rectangular (orthogonal) lattice for which $k - mR$ is an integer multiple of D. Note that the original rectangular lattice arises for $R = 0$ and $D = 1$, see Fig. 6.1(a), and that a hexagonal lattice occurs for $R = 1$ and $D = 2$, see Fig. 6.1(b).

6.3.6 Gabor's Signal Expansion on a Non-orthogonal Lattice

If we define the two-dimensional array λ_{mk} as

$$\lambda_{mk} = \sum_{n=-\infty}^{\infty} \delta_{k-mR-nD}, \tag{6.3.17}$$

Gabor's signal expansion on a non-orthogonal lattice can be expressed as [cf. Eq. (6.3.1)]

$$\varphi(t) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \lambda_{mk}\, a_{mk}\, g_{mk}(t) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a'_{mk}\, g_{mk}(t), \tag{6.3.18}$$

while – with a different analysis window $w(t)$, though! – the expansion coefficients a_{mk} are still determined by the Gabor transform (6.3.2). Of course, since we only need the limited array $a'_{mk} = \lambda_{mk}\, a_{mk}$ – which is, in fact, a properly sampled version of the full array a_{mk} – we need only calculate the coefficients a_{mk} for those values of m and k for which $k - mR$ is an integer multiple of D. We note that the Fourier transform $\bar{a}'(x,y)$ of the limited array a'_{mk} is related to the Fourier transform $\bar{a}(x,y)$ of the full array a_{mk} via the periodization relation

$$\bar{a}'(x,y) = \frac{1}{D} \sum_{n=0}^{D-1} \bar{a}\left(x - \frac{n}{D}, y - \frac{nR}{D}\right) \tag{6.3.19}$$

and thus

$$a'_r(x,y) = \frac{1}{D} \sum_{n=0}^{D-1} a_r \left(x - \frac{n}{D}, y - \frac{nR}{D}\right). \tag{6.3.20}$$

In the non-orthogonal case, the biorthogonality condition takes the form [cf. Eq. (6.3.3)]

$$\sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \lambda_{mk}\, g_{mk}(t_1)\, w^*_{mk}(t_2) = \delta(t_1 - t_2) \tag{6.3.21}$$

and leads to the equivalent but simpler expression [cf. Eq. (6.3.4)]

$$\frac{2\pi}{D\Omega} \sum_{m=-\infty}^{\infty} g(t - mT)\, w^*\left(t - \left[mT + n\frac{2\pi}{D\Omega}\right]\right) \exp\left(j2\pi m \frac{nR}{D}\right) = \delta_n. \tag{6.3.22}$$

Note that for $R = 0$ and $D = 1$, for which we have a rectangular lattice [see Fig. 6.1(a)], Eq. (6.3.22) reduces to Eq. (6.3.4), and that for $R = 1$ and $D = 2$, for which we have a hexagonal lattice [see Fig. 6.1(b)], Eq. (6.3.22) takes the form

$$\frac{\pi}{\Omega} \sum_{m=-\infty}^{\infty} g(t - mT)\, w^*\left(t - \left[mT + n\frac{\pi}{\Omega}\right]\right) (-1)^{mn} = \delta_n. \tag{6.3.23}$$

The biorthogonality condition expressed in terms of the Zak transforms of the window functions now takes the form, cf. Eq. (6.3.13),

$$\frac{T}{Dq} \sum_{r=0}^{p-1} g_{s_1,r}(x,y)\, w^*_{s_2,r}\left(x - \frac{n}{D}, y - \frac{nR}{D}\right) = \delta_n \delta_{s_1 - s_2}, \tag{6.3.24}$$

with $s_1, s_2 = 0, 1, \ldots, q-1$ and $n = 0, 1, \ldots, D-1$, and allows an easy determination of the analysis window $w(t)$ for a given synthesis window $g(t)$. For $R = 0$ and $D = 1$, for instance, relation (6.3.24) reduces to Eq. (6.3.13), while for $R = 1$, $D = 2$, $q = 1$, and p an even integer – which corresponds to the integer ($p/2$-times) oversampled hexagonal case – it reduces to

$$\frac{T}{2} \sum_{r=0}^{p-1} g_{0,r}(x,y)\, w^*_{0,r-np/2}(x,y)\, (-1)^{nr} = \delta_n \quad (n = 0, 1;\ p \text{ even}), \tag{6.3.25}$$

from which the Zak transform $\tilde{w}(t, \omega; T)$ and hence the window function $w(t)$ can easily be determined.

Since we have related Gabor's signal expansion on a non-orthogonal lattice to sampling on a denser but orthogonal lattice, followed by restriction to a sub-lattice that corresponds to the non-orthogonal lattice, we can still use all the techniques that are developed for rectangular lattices, in particular the technique of determining Gabor's expansion coefficients via the Zak transform, cf. Eq. (6.3.12).

6.3.7 Summary and Conclusions

Gabor's signal expansion and the Gabor transform on a rectangular lattice have been introduced, along with the Fourier transform of the array of expansion coefficients and the Zak transforms of the signal and the window functions. Based on these Fourier and Zak transforms, the sum-of-products forms for the Gabor expansion and the Gabor transform, which hold in the rationally oversampled case, have been derived.

We have then studied Gabor's signal expansion and the Gabor transform based on a non-orthogonal sampling geometry. We have done this by considering the non-orthogonal lattice as a sub-lattice of an orthogonal lattice. This procedure allows us to use all the formulas that hold for the orthogonal sampling geometry. In particular we can use the sum-of-products forms that hold in the case of a rationally oversampled rectangular lattice.

We finally note that if everything remains to be based on a rectangular sampling geometry, it will be easier to extend the theory of the Gabor scheme to higher-dimensional signals; see, for instance, [6], where the multi-dimensional case is treated for continuous-time as well as discrete-time signals.

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