On the Moments of the Wigner Distribution and the Fractional Fourier Transform

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Abstract— **The connection between the Wigner distribution and the squared modulus of the fractional Fourier transform – which are both well-known time-frequency representations of a signal – is established. In particular the Radon-Wigner transform is used, which relates projections of the Wigner distribution to the squared modulus of the fractional Fourier transform. Moments of the Wigner distribution are then expressed in terms of moments of the fractional Fourier transform. Relations for the fractional Fourier transform moments are derived, and some new procedures are presented with the help of which Wigner distribution moments can be determined by using the fractional Fourier transform.**

Keywords— **Wigner distribution, fractional Fourier transform, time-frequency signal analysis**

I. Introduction

A Fourier transformation maps a one-dimensional time signal $x(t)$ into a one-dimensional frequency function $X_{\pi/2}(f)$, the signal spectrum. Although the Fourier transform (FT) provides the signal's spectral content, it fails to indicate the time location of the spectral components, which is important, for example, when we consider non-stationary or time-varying signals. In order to describe these signals, time-frequency representations (TFRs) are used. A TFR maps a one-dimensional time signal into a two-dimensional function of time *and* frequency. In this paper we consider the fractional FT, which belongs to the class of *linear* TFRs, and establish its connection to the Wigner distribution (WD), which is one of the most widely used *quadratic* TFRs in electrical engineering. In particular we will use the Radon-Wigner transform (RWT), which relates projections of TFRs to the squared modulus of the fractional FT.

Since the application of the different TFRs often depends on how informative their moments are and how easily these moments can be measured or calculated, it is worthwhile to look for ways in which the moments can be determined. The connection between the WD and the RWT permits to find an optimal way for the calculation of the known WD moments and permits to introduce fractional FT moments that can be useful for signal analysis. We conclude that all frequently used moments of the WD can be obtained from the RWT.

II. Wigner distribution and ambiguity function

The *Wigner distribution* is defined as [1, Chapter 12]

$$\begin{aligned} W_x(t,f) &= \int_{-\infty}^{\infty} x(t+\tfrac{1}{2}\tau)x^*(t-\tfrac{1}{2}\tau)\exp(-j2\pi f\tau)d\tau \\ &= \int_{-\infty}^{\infty} X_{\pi/2}(f+\tfrac{1}{2}\nu)X^*_{\pi/2}(f-\tfrac{1}{2}\nu)\exp(j2\pi\nu t)\,d\nu, \end{aligned} \tag{1}$$

where $x(t)$ is a time signal and $X_{\pi/2}(f)$ its FT. The WD is always real-valued, but not necessarily positive; it preserves time and frequency shifts, and satisfies the marginal properties, which means that the frequency and time integrals of the WD, $\int W_x(t,f)df$ and $\int W_x(t,f)dt$, correspond to the signal's instantaneous power $|x(t)|^2$ and its spectral energy density $|X_{\pi/2}(f)|^2$, respectively. The WD can roughly be considered as the signal's energy distribution over the time-frequency plane, although the uncertainty principle prohibits the interpretation as a point time-frequency energy density.

If in Eqs. (1) the integrations are carried out over the common variable (t or f) instead of over the difference variable (τ or ν), we get the *ambiguity function* $A_x(\tau,\nu)$, which is related to the WD by means of a Fourier transformation [1, Chapter 12]:

$$A_x(\tau,\nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_x(t,f)\exp[-j2\pi(\nu t - f\tau)]dtdf. \tag{2}$$

The ambiguity function (AF) is another widely-used quadratic TFR.

III. Fractional Fourier transform

The *fractional Fourier transform* of a signal $x(t)$ is defined as [2], [3]

$$X_\alpha(u) = \mathcal{R}^\alpha \left[x(t)\right](u) = \int_{-\infty}^{\infty} K(\alpha,t,u)x(t)dt, \quad (3)$$

where the kernel $K(\alpha,t,u)$ is given by

$$\begin{aligned} K(\alpha,t,u) &= \frac{\exp(j\frac{1}{2}\alpha)}{\sqrt{j\sin\alpha}} \exp\left(j\pi\frac{(t^2+u^2)\cos\alpha - 2ut}{\sin\alpha}\right) \\ &= K(\alpha,u,t). \end{aligned} \quad (4)$$

The fractional FT with parameter α can be considered as a generalization of the ordinary FT; thus the fractional FT for $\alpha = \frac{1}{2}\pi$ and $\alpha = -\frac{1}{2}\pi$ reduces to the ordinary and inverse FT, respectively. For $\alpha = 0$ the fractional FT corresponds to the identity operation, $X_0(u) = \mathcal{R}^0[x(t)](u) = x(u)$, and for $\alpha = \pm\pi$ to the axis reversal operation, $X_{\pm\pi}(u) = \mathcal{R}^{\pm\pi}[x(t)](u) = x(-u)$. With respect to the parameter α, the fractional FT is continuous, periodic [$\mathcal{R}^{\alpha+2\pi n} = \mathcal{R}^\alpha$, with n an integer] and additive [$\mathcal{R}^\alpha\mathcal{R}^\beta = \mathcal{R}^{\alpha+\beta}$]. The inverse fractional FT can thus be written as

$$x(t) = \mathcal{R}^{-\alpha}\left[X_\alpha(u)\right](t) = \int_{-\infty}^{\infty} K(-\alpha,t,u)X_\alpha(u)du. \quad (5)$$

Since the Hermite-Gauss functions $\Psi_n(t) = (2^{n-1/2}n!)^{-1/2}\exp(-\pi t^2)H_n(\sqrt{2\pi}t)$, with $H_n(t)$ the Hermite polynomials, are eigenfunctions of the fractional FT with eigenvalues $\exp(-jn\alpha)$, and since they compose a complete orthonormal set, it is possible to write the fractional FT kernel in the alternative form

$$K(\alpha,t,u) = \sum_{n=0}^{\infty} \exp(-jn\alpha)\Psi_n(t)\Psi_n(u) = K(\alpha,u,t). \quad (6)$$

The important property of the fractional FT, which allows us to establish a connection between it and the WD, the AF, and other members of Cohen's class [1, Chapter 12] of quadratic TFRs, is that a fractional Fourier transformation produces a *rotation* of these functions in the time-frequency plane [2]:

$$\begin{array}{lcl} x(t) & \longleftrightarrow & W_x(t,f) \text{ and } A_x(\tau,\nu) \\ \downarrow \text{fractional FT} & & \downarrow \text{rotation of WD and AF} \\ X_\alpha(t) = \mathcal{R}^\alpha[x] & \longleftrightarrow & W_{X_\alpha}(t,f) \text{ and } A_{X_\alpha}(\tau,\nu), \end{array}$$

with $W_{X_\alpha}(t,f) = W_x(t\cos\alpha - f\sin\alpha, t\sin\alpha + f\cos\alpha)$ and $A_{X_\alpha}(\tau,\nu) = A_x(\tau\cos\alpha - \nu\sin\alpha, \tau\sin\alpha + \nu\cos\alpha)$. Hence, we conclude that $W_{X_\alpha}(u,v) = W_x(t,f)$, where the coordinates (u,v) in the rotated frame are related to (t,f) via the matrix relationship

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} t \\ f \end{pmatrix}. \quad (7)$$

A similar relation holds for the AF.

The main properties of the fractional FT are listed in Table 1, and the fractional FT of some common functions are given in Table 2.

IV. Fractional power spectrum and Radon-Wigner transform

If we introduce the *fractional power spectrum* $|X_\alpha(t)|^2$ as the squared modulus of the corresponding fractional FT, we obtain that these fractional power spectra are the projections of the WD upon a direction at an angle α in the time-frequency plane [4],

$$|X_\alpha(t)|^2 = \int_{-\infty}^{\infty} W_x(t\cos\alpha - f\sin\alpha, t\sin\alpha + f\cos\alpha)df, \quad (8)$$

and that they are related to the AF by a Fourier transformation:

$$|X_\alpha(t)|^2 = \int_{-\infty}^{\infty} A_x(f\sin\alpha, -f\cos\alpha)\exp(-j2\pi ft)df. \quad (9)$$

The set of fractional power spectra for the angles $\alpha \in [0,\pi)$ is called the *Radon-Wigner transform*, because it defines the Radon transform [1, Chapter 8] of the WD. The WD can be obtained from the RWT by applying an inverse Radon transformation. Note also that the AF can be reconstructed from the RWT by a simple inverse Fourier transformation, see Eq. (9), and that other members of Cohen's class of TFRs can be constructed subsequently.

The RWT can be considered as a quadratic TFR of $x(t)$, which has very advantageous properties. It is positive, invertible up to a constant phase factor, and ideally combines the concepts of the instantaneous power $|x(t)|^2$ and the spectral energy density $|X_{\pi/2}(f)|^2$. The association of the RWT with the power distributions allows its direct measurement in optics and quantum mechanics, which opens new perspectives for optical signal processing and quantum state characterization.

V. Fractional Fourier transform moments

The application of the different TFRs often depends on how informative their moments are and how easily these moments can be measured or calculated. The established connection between the WD and the RWT permits to find an optimal way for the calculation of the known WD moments and permits to introduce *fractional FT moments* that can be useful for signal analysis.

By analogy with time and frequency moments [5],

$$\begin{aligned} \int_{-\infty}^{\infty} t^n |x(t)|^2 dt &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t^n W_x(t,f) dtdf \\ \int_{-\infty}^{\infty} f^n \left|X_{\pi/2}(f)\right|^2 df &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^n W_x(t,f) dtdf, \end{aligned}$$

the fractional FT moments can be introduced:

$$\int_{-\infty}^{\infty} t^n |X_\alpha(t)|^2 dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t^n W_x(t\cos\alpha - f\sin\alpha, t\sin\alpha + f\cos\alpha) dtdf.$$

The *zero-order* fractional FT moment E,

$$\begin{aligned} E &= \int_{-\infty}^{\infty} |X_\alpha(t)|^2 dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_x(t,f) dtdf \\ &= \int_{-\infty}^{\infty} |x(t)|^2 dt, \end{aligned} \tag{10}$$

is invariant under fractional Fourier transformation, which expresses the energy conservation law of a unitary transformation, also known as Parseval's relation.

The normalized *first-order* fractional FT moment m_α,

$$\begin{aligned} m_\alpha &= \frac{1}{E}\int_{-\infty}^{\infty} t |X_\alpha(t)|^2 dt \\ &= \frac{1}{E}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t W_x(t\cos\alpha - f\sin\alpha, \\ &\qquad t\sin\alpha + f\cos\alpha) dtdf, \end{aligned} \tag{11}$$

is related to the center of gravity of the fractional power spectrum. One can write the simple connection

$$m_\alpha = m_0\cos\alpha + m_{\pi/2}\sin\alpha \tag{12}$$

between the first-order fractional FT moments. It is easy to see that the pair $(m_\alpha, m_{\alpha+\pi/2})$ is connected to $(m_0, m_{\pi/2})$ through the rotation transformation (7), and that $m_\alpha^2 + m_{\alpha+\pi/2}^2 = m_0^2 + m_{\pi/2}^2$ is invariant under fractional Fourier transformation. The fractional domain corresponding to the zero-centered fractional power spectrum can be found as $\tan\alpha = -m_0/m_{\pi/2}$.

The normalized *second-order* fractional FT moments w_α,

$$\begin{aligned} w_\alpha &= \frac{1}{E}\int_{-\infty}^{\infty} t^2 |X_\alpha(t)|^2 dt \\ &= \frac{1}{E}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t^2 W_x(t\cos\alpha - f\sin\alpha, \\ &\qquad t\sin\alpha + f\cos\alpha) dtdf, \end{aligned} \tag{13}$$

is related to the effective width of the signal in the fractional FT domain. The normalized *mixed* second-order fractional FT moments are given by

$$\begin{aligned} \mu_\alpha &= \frac{1}{E}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} tf W_x(t\cos\alpha - f\sin\alpha, \\ &\qquad t\sin\alpha + f\cos\alpha) dtdf \\ &= \frac{1}{4\pi jE}\int_{-\infty}^{\infty}\left[X_\alpha^*(t)\frac{\partial X_\alpha(t)}{\partial t}\right. \\ &\qquad \left. - X_\alpha(t)\frac{\partial X_\alpha^*(t)}{\partial t}\right] t dt. \end{aligned} \tag{14}$$

The following relationships between the second-order fractional FT moments hold:

$$\begin{aligned} w_\alpha &= w_0\cos^2\alpha + w_{\pi/2}\sin^2\alpha - \mu_0\sin 2\alpha \\ \mu_\alpha &= \tfrac{1}{2}(w_0 - w_{\pi/2})\sin 2\alpha + \mu_0\cos 2\alpha. \end{aligned} \tag{15}$$

In general all second-order moments w_α and μ_α can be obtained from any three second-order moments w_α taken for three different angles α from the region $[0,\pi)$. The mixed moment μ_0, for instance, can be expressed as $\mu_0 = \frac{1}{2}(w_0 + w_{\pi/2}) - w_{\pi/4}$.

From Eqs. (15) we conclude that the sum of the signal widths in the position and the Fourier domain is invariant under fractional Fourier transformation:

$$w_\alpha + w_{\alpha+\pi/2} = w_0 + w_{\pi/2}. \tag{16}$$

We also conclude that the fractional domain corresponding to the extremum signal width w_α, can be found by solving the equation $\tan 2\alpha = -2\mu_0/(w_0 - w_{\pi/2})$. Due to the invariance relationship (16), the solution of this equation corresponds to the domain with the smallest w_0 and the largest $w_{\pi/2}$, or vice versa.

For the product of the signal widths we find

$$\begin{aligned} w_\alpha w_{\alpha+\pi/2} &= w_0 w_{\pi/2} \\ &\quad + \tfrac{1}{4}\left[\left(w_0 - w_{\pi/2}\right)^2 - 4\mu_0^2\right]\sin^2 2\alpha \\ &\quad + \tfrac{1}{2}\mu_0\left(w_0 - w_{\pi/2}\right)\sin 4\alpha, \end{aligned} \tag{17}$$

which expression is, in general, not invariant under fractional Fourier transformation; invariance does occur, for instance, in the case of eigenfunctions of the Fourier transformation $g(t)$ with $G_{\alpha+\pi/2}(t) = \exp(-j\frac{1}{2}\pi)G_\alpha(t)$, for which $w_\alpha = w_0 = w_{\pi/2}$ and $\mu_\alpha = 0$. Note that, due to the uncertainty principle, we have $w_\alpha w_{\alpha+\pi/2} \ge \frac{1}{4}$. The fractional FT domain where the product $w_\alpha w_{\alpha+\pi/2}$ has an extremum value, can be found by solving the equation $\tan 4\alpha = 4\mu_0(w_0 - w_{\pi/2})/[4\mu_0^2 - (w_0 - w_{\pi/2})^2]$.

From Eq. (15) we finally conclude the following property for the mixed moment μ_α:

$$\mu_\alpha = \tfrac{1}{2}(w_{\alpha-\pi/4} - w_{\alpha+\pi/4}) = -\mu_{\alpha\pm\pi/2}. \tag{18}$$

Instead of *global* moments, which we considered above, one can consider *local* fractional FT moments, which are related to such things as the instantaneous power and instantaneous frequency (for $\alpha = 0$) or the spectral energy density and group delay (for $\alpha = \frac{1}{2}\pi$) in the different fractional FT domains.

The local frequency in the fractional FT domain with parameter α is defined as

$$U_{X_\alpha}(t)$$

$$= \frac{\int_{-\infty}^{\infty} f W_x(t\cos\alpha - f\sin\alpha, t\sin\alpha + f\cos\alpha) df}{\int_{-\infty}^{\infty} W_x(t\cos\alpha - f\sin\alpha, t\sin\alpha + f\cos\alpha) df}$$
$$= \frac{1}{2\,|X_\alpha(t)|^2} \int_{-\infty}^{\infty} \left.\frac{\partial |X_\beta(\tau)|^2}{\partial\beta}\right|_{\beta=\alpha} \operatorname{sgn}(\tau - t) d\tau. \quad (19)$$

The local frequency $U_{X_\alpha}(t)$ is related to the phase $\varphi_\alpha(t) = \arg X_\alpha(t)$ of the fractional FT $X_\alpha(t)$ through $U_{X_\alpha}(t) = (1/2\pi)d\varphi_\alpha(t)/dt$. This implies that the derivative of the fractional power spectra with respect to the angle α defines the local frequency in the fractional domain, and that it can be used for solving the phase retrieval problem by measuring intensity functions only.

We finally mention the relationship between the central local fractional second-order moment and the instantaneous power in the fractional FT domain:

$$V_{X_\alpha}(t) = \frac{\int_{-\infty}^{\infty} \left[f - U_{X_\alpha}(t)\right]^2 f W_x(t\cos\alpha - f\sin\alpha, t\sin\alpha + f\cos\alpha) df}{\int_{-\infty}^{\infty} W_x(t\cos\alpha - f\sin\alpha, t\sin\alpha + f\cos\alpha) df} = -\frac{1}{4}\frac{d^2 \ln |X_\alpha(t)|^2}{dt^2}. \quad (20)$$

We conclude that all frequently used moments of the WD can be obtained from the RWT. The fractional FT moments may be helpful in the search for the most appropriate fractional domain to perform a filtering operation; in the special case of noise that is equally distributed throughout the time-frequency plane, for instance, the fractional domain with the smallest signal width is then evidently the most preferred one.

VI. Applications

The fractional FT and the WD are applied in such diverse fields as quantum mechanics, optics, and signal processing [1], [4], [5], [6].

The wide application of the fractional FT in optics is based on the fact that – in the paraxial approximation of the scalar diffraction theory – it describes the optical field evolution during propagation through a quadratic refractive index (lens-like) medium. The RWT, associated with the intensity distributions, is used in particular for the reconstruction of the WD and subsequently of the complex field amplitude (in the case of coherent light) or the two-point correlation function (in the case of partially coherent light).

In signal processing the RWT was primarily developed for detection and classification of multi-component linear FM signals in noise [6]. Since the fractional FT of the chirp-type signal $\exp(-j\pi t^2/\tan\beta)$ reads

$$\exp(j\tfrac{1}{2}\alpha)\sqrt{\sin\beta/\sin(\beta-\alpha)}\exp[-j\pi u^2/\tan(\beta-\alpha)],$$

it becomes proportional to a Dirac-function $\delta(u)$ for $\alpha \to \beta$, and it can be detected as a local maximum on the RWT map. Analogously, in order to remove chirp-type noise, a notch filter, which minimizes the signal information loss, can be placed at the proper point of the corresponding fractional FT domain [4].

Instead of performing, as usual, filtering operations in the frequency or the time domain, it can be done in a more appropriate fractional domain, for instance, the one that corresponds to the best signal/noise time-frequency separation [4].

The complexity of computation for the fractional FT is $O(N \log N)$ [7], where N is the time-bandwidth product of the signal.

References

[1] A. D. Poularikas, ed., *The Transforms and Applications Handbook*, CRC Press, Alabama, 1996. Chapter 8, S. R. Deans, “Radon and Abel Transforms,” and Chapter 12, G. F. Boudreaux-Bartels, “Mixed Time-frequency Signal Transformations.”

[2] L. B. Almeida, “The fractional Fourier transform and time-frequency representations,” *IEEE Trans. Signal Process.*, vol. **42**, pp. 3084-3091, 1994.

[3] H. M. Ozaktas, M. A. Kutay, and D. Mendlovic, “Introduction to the fractional Fourier transform and its applications,” in *Advances in Imaging and Electron Physics*, vol. **106**, ed. P. W. Hawkes, Academic Press, San Diego, CA, pp. 239-291, 1999.

[4] A. W. Lohmann, D. Mendlovic, Z. Zalevsky, “Fractional transformation in optics,” in *Progress in Optics*, vol. **XXXVIII**, ed. E. Wolf, Elsevier, Amsterdam, pp. 263-342, 1998.

[5] W. Mecklenbräuker and F. Hlawatsch, eds., *The Wigner Distribution – Theory and Applications in Signal Processing*, Elsevier, Amsterdam, 1997.

[6] J. C. Wood and D. T. Barry, “Tomographic time-frequency analysis and its application toward time-varying filtering and adaptive kernel design for multicomponent linear FM signals,” *IEEE Trans. Signal Process.*, vol. **42**, pp. 2094-2104, 1994.
[7] H. M. Ozaktas, O. Arıkan, M. A. Kutay, and Gözde Bozdağı, “Digital computation of the fractional Fourier transform,” *IEEE Trans. Signal Process.*, vol. **44**, pp. 2141-2150, 1996.

Table 1: Fractional Fourier transform properties

Linearity:

$$\mathcal{R}^\alpha\left[ax(t)+by(t)\right](u) = a\mathcal{R}^\alpha\left[x(t)\right](u) + b\mathcal{R}^\alpha\left[y(t)\right](u)$$

Parseval's equality:

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X_\alpha(u)Y_\alpha^*(u)du$$

Shift theorem (real τ):

$$\mathcal{R}^\alpha\left[x(t-\tau)\right](u) = X_\alpha(u-\tau\cos\alpha)\exp\left(j\pi\sin\alpha\left(\tau^2\cos\alpha - 2u\tau\right)\right)$$

Modulation theorem (real ν):

$$\mathcal{R}^\alpha\left[x(t)\exp(j2\pi\nu t)\right](u) = X_\alpha(u-\nu\sin\alpha)\exp\left(-j\pi\cos\alpha\left(\nu^2\sin\alpha - 2u\nu\right)\right)$$

Scaling theorem (real c and β):

$$\mathcal{R}^\alpha\left[x(ct)\right](u) = \sqrt{\frac{\cos\beta}{\cos\alpha}\frac{\exp(j\frac{1}{2}\alpha)}{\exp(j\frac{1}{2}\beta)}}\exp\left(j\pi u^2\cot\alpha\left(1-\frac{\cos^2\beta}{\cos^2\alpha}\right)\right)X_\beta\left(\frac{u}{c}\frac{\sin\beta}{\sin\alpha}\right),$$

where $\tan\beta = c^2\tan\alpha$

Table 2: Fractional Fourier transforms of some common functions

$x(t)$	$X_\alpha(u)$
$\delta(t-\tau)$	$\frac{\exp(j\frac{1}{2}\alpha)}{\sqrt{j\sin\alpha}}\exp\left(j\pi\frac{(\tau^2+u^2)\cos\alpha - 2u\tau}{\sin\alpha}\right)$
$\exp(j2\pi t\nu)$	$\frac{\exp(j\frac{1}{2}\alpha)}{\sqrt{\cos\alpha}}\exp\left(-j\pi(\nu^2+u^2)\tan\alpha + j2\pi u\nu\sec\alpha\right)$
$\exp\left(jc\pi t^2\right)$	$\frac{\exp(j\frac{1}{2}\alpha)}{\sqrt{\cos\alpha + c\sin\alpha}}\exp\left(j\pi u^2\frac{c-\tan\alpha}{1+c\tan\alpha}\right)$
$H_n(\sqrt{2\pi}t)\exp(-\pi t^2)$	$H_n(\sqrt{2\pi}u)\exp(-\pi u^2)\exp(-jn\alpha)$, H_n are the Hermite polynomials
$\exp\left(-c\pi t^2\right) \quad (c \geq 0)$	$\frac{\exp(j\frac{1}{2}\alpha)}{\sqrt{\cos\alpha + jc\sin\alpha}}\exp\left(\pi u^2\frac{j(c^2-1)\cot\alpha - c\csc^2\alpha}{c^2+\cot^2\alpha}\right)$