Phase-space distributions in quasi-polar coordinates and the fractional Fourier transform

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Abstract

The ambiguity function and Cohen's class of bilinear phase-space distributions are represented in a quasi-polar coordinate system instead of in a Cartesian system. Relationships between these distributions and the fractional Fourier transform are derived; in particular, derivatives of the ambiguity function are related to moments of the fractional power spectra. A simplification is achieved for the description of underspread signals, for optical beam characterization, and for the generation of signal-adaptive phase-space distributions.

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1 Introduction

The application of the ambiguity function (AF) and the Wigner distribution (WD) [1] for the description of coherent and partially coherent optical fields was proposed more then 20 years ago [2, 3]. The introduction of the fractional Fourier transform (FT) [4] in optics made the AF and the WD more attractive for the analysis of optical signals, due to the rather simple technique of reconstructing these functions from the fractional FT squared modulus [5, 6]. Apart from the AF and the WD, a long list of phase-space distributions has been developed in digital signal processing. Our goal is to establish some optical or optoelectronic procedures that permit us to treat optical signals and images in the same manner as it is done in digital signal and image processing. The measurement of the fractional FT squared modulus, which can be done by an optical setup – as described by Lohmann *et al.* [7] and Ozaktas *et al.* [8], for instance – is considered here as a basic operation. In general, the unified approach for the description of signals and images in optics, electrical engineering and computer science might enrich the fields of optoelectronics and optical computing.

Phase-space distributions – like the short-time Fourier transform (STFT), the Wigner distribution [1], the ambiguity function, and other distributions belonging to the Cohen class [9] – are usually represented in a Cartesian system of coordinates (x, u), where x is a time or space variable and u is a (temporal or spatial) frequency variable. For the description of field propagation through a quadratic refractive index medium, it is more convenient to represent these distributions in a quasi-polar coordinate system (ρ, θ), where $x = \rho \cos\theta$ and $u = \rho \sin\theta$, with $\rho \in (-\infty, \infty)$ and $\theta \in [0, \pi)$. In this paper, based on the rotation of the AF and the WD due to fractional Fourier transformation [4], we express the AF in quasi-polar coordinates and derive some useful relationships for the description of optical signals. In particular the moments of the fractional FT of an optical signal are analyzed. The main reason to use *quasi*-polar coordinates instead of normal polar coordinates is that the relationship between the AF and the fractional FT squared modulus then takes the form of a simple 1-D Fourier transformation, see Eq. (11).

Once we have represented the AF in quasi-polar coordinates, it is easy to relate this function and the fractional FT to Cohen's class of phase-space distributions represented in quasi-polar coordinates. In particular the quasi-polar forms for the STFT and the WD are considered.

2 Ambiguity function in quasi-polar coordinates and fractional power spectra

The (cross-)ambiguity function $A_{fh}(x, u)$ of the optical signals $f(x)$ and $h(x)$ is defined as

$$A_{fh}(x,u) = \int_{-\infty}^{\infty} f(x' + \tfrac{1}{2}x) h^*(x' - \tfrac{1}{2}x) \exp(-i2\pi u x') dx' \tag{1}$$

or, equivalently, expressed in terms of the Fourier transforms $F_{\pi/2}(u)$ and $H_{\pi/2}(u)$ of $f(x)$ and $h(x)$, respectively, as

$$A_{fh}(x,u) = \int_{-\infty}^{\infty} F_{\pi/2}(u' + \tfrac{1}{2}u) H^*_{\pi/2}(u' - \tfrac{1}{2}u) \exp(i2\pi x u') du' \tag{2}$$

Note that if we put $x = 0$ in Eq. (1), we have the following relationship between $A_{fh}(x, u)$ along the line $x = 0$ on the one hand and the functions $f(x)$ and $g(x)$ on the other:

$$A_{fh}(0, u) = \int_{-\infty}^{\infty} f(x')h^*(x') \exp(-i2\pi ux')dx'. \tag{3}$$

Analogously, putting $u = 0$ in Eq. (2) one gets

$$A_{fh}(x, 0) = \int_{-\infty}^{\infty} F_{\pi/2}(u')H^*_{\pi/2}(u') \exp(i2\pi xu')du'. \tag{4}$$

Let us consider the transformation of the AF under fractional Fourier transformation. The fractional FT of a function $f(x)$ can be defined as [4]

$$R^\alpha [f(x)] (u) = F_\alpha(u) = \int_{-\infty}^{\infty} K(\alpha, x, u)f(x)dx, \tag{5}$$

where the kernel $K(\alpha, x, u)$ is given by

$$K(\alpha, x, u) = \frac{\exp(i\frac{1}{2}\alpha)}{\sqrt{i \sin\alpha}} \exp\left(i\pi \frac{(x^2 + u^2)\cos\alpha - 2ux}{\sin\alpha}\right). \tag{6}$$

Note that, in particular, $F_0(u) = f(u)$, $F_\pi(u) = f(-u)$, and that $F_{\pi/2}(u)$ corresponds to the normal FT of $f(x)$.

It is well known (see for example [2], [4] or [10]) that the fractional FT corresponds to a rotation of the AF and the WD in the phase plane. This rotation can best be described by introducing quasi-polar coordinates $R \in (-\infty, \infty)$ and $\beta \in [0, \pi)$ through

$$\begin{aligned} x &= R\cos\beta \\ u &= R\sin\beta, \end{aligned} \tag{7}$$

in which coordinates we denote the AF as $\widetilde{A}_{fh}(R, \beta) = A_{fh}(R\cos\beta, R\sin\beta)$. We remark that $\widetilde{A}_{fh}(R, 0) = A_{fh}(R, 0)$. Since the fractional FT corresponds to a rotation in the phase plane for the angle α,

$$\begin{array}{ccc} f(x),\ h(x) & \stackrel{\text{cross-ambiguity}}{\longrightarrow} & \widetilde{A}_{fh}(R,\beta) \\ \downarrow \text{fractional FT} & & \downarrow \text{rotation of AF} \\ R^\alpha[f],\ R^\alpha[h] & \stackrel{\text{cross-ambiguity}}{\longrightarrow} & \widetilde{A}_{R^\alpha[f],R^\alpha[h]}(R,\beta) = \widetilde{A}_{fh}(R,\beta+\alpha), \end{array} \tag{8}$$

and choosing $\beta = 0$ in this scheme, we obtain that

$$\widetilde{A}_{fh}(R, \alpha) = \widetilde{A}_{F_\alpha, H_\alpha}(R, 0) = A_{F_\alpha, H_\alpha}(R, 0). \tag{9}$$

Thus by analogy with Eq. (4) it follows that

$$\widetilde{A}_{fh}(R, \alpha) = \int_{-\infty}^{\infty} F_{\alpha+\pi/2}(x)H^*_{\alpha+\pi/2}(x) \exp(i2\pi Rx)dx. \tag{10}$$

A similar relationship was obtained in [11] and, for the case of the auto-AF, in [6].

We remark that Eq. (10) shows a simple FT relationship between the AF and the fractional FTs of two functions. In particular, for the auto-AF $A_f(x,u)$ of a signal $f(x)$, defined, for instance, by Eq. (1) with $h(x) = f(x)$, we obtain that it can be represented in quasi-polar coordinates as

$$\widetilde{A}_f(R, \alpha - \tfrac{1}{2}\pi) = \int_{-\infty}^{\infty} |F_\alpha(x)|^2 \exp(i2\pi Rx)dx, \tag{11}$$

from which relation we conclude that the fractional power spectrum $|F_\alpha(x)|^2$ is the 1-D Fourier transform of the AF with respect to the quasi-polar variable R. Note that Eq. (11) is very important for the experimental determination of the AF in optics, where the fractional power spectra related to intensity distributions can be measured by a simple optical setup [6].

The quasi-polar form of the AF permits to represent underspread processes, introduced and investigated in [12], in a very simple form. A nonstationary process is called underspread, if its expected AF satisfies the relationship

$$\left|\widetilde{A}_f(R,\alpha)\right| \to 0, \text{ for } R \geq R_0, \tag{12}$$

where the parameter R_0 characterizing the measure of the AF localization has to be sufficiently small. The connection (11) between the AF and the fractional power spectra might thus be useful for the description of such underspread processes.

3 Fractional Fourier transform moments

In this section we elaborate on Eq. (11) and relate the derivatives of the AF $\widetilde{A}_f(R, \alpha - \frac{1}{2}\pi)$ in the origin (i.e., for $R = 0$) to the fractional FT moments.

For the zero-order moment E we have

$$E = \int_{-\infty}^{\infty} |F_\alpha(x)|^2 dx = \widetilde{A}_f(R, \alpha - \tfrac{1}{2}\pi)\Big|_{R=0} = A_f(0,0). \tag{13}$$

Note that the zero-order moment E represents the signal's energy and that – in accordance with Parseval's theorem for a unitary transformation – it does not depend on α.

For the (normalized) first-order moments m_α we have

$$m_\alpha = \frac{1}{E}\int_{-\infty}^{\infty} |F_\alpha(x)|^2 x dx = \frac{1}{E}\frac{1}{2\pi i}\left.\frac{\partial \widetilde{A}_f(R, \alpha - \frac{1}{2}\pi)}{\partial R}\right|_{R=0}. \tag{14}$$

Note that the moments m_α are related to the centers of gravity of the fractional power spectra and that they are determined by the first-order derivative of the AF in the direction $\alpha - \frac{1}{2}\pi$. We remark that Eq. (14) is a generalization of the two well-known special cases [2] for $\alpha = \frac{1}{2}\pi$ and $\alpha = \pi$:

$$\begin{aligned}
\left.\frac{\partial \widetilde{A}_f(R, \alpha - \frac{1}{2}\pi)}{\partial R}\right|_{R=0,\alpha=\pi/2} &= \left.\frac{\partial A_f(x,u)}{\partial x}\right|_{x=0,u=0} &&= 2\pi i \int_{-\infty}^{\infty} \left|F_{\pi/2}(u)\right|^2 u du, \\
\left.\frac{\partial \widetilde{A}_f(R, \alpha - \frac{1}{2}\pi)}{\partial R}\right|_{R=0,\alpha=\pi} &= \left.\frac{\partial A_f(x,u)}{\partial u}\right|_{x=0,u=0} &&= 2\pi i \int_{-\infty}^{\infty} |f(-x)|^2 x dx.
\end{aligned}$$

From the relationship

$$\frac{\partial \widetilde{A}_f(R,\alpha-\frac{1}{2}\pi)}{\partial R} = \frac{\partial A_f(x,u)}{\partial x}\frac{\partial x}{\partial R} + \frac{\partial A_f(x,u)}{\partial u}\frac{\partial u}{\partial R} = \frac{\partial A_f(x,u)}{\partial x}\sin\alpha - \frac{\partial A_f(x,u)}{\partial u}\cos\alpha,$$

we have

$$m_\alpha = m_0\cos\alpha + m_{\pi/2}\sin\alpha. \tag{15}$$

The (normalized) second-order moments w_α, defined by

$$w_\alpha = \frac{1}{E}\int_{-\infty}^{\infty}|F_\alpha(x)|^2x^2dx = \frac{1}{E}\left(\frac{1}{2\pi i}\right)^2 \left.\frac{\partial^2\widetilde{A}_f(R,\alpha-\frac{1}{2}\pi)}{\partial R^2}\right|_{R=0}, \tag{16}$$

are related to the effective widths in the fractional FT domain and are determined by the second-order derivative of the AF in the direction $\alpha-\frac{1}{2}\pi$. Note that, due to the uncertainty principle, $w_0w_{\pi/2}\geq\frac{1}{4}$.

On the analogy of the relationship

$$\begin{aligned}\left.\frac{\partial^2 A_f(x,u)}{\partial x\partial u}\right|_{x=0,u=0} &= \pi i\int_{-\infty}^{\infty}\left[\frac{\partial F_{\pi/2}(u)}{\partial u}F^*_{\pi/2}(u) - F_{\pi/2}(u)\frac{\partial F^*_{\pi/2}(u)}{\partial u}\right]udu\\ &= \pi i\int_{-\infty}^{\infty}\left[\frac{\partial f(x)}{\partial x}f^*(x) - f(x)\frac{\partial f^*(x)}{\partial x}\right](-x)dx,\end{aligned}$$

we introduce the *mixed* second-order derivative of the AF, where we have a first-order derivative in the direction $\alpha-\frac{1}{2}\pi$ combined with a first-order derivative in the direction α:

$$\left.\frac{\partial^2\widetilde{A}_f(R,\alpha-\frac{1}{2}\pi)}{\partial R\partial R_\perp}\right|_{R=0} = \pi i\int_{-\infty}^{\infty}\left[\frac{\partial F_\alpha(x)}{\partial x}F^*_\alpha(x) - F_\alpha(x)\frac{\partial F^*_\alpha(x)}{\partial x}\right]xdx; \tag{17}$$

$R_\perp$ is a local coordinate for a given angle coordinate orthogonal to R. The (normalized) mixed second-order moments μ_α, associated with the mixed second-order derivative, are now defined as

$$\mu_\alpha = \frac{\pi i}{E}\left(\frac{1}{2\pi i}\right)^2\int_{-\infty}^{\infty}\left[\frac{\partial F_\alpha(x)}{\partial x}F^*_\alpha(x) - F_\alpha(x)\frac{\partial F^*_\alpha(x)}{\partial x}\right]xdx. \tag{18}$$

From the relationship

$$\frac{\partial^2\widetilde{A}_f(R,\alpha-\frac{1}{2}\pi)}{\partial R^2} = \frac{\partial^2 A_f(x,u)}{\partial x^2}\sin^2\alpha + \frac{\partial^2 A_f(x,u)}{\partial u^2}\cos^2\alpha - \frac{\partial^2 A_f(x,u)}{\partial x\partial u}\sin 2\alpha$$

we have

$$w_\alpha = w_o\cos^2\alpha + w_{\pi/2}\sin^2\alpha - \mu_0\sin 2\alpha. \tag{19}$$

This relationship is in accordance with the relationships between second-order moments of signals whose AFs are related through a canonical transformation described by a real, symplectic 2×2 $ABCD$ matrix, as described in [2]. In the particular case of a fractional Fourier transformation, corresponding with a rotation of the AF, we thus have [13]

$$\begin{bmatrix} w_\alpha & \mu_\alpha \\ \mu_\alpha & w_{\alpha+\pi/2}\end{bmatrix} = \begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix} w_0 & \mu_0\\ \mu_0 & w_{\pi/2}\end{bmatrix}\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix},$$

from which we, apart from Eq. (19), also get

$$\mu_\alpha = \tfrac{1}{2}(w_0 - w_{\pi/2})\sin 2\alpha + \mu_0 \cos 2\alpha. \tag{20}$$

Eqs. (19) and (20) show that mixed second order moments μ_α can be obtained from three moments w_α taken for three different angles α from the region $[0, \pi)$. This implies that the corresponding three fractional power spectra define all second-order moments.

One can easily express the well-known expression [14] for the local spatial frequency $U_0(r)$ at the position r,

$$U_0(r) = \frac{1}{2\pi i}\frac{1}{|f(r)|^2}\int_{-\infty}^{\infty} \left.\frac{\partial A_f(x,u)}{\partial x}\right|_{x=0} \exp(i2\pi r u)du, \tag{21}$$

in terms of the local moments of the fractional power spectra. Indeed, using the relationship

$$\begin{aligned}\left.\frac{\partial A_f(x,u)}{\partial x}\right|_{x=0} &= \left.\frac{\partial \widetilde{A}_f(R,\alpha-\frac{1}{2}\pi)}{\partial R}\frac{\partial R}{\partial x}\right|_{x=0} + \left.\frac{\partial \widetilde{A}_f(R,\alpha-\frac{1}{2}\pi)}{\partial \alpha}\frac{\partial \alpha}{\partial x}\right|_{x=0} \\ &= \left.-\frac{1}{u}\,\frac{\partial \widetilde{A}_f(-u,\alpha-\frac{1}{2}\pi)}{\partial \alpha}\right|_{\alpha=0},\end{aligned}$$

we have, after substituting from Eq. (11),

$$\left.\frac{\partial A_f(x,u)}{\partial x}\right|_{x=0} = -\frac{1}{u}\int_{-\infty}^{\infty} \left.\frac{\partial\,|F_\alpha(x)|^2}{\partial \alpha}\right|_{\alpha=0} \exp(-i2\pi u x)dx$$

and thus

$$U_0(r) = \frac{1}{2\,|F_0(r)|^2}\int_{-\infty}^{\infty} \left.\frac{\partial\,|F_\alpha(x)|^2}{\partial \alpha}\right|_{\alpha=0} \left(-\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{1}{u}\exp[i2\pi u(r-x)]du\right)dx.$$

Finally, with $-\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{1}{u}\exp[i2\pi u(r-x)]du = \text{sgn}(x-r)$, we get

$$U_0(r) = \frac{1}{2\,|F_0(r)|^2}\int_{-\infty}^{\infty} \left.\frac{\partial\,|F_\alpha(x)|^2}{\partial \alpha}\right|_{\alpha=0} \text{sgn}(x-r)dx, \tag{22}$$

which relationship can easily be generalized to

$$U_\beta(r) = \frac{1}{2\,|F_\beta(r)|^2}\int_{-\infty}^{\infty} \left.\frac{\partial\,|F_\alpha(x)|^2}{\partial \alpha}\right|_{\alpha=\beta} \text{sgn}(x-r)dx. \tag{23}$$

We remark that, with $F_\beta(r) = |F_\beta(r)|\exp[i\varphi_\beta(r)]$, the local spatial frequency $U_\beta(r)$ is related to the phase $\varphi_\beta(r)$ of the fractional FT through

$$U_\beta(r) = \frac{1}{2\pi}\frac{d\varphi_\beta(r)}{dr}. \tag{24}$$

In general, the complex-valued fractional FT $F_\beta(r)$ – and in particular the signal $f(r) = F_0(r)$ – can be completely reconstructed – except for a constant phase shift and the possible occurence of an additional π phase shift – from its intensity distribution $|F_\beta(r)|^2$ and its local spatial frequency $U_\beta(r)$. Since the latter quantity is determined by the derivative of the fractional power spectra, see Eq. (23), only two fractional power spectra for close angles suffice to solve the phase retrieval problem. Note that this result resembles the so-called 'transport of intensity equation' [17, 18, 19], which deals with the propagation of optical signals through free space. Such a resemblance is not surprising, since both the Fresnel transform and the fractional FT [20] belong to the class of canonical integral transforms, and the properties of any member of this class are related to and can be obtained from the properties of any other member.

We conclude that moments of phase-space distributions – like the ones for the WD, for instance, which are frequently used in signal processing – can be obtained from knowledge of the fractional power spectra. Introducing fractional FT moments might then be helpful, for example in the search for an appropriate fractional domain – i.e., a proper choice for α – in which filtering operations will be performed; in the special case of noise that is equally distributed in the phase plane, for instance, the fractional domain with the smallest signal width w_α is then evidently the most preferred choice.

4 Cohen's class of bilinear phase-space distributions in quasi-polar coordinates

The auto-AF $A_f(x,u)$ can be used for the generation of Cohen's class of bilinear phase-space distributions [14]

$$P_f(y,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(x,u) A_f(x,u) \exp[i2\pi(yu - vx)]dxdu, \tag{25}$$

where the kernel $\Phi(x,u)$ characterizes the particular distribution. Introducing the quasi-polar coordinates (ρ,φ) and (R,α) according to

$$\begin{matrix} y = \rho\cos\varphi \\ v = \rho\sin\varphi \end{matrix} \quad \text{and} \quad \begin{matrix} x = R\cos\alpha \\ u = R\sin\alpha, \end{matrix} \tag{26}$$

where $R,\rho \in (-\infty,\infty)$ and $\alpha,\varphi \in [0,\pi)$, Eq. (25) can be written as

$$\widetilde{P}_f(\rho,\varphi) = \int_{-\infty}^{\infty} RdR \int_0^{\pi} d\alpha\, \widetilde{\Phi}(R,\alpha)\widetilde{A}_f(R,\alpha)\exp[i2\pi R\rho\sin(\alpha-\varphi)]. \tag{27}$$

Since the AF is related to the fractional power spectrum by a 1-D Fourier transformation [see Eq. (11)] and since all members of Cohen's class – with a proper choice of the kernel – can be expressed in terms of the AF [see Eqs. (26) and (27)], we can conclude that all these members may also be expressed in terms of fractional power spectra. In particular for those kernels that have a simple form in quasi-polar coordinates, the quasi-polar description of the corresponding phase-space distribution is beneficial.

A quasi-polar representation of the kernel might be very useful, for instance, for the design of signal-adaptive, radially-Gaussian kernel distributions [15], $\widetilde{\Phi}(R,\alpha) = \exp[-R^2/2\sigma^2(\alpha)]$, where the kernel parameter $\sigma(\alpha)$ is found by solving the optimization problem in order to optimally suppress cross-components and to pass auto-components with as little distortion as possible. Also the special case of shift-scale invariant distributions, which are characterized by kernels $\Phi(x,u)$ of the form $\Psi(xu)$, might be treated in quasi-polar coordinates; in these coordinates a kernel $\widetilde{\Phi}(R,\alpha)$ then takes the form $\widetilde{\Psi}(\frac{1}{2}R^2\sin 2\alpha)$.

Some particular cases of the Cohen's class distributions, viz. the WD and the STFT, will be considered below.

4.1 Wigner distribution in quasi-polar coordinates

The (cross-)Wigner distribution $W_{fh}(x,u)$ of the two signals $f(x)$ and $h(x)$ is defined as

$$W_{fh}(x,u) = \int_{-\infty}^{\infty} f(x+\tfrac{1}{2}x')h^*(x-\tfrac{1}{2}x')\exp(-i2\pi ux')dx', \tag{28}$$

or, equivalently, expressed in terms of the Fourier transforms $F_{\pi/2}(u)$ and $H_{\pi/2}(u)$ of $f(x)$ and $h(x)$, respectively, as

$$W_{fh}(x,u) = \int_{-\infty}^{\infty} F_{\pi/2}(u+\tfrac{1}{2}u')H^*_{\pi/2}(u-\tfrac{1}{2}u')\exp(i2\pi u'x)du', \tag{29}$$

and relates to the AF through the two-dimensional FT:

$$W_{fh}(x,u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_{fh}(y,v)\exp[i2\pi(xv-uy)]dydv. \tag{30}$$

Using the property of the WD rotation under fractional Fourier transformation [4, 16], one obtains by analogy with the considerations in Section 2 that

$$\widetilde{W}_{fh}(R,\alpha) = \int_{-\infty}^{\infty} F_{\alpha+\pi/2}(\tfrac{1}{2}x)H^*_{\alpha+\pi/2}(-\tfrac{1}{2}x)\exp(i2\pi xR)dx. \tag{31}$$

If $h(x) = f(x)$, the latter equation – unlike for the case of the auto-AF [see Eq. (11)] – does not yield a direct connection between the auto-WD and the fractional power spectra. Only for the particular class of symmetric (even or odd) signals, $f(x) = \pm f(-x)$, we obtain a simple relationship between the WD and the intensity distribution of the fractional Fourier transform. Indeed, in this case the fractional FT is also symmetric, $F_\alpha(x) = \pm F_\alpha(-x)$, and hence $|F_\alpha(x)|^2 = |F_\alpha(-x)|^2$, and the WD in quasi-polar coordinates is the inverse 1-D FT of the fractional power spectra:

$$\begin{aligned}\widetilde{W}_f(\rho,\alpha) &= \pm\int_{-\infty}^{\infty}\left|F_{\alpha+\pi/2}(\tfrac{1}{2}x)\right|^2\exp(i2\pi x\rho)dx \\ &= \pm\int_{0}^{\infty}\left|F_{\alpha+\pi/2}(\tfrac{1}{2}x)\right|^2\cos(2\pi x\rho)dx.\end{aligned} \tag{32}$$

In the general case, the reconstruction of the WD from the fractional power spectra can be achieved by using Eqs. (11) and (30) in quasi-polar coordinates [cf. Eq. (27) with $\widetilde{\Phi}(R,\alpha)=1$]:

$$\widetilde{W}_f(\rho,\varphi) = \int_{-\infty}^{\infty} RdR\int_0^{\pi} d\alpha\,\widetilde{A}_f(R,\alpha)\exp[i2\pi R\rho\sin(\alpha-\varphi)]. \tag{33}$$

4.2 Short-time Fourier transform and the spectrogram in quasi-polar coordinates

In order to represent the short-time FT in quasi-polar coordinates, let us recall the connection between the STFT and the cross-AF. With $h(x)$ the window function, the STFT $G_{fh}(x,u)$ of a signal $f(x)$ is given by

$$G_{fh}(x,u) = \int_{-\infty}^{\infty} f(x')h^*(x'-x)\exp(-i2\pi ux')dx' \tag{34}$$

and is related to the AF by

$$G_{fh}(x,u) = \exp(-i\pi ux)A_{fh}(x,u). \tag{35}$$

Using quasi-polar coordinates [see Eq. (26)] and substituting from Eq. (10), we obtain the following relationship between the STFT and the fractional FTs of the signal and the window function:

$$\begin{aligned}\widetilde{G}_{fh}(R,\alpha) &= \exp(-i\tfrac{1}{2}\pi R^2\sin 2\alpha)\widetilde{A}_{fh}(R,\alpha)\\ &= \exp(-i\tfrac{1}{2}\pi R^2\sin 2\alpha)\int_{-\infty}^{\infty} F_{\alpha+\pi/2}(x')H^*_{\alpha+\pi/2}(x')\exp(i2\pi Rx')dx'.\end{aligned} \tag{36}$$

The spectrogram, which is the squared modulus of the corresponding STFT, equals the squared modulus of the corresponding AF and is related to the fractional FTs by

$$|\widetilde{G}_{fh}(R,\alpha)|^2 = |\widetilde{A}_{fh}(R,\alpha)|^2 = \left|\int_{-\infty}^{\infty} F_{\alpha+\pi/2}(x')H^*_{\alpha+\pi/2}(x')\exp(i2\pi Rx')dx'\right|^2. \tag{37}$$

Let us consider the particular case of the STFT with the Gaussian window function $h(x) = \exp(-\pi x^2)$. Since this function is an eigenfunction of the fractional FT, $H^*_{\alpha+\pi/2}(x) = h(x)$, and the STFT along a certain α-slice can be represented as a filter operation of the corresponding fractional FT $F_\alpha(x)$ with a Gaussian mask:

$$\widetilde{G}_{fh}(R,\alpha) = \exp\left(-i\tfrac{1}{2}\pi R^2\sin 2\alpha\right)\int_{-\infty}^{\infty} F_{\alpha+\pi/2}(x')\exp(-\pi x'^2)\exp(i2\pi Rx')dx'. \tag{38}$$

5 Conclusion

We have represented the AF in a quasi-polar coordinate system and we have shown how this AF is related to the fractional power spectra; in particular we have shown how its derivatives are related to the fractional FT moments. Based on the quasi-polar form of the AF, various phase-space distributions from Cohen's class can be represented in quasi-polar coordinates and their relationships with the fractional FT can be established. In particular, the quasi-polar form of the WD and the STFT have been considered. The advantages of the quasi-polar coordinate system for the description of underspread signals, the characterization of optical beams, and the construction of signal-adaptive distributions were discussed.

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