On the discrete version of Gabor's signal expansion, the Gabor transform, and the Zak transform

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Abstract— **Gabor's expansion of a discrete-time signal into a set of shifted and modulated versions of an elementary signal or synthesis window is introduced, along with the inverse operation, i.e., the Gabor transform, which uses an analysis window that is related to the synthesis window and with the help of which Gabor's expansion coefficients can be determined. The restriction to a signal and an analysis window that both have finite-support, leads to the concept of a discrete Gabor expansion and a discrete Gabor transform.**

After introduction of the discrete Fourier transform and the discrete Zak transform, it is possible to express the discrete Gabor expansion and the discrete Gabor transform as matrix-vector products. Using these matrix-vector products, a relationship between the analysis window and the synthesis window is derived. It is shown how this relationship enables us to determine the optimum synthesis window in the sense that it has minimum L_2 norm, and it is shown that this optimum synthesis window resembles best the analysis window.

Keywords— **time-frequency signal analysis, Gabor transform, Gabor's signal expansion, Zak transform**

I. Introduction

It is sometimes convenient to describe a (discrete-time) signal $\varphi(n)$ say, not in the time domain, but in the frequency domain by means of its *frequency spectrum*, i.e., the Fourier transform of $\varphi(n)$. This frequency spectrum shows us the *global* distribution of the energy of the signal as a function of frequency. However, one is often more interested in the momentary or *local* distribution of the energy as a function of frequency. This leads to the concept of a *local frequency spectrum*, where the signal is described in time and frequency, simultaneously.

A candidate for a local frequency spectrum is Gabor's signal expansion. In 1946 Gabor [1] suggested the expansion of a (continuous-time) signal into a discrete set of properly shifted and modulated versions of a Gaussian elementary signal [1], [2], [3]. Although Gabor restricted himself to a Gaussian-shaped elementary signal, his signal expansion holds for rather arbitrarily shaped elementary signals. It was shown that for an arbitrary elementary signal (or synthesis window, as it if often called) an analysis window could be found such that Gabor's expansion coefficients can be found as sampling values of a windowed Fourier transform.

In his original paper, Gabor restricted himself to a *critical* sampling of the time-frequency domain, in which case the expansion coefficients can be interpreted as independent data, i.e., degrees of freedom of a continuous-time signal. It is the aim of this paper to apply Gabor's concepts to discrete-time signals and to the case of *oversampling* (see, for instance, [4], [5]); in the case of oversampling the expansion coefficients are no longer independent, of course. Moreover, we will restrict ourselves to the case of a *finite-support* analysis window, which enables us to treat all Fourier transforms as discrete Fourier transforms, for which fast algorithms exist.

After a short review of Gabor's expansion and the Gabor transform, we introduce the discrete Gabor expansion and its companion, the discrete Gabor transform with which the expansion coefficients can be determined. We then introduce the mathematical tools that we will use: the discrete Fourier transform and the discrete Zak transform. We will use these mathematical tools to transform the discrete Gabor expansion and the discrete Gabor transform into another, mathematically more attractive form. We then present an elegant relationship between the synthesis window, which appears in the discrete Gabor expansion, and the analysis window, which appears in the discrete Gabor transform. We will show how, for an arbitrary finite-support analysis window, an optimum synthesis window can be found.

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II. Gabor expansion and Gabor transform

We start with *Gabor's signal expansion* [1], [6], [7], [8] for a discrete-time signal $\varphi(n)$,

$$\varphi(n) = \sum_{m=-\infty}^{\infty} \sum_{k=<K>} a_{mk} g(n-mN) e^{j2\pi kn/K}, \quad (1)$$

where the array a_{mk} is periodic in k with period K and where the expression $k = < K >$ throughout denotes a finite interval of K successive integers k ($k = 0, 1, ..., K-1$, for instance). The sequence $g(n)$ is known as the *elementary signal* or *synthesis window*. The array of *Gabor coefficients* a_{mk} can be found via the *Gabor transform*

$$a_{mk} = \sum_{n=-\infty}^{\infty} \varphi(n) w^*(n-mN) e^{-j2\pi kn/K}, \quad (2)$$

where the sequence $w(n)$ is known as the *analysis window*.

In Gabor's original case of critical sampling ($K = N$), it was possible (see, for instance, [6], [7], [8]) to formulate the relationship between the synthesis window $g(n)$ and the analysis window $w(n)$. It is the aim of this paper to show how, for a given analysis window $w(n)$, a synthesis window $g(n)$ can be found in the case of oversampling $K > N$ where, moreover, the analysis window $w(n)$ has a finite support N_w. For convenience, we consider signals $\varphi(n)$ that have a finite support N_φ, too; we remark that in the case that we are dealing with signals of longer (or even infinite) support, we can always split the signal in parts that do have a finite support N_φ, treat all these parts separately and apply an overlap-add technique. Under the conditions of finite support, the array a_{mk} has a finite support M in the m-variable, where the support M satisfies the condition $MN \geq N_\varphi + N_w - 1$.

III. Discrete Gabor expansion and discrete Gabor transform

We now introduce the periodized version A_{mk} of the array a_{mk} according to

$$A_{mk} = \sum_{r=-\infty}^{\infty} a_{m+rM,k}. \quad (3)$$

Note that the periodized array A_{mk} is periodic in m with period M, and that we can identify a_{mk} as one period of A_{mk}. We also introduce the periodized version $W(n)$ of the (finite support) analysis window $w(n)$ according to

$$W(n) = \sum_{r=-\infty}^{\infty} w(n+rMN). \quad (4)$$

Note that the periodized analysis window $W(n)$ is periodic with period MN and that we can identify $w(n)$ as one period of $W(n)$. We also periodize [cf. Eq. (4)] the (finite support) signal $\varphi(n)$ and the synthesis window $g(n)$ to get the periodized signal $\Phi(n)$ and the periodized synthesis window $G(n)$, respectively. Under the condition that K is a divisor of MN, it is not difficult to derive the relationships

$$\Phi(n) = \sum_{m=<M>} \sum_{k=<K>} A_{mk} G(n-mN) e^{j2\pi kn/K} \quad (5)$$

and

$$A_{mk} = \sum_{n=<MN>} \Phi(n) W^*(n-mN) e^{-j2\pi kn/K}, \quad (6)$$

which are fully periodized versions of Gabor's signal expansion (1) and the Gabor transform (2), respectively. Equation (5) is known as the *discrete Gabor expansion*, while Eq. (6) is known as the *discrete Gabor transform* [8], [9], [10], [11].

IV. Discrete Fourier transform and discrete Zak transform

We introduce the Fourier transform of a (periodized) two-dimensional array and the Zak transform [12], [13], [14] of a (periodized) discrete-time sequence. For convenience, we introduce two integers p and q ($p \geq q \geq 1$) that do not have common factors and for which the relationship $pN = qK$ holds; note that $K/N = p/q \geq 1$ represents the degree of oversampling.

The *discrete Fourier transform* $\bar{a}(n/K, l/M)$ of the periodized array A_{mk} is defined according to

$$\bar{a}\left(\frac{n}{K}, \frac{l}{M}\right) = \sum_{m=<M>} \sum_{k=<K>} A_{mk} e^{-j2\pi(ml/M - kn/K)}; \quad (7)$$

we will throughout denote the discrete Fourier transform of an array by the same symbol as the array itself, but marked by a bar on top of it. Note that the discrete Fourier transform $\bar{a}(n/K, l/M)$ is periodic in the variables n and l with periods K and M, respectively.

The *discrete Zak transform* $\tilde{w}(n, l/MN; N)$ of the periodized analysis window $W(n)$ is defined as a one-dimensional discrete Fourier transform of the sequence $W(n+mN)$ (with $m = < M >$ and n being a mere parameter), hence

$$\tilde{w}\left(n, \frac{l}{MN}; N\right) =$$

$$\sum_{m=<M>} W(n+mN)e^{-j2\pi mN(l/MN)}; \qquad (8)$$

we will throughout denote the discrete Zak transform of a sequence by the same symbol as the sequence itself, but marked by a tilde on top of it. We remark that the discrete Zak transform $\tilde{w}(n, l/MN; N)$ is *periodic* in the frequency variable l with period M and *quasi-periodic* in the time variable n with quasi-period N: $\tilde{w}(n+mN, [l+kM]/MN; N) = \tilde{w}(n, l/MN; N)\exp[j2\pi mN(l/MN)]$.

Under the condition that p is a divisor of M, we can also introduce the discrete Zak transform $\tilde{\varphi}(n, l/MN; pN)$ of $\Phi(n)$. Due to the condition that p is a divisor of M, we can write $M = pL$, where L is an integer. From $pN = qK$ and $M = pL$ we conclude that $MN = qKL$, which implies that K is a divisor of MN. The latter condition is exactly the condition that should hold to be able to derive Eq. (5). Moreover, from $pN = qK$ and assuming that p and q do not have common factors, we also conclude that p is a divisor of K; hence we can write $K = pJ$, where J is an integer. We thus conclude that K, M, and N can be expressed in terms of the integers p, q (with $p \geq q \geq 1$, and p and q not having common factors), J and L: $K = pJ$, $M = pL$, and $N = qJ$.

V. Discrete Gabor expansion and discrete Gabor transform expressed as matrix-vector products

Using the discrete Fourier transform and the discrete Zak transform, it can be shown [15] that the discrete Gabor expansion (5) and the discrete Gabor transform (6) can be transformed into the *sum-of-products forms*,

$$\tilde{\varphi}\left(n+sK, \frac{l}{MN}; pN\right) = \frac{1}{p}\sum_{r=<p>}$$

$$\bar{a}\left(\frac{n}{K}, \frac{l+rM/p}{M}\right)\tilde{g}\left(n+sK, \frac{l+rM/p}{MN}; N\right) \qquad (9)$$

and

$$\bar{a}\left(\frac{n}{K}, \frac{l+rM/p}{M}\right) = K\sum_{s=<q>}$$

$$\tilde{\varphi}\left(n+sK, \frac{l}{MN}; pN\right)\tilde{w}^*\left(n+sK, \frac{l+rM/p}{MN}; N\right), \qquad (10)$$

respectively.

It is not difficult to express the sum-of-products forms (9) and (10) as *matrix-vector products*, yielding [15]

$$\boldsymbol{\phi} = \frac{1}{p}\,\mathbf{G}\mathbf{a} \qquad (11)$$

and

$$\mathbf{a} = K\,\mathbf{W}^*\boldsymbol{\phi}, \qquad (12)$$

respectively, where, as usual, the asterisk in connection with vectors and matrices denotes complex conjugation *and* transposition. We remark that the p-dimensional column vector $\mathbf{a} = \mathbf{a}(n,l)$ represents the Fourier transform of the Gabor coefficients A_{mk}, the q-dimensional column vector $\boldsymbol{\phi} = \boldsymbol{\phi}(n,l)$ represents the Zak transform of the signal $\Phi(n)$, and the $(q\times p)$-dimensional matrices $\mathbf{W} = \mathbf{W}(n,l)$ and $\mathbf{G} = \mathbf{G}(n,l)$ represent the Zak transforms of the analysis window $W(n)$ and the synthesis window $G(n)$, respectively. Note that Eq. (11) represents q equations in p unknowns, whereas Eq. (12) represents p equations in q unknowns. In the case of oversampling ($p > q \geq 1$) the former set of equations is thus *underdetermined*.

VI. Relationship between the analysis and the synthesis window

We now prove that the discrete Gabor expansion (5) and the discrete Gabor transform (6) form a transform pair, by showing that for any analysis window $W(n)$ a synthesis window $G(n)$ can be constructed. Instead of doing this by directly combining Gabor's signal expansion and the Gabor transform, we use the matrix-vector products (11) and (12).

If we substitute from Eq. (12) into Eq. (11) we get $\boldsymbol{\phi} = (K/p)\mathbf{G}\mathbf{W}^*\boldsymbol{\phi}$, which relation should hold for any arbitrary vector $\boldsymbol{\phi}$ [i.e., for any arbitrary signal $\Phi(n)$]. This condition immediately leads to the relationship

$$\frac{K}{p}\mathbf{G}\mathbf{W}^* = \frac{K}{p}\mathbf{W}\mathbf{G}^* = \mathbf{I}_q, \qquad (13)$$

where $\mathbf{I}_q$ is the $(q\times q)$-dimensional identity matrix.

Let us now consider Eq. (13) in the general case of oversampling. In that case we have $q < p$, which implies that $\mathbf{W}$ is not a square matrix and does not have a normal inverse $\mathbf{W}^{-1}$, and that Eq. (13) does not have a unique solution. It is well known that, under the condition that $\mathrm{rank}(\mathbf{W}) = q$, the *optimum solution* in the sense of the *minimum L_2 norm* can now be found with the help of the so-called *generalized (Moore-Penrose) inverse* $\mathbf{W}^\dagger = \mathbf{W}^*(\mathbf{W}\mathbf{W}^*)^{-1}$. The optimum solution $\mathbf{G}_{opt}$ then reads

$$\mathbf{G}_{opt} = \frac{p}{K}\left(\mathbf{W}^\dagger\right)^* = \frac{p}{K}(\mathbf{W}\mathbf{W}^*)^{-1}\mathbf{W}. \qquad (14)$$

Of course, if we proceed in this way, we will find, for any n and l, the minimum L_2 norm solution for the matrix $\mathbf{G}$. It is not difficult to show, however, that the minimum L_2 norm of $\mathbf{G}$ corresponds to the minimum L_2 norm of the discrete Zak transform $\tilde{g}(n, l/MN; N)$, and thus, with the help of Parseval's energy theorem, to the minimum L_2 norm of the synthesis window $G(n)$.

Instead of looking for the optimum solution $\mathbf{G}_{opt}$ in the sense of the minimum L_2 norm of $\mathbf{G}$, we could as well look for the optimum solution $\mathbf{G}_F$ in the sense of the minimum L_2 norm of the difference $\mathbf{G} - \mathbf{F}$; in this way we would find the matrix $\mathbf{G}$ that resembles best the matrix $\mathbf{F}$. As a result we then find

$$\mathbf{G}_F^* = \mathbf{G}_{opt}^* + \left(\mathbf{I}_p - \mathbf{W}^\dagger \mathbf{W}\right) \mathbf{F}^*. \qquad (15)$$

An obvious choice for the matrix $\mathbf{F}$ would be a matrix that is proportional to the matrix $\mathbf{W}$. From Eq. (15) we then have

$$\mathbf{G}_W^* = \mathbf{G}_{opt}^* + \left[\mathbf{I}_p - \mathbf{W}^* (\mathbf{W}\mathbf{W}^*)^{-1} \mathbf{W}\right] \mathbf{W}^*, \qquad (16)$$

but the second term in the right-hand side of this relationship vanishes. We thus reach the important conclusion that $\mathbf{G}_W = \mathbf{G}_{opt}$; hence, the synthesis window $G_{opt}(n)$ that has the minimum L_2 norm is the same as the synthesis window $G_W(n)$ whose difference from the analysis window $W(n)$ has the minimum L_2 norm, and resembles best this analysis window.

As an example, we consider a Gaussian analysis window $w(n)$ that is symmetrical around the point $\frac{1}{2}(N-1)$:

$$w(n) = e^{-(\pi/pN^2)(n - \frac{1}{2}[N-1])^2}. \qquad (17)$$

The discrete Zak transform $\tilde{w}(n, l/MN; N)$ of the analysis window reads

$$\tilde{w}\left(n, \frac{l}{MN}; N\right) =$$
$$e^{-(\pi/pN^2)(n - \frac{1}{2}[N-1])^2} \theta_3\left(z; e^{-\pi/p}\right), \qquad (18)$$

where $\theta_3(z; e^{-\pi/p})$ is a *theta function* [16] with *nome* $e^{-\pi/p}$ and where we have set $z = \pi(l/M) - j\pi(n - \frac{1}{2}[N-1])/pN$. We remark that the function $\theta_3(z; e^{-\pi/p})$ has zeros for $z = \pi(k + \frac{1}{2}) - j\pi(m + \frac{1}{2})/p$; hence, although a zero will not be reached for integer values of n, the value of $\tilde{w}(n, l/MN; N)$ will be very small for l in the neighbourhood of $(k + \frac{1}{2})M$ and n in the neighbourhood of $mN - \frac{1}{2}$. The discrete Zak transform $\tilde{w}(n, l/MN; N)$ for several values of the parameter p has been depicted in Fig. 1. We remark that for increasing values of p, the Zak transform becomes less dependent upon n.

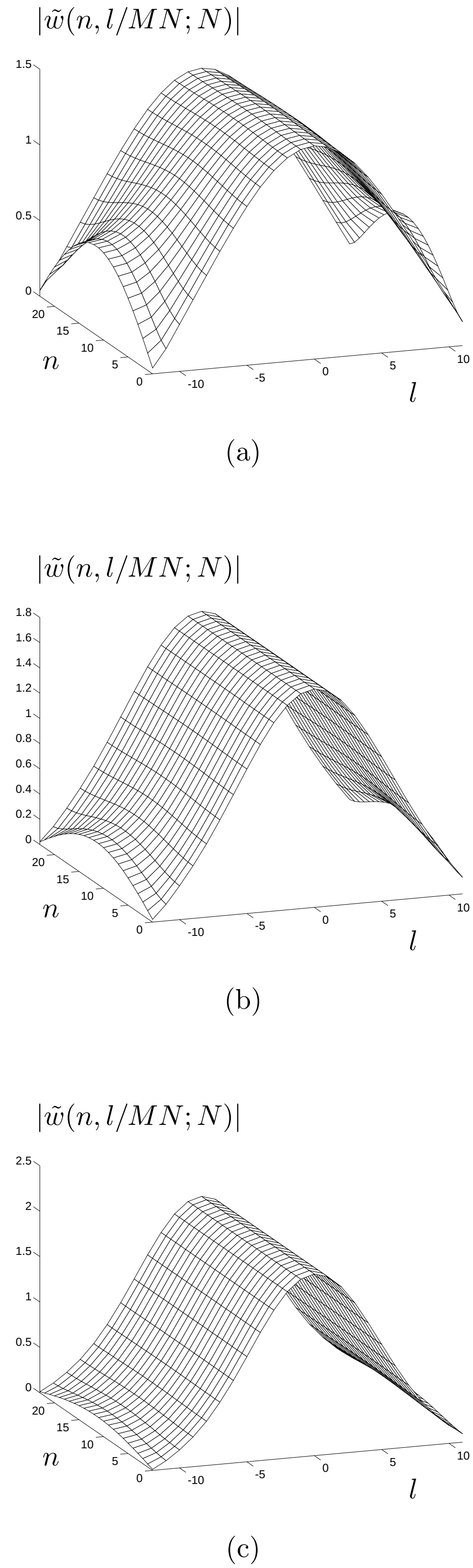


Fig. 1. The discrete Zak transform $\tilde{w}(n, l/MN; N)$ in the case of a Gaussian window function $w(n) = \exp[-(\pi/pN)^2(n - \frac{1}{2}[N-1])^2]$ (with $N = M = 24$) for (a) $p = 2$, (b) $p = 3$, and $p = 4$.

The width of the Gaussian analysis window (17) is, roughly, $N\sqrt{p}$. It will be clear that when we truncate the analysis window to an interval of length N_w where N_w is much larger than $N\sqrt{p}$, the discrete Zak transform of this truncated analysis window will almost be equal to the one of the untruncated analysis window.

Finally we apply the techniques outlined in this paper to determine the synthesis window $g_{opt}(n)$ that corresponds to a truncated Gaussian analysis window $w(n)$ [cf. Eq. (17)]. The discrete Zak transforms $\tilde{g}_{opt}(n, l/MN; N)$ of the optimum synthesis windows $g_{opt}(n)$ for different degrees of oversampling have been depicted in Fig. 2, while the synthesis windows themselves have been depicted in Fig. 3. We observe that the resemblance between (the Zak transforms of) the synthesis window and the analysis window increases with increasing degree of oversampling.

VII. Conclusion

We have studied Gabor's expansion of a discrete-time signal into a set of shifted and modulated versions of an elementary signal (or synthesis window). We have also considered the inverse operation – the Gabor transform – with which Gabor's expansion coefficients can be determined. In particular we have considered the discrete Gabor transform, in which the analysis window and the signal must have a finite support. It will be clear that the discrete Gabor transform can also be used to determine Gabor's expansion coefficients for a signal whose support is not finite, if we apply an overlap-add technique well-known in digital signal processing.

We have introduced the discrete Zak transform, and we have shown how this transform, together with the discrete Fourier transform, can be used to represent the discrete Gabor transform and the discrete Gabor expansion in mathematically more attractive sum-of-products forms. The sum-of-products form of the discrete Gabor transform enables us to determine Gabor's expansion coefficients in a different way, in which fast algorithms can be applied. This way of determining the expansion coefficients resembles the well-known procedure in which a convolution is transformed into product form by means of a Fourier transformation and which allows the determination of the convolution product by performing a normal product in the frequency domain; the use of a *fast Fourier transform* algorithm would then lead to an algorithm known as the *fast convolution.* The analogous procedure to determine Gabor's expansion coefficients might thus be called the *fast Gabor transform.*

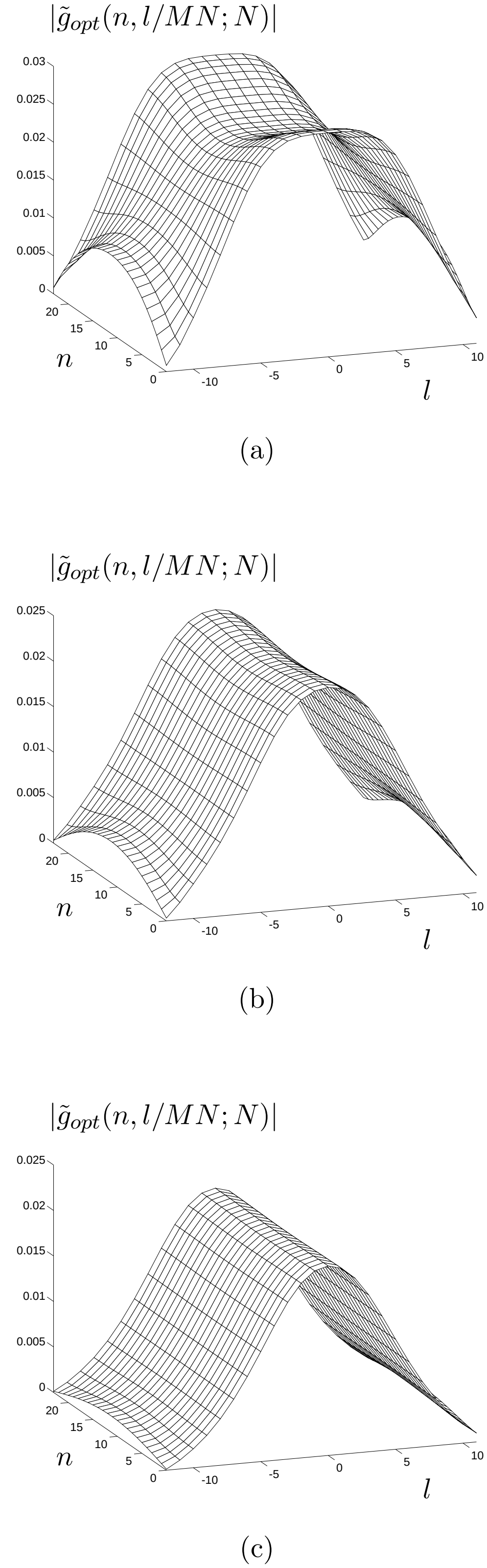
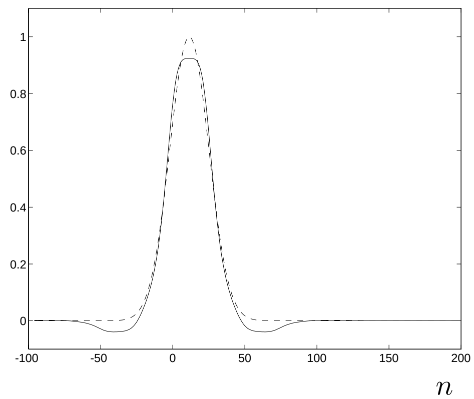
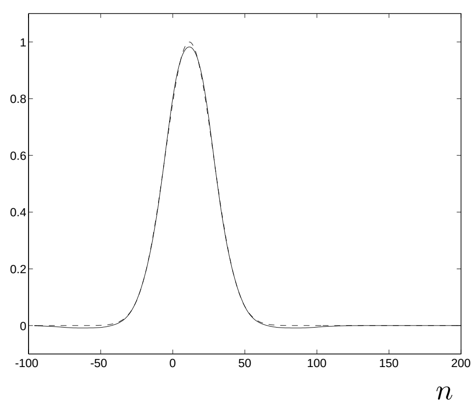


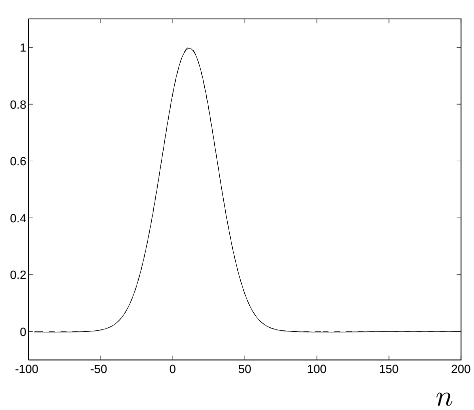
Fig. 2. The discrete Zak transform $\tilde{g}_{opt}(n, l/MN; N)$ in the case of a Gaussian window function $w(n) = \exp[-(\pi/pN)^2(n - \frac{1}{2}[N-1])^2]$ (with $N = M = 24$) for (a) $p = 2$, (b) $p = 3$, and (c) $p = 4$.



(a)



(b)



(c)

Fig. 3. A Gaussian analysis window $w(n) = \exp[-(\pi/pN)^2(n-\frac{1}{2}[N-1])^2]$ (with $N = 24$, dashed line) and its corresponding synthesis window $g_{opt}(n)$ (solid line) for (a) $p = 2$, (b) $p = 3$, and (c) $p = 4$.

Using the sum-of-products forms of the discrete Gabor transform and the discrete Gabor expansion enabled us to formulate a relationship between the analysis window and the synthesis window. In the general case of oversampling, this relationship leads to a set of equations that is underdetermined, which implies that the synthesis window that corresponds to a given analysis window is not unique. We have shown an easy way to determine the optimum synthesis window in the sense that it has minimum L_2 norm, and we have shown that this optimum synthesis window resembles best (in the sense of minimum L_2 norm, again) the analysis window.

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