First-order optical systems with real eigenvalues

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Abstract

It is shown that a lossless first-order optical system whose real symplectic ray transformation matrix can be diagonalized and has only real eigenvalues, is similar to a separable hyperbolic expander in the sense that the respective ray transformation matrices are related by means of a similarity transformation. Moreover, it is shown how eigenfunctions of such a system can be determined, based on the fact that simple powers are eigenfunctions of a separable magnifier. As an example, a set of eigenfunctions of a hyperbolic expander is determined and the resemblance between these functions and the well-known Hermite-Gauss modes is exploited.

Key words: first-order optical systems, symplectic ray transformation matrix
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Recently it was shown [1] that any lossless first-order optical system whose real symplectic ray transformation matrix can be diagonalized and has only *unimodular* eigenvalues, is similar to a separable fractional Fourier transformer [2]. In the present paper we will treat the case when the ray transformation matrix has only *real* eigenvalues.

Any lossless first-order optical system can be described by its ray transformation matrix [3,4], which relates the position $\vec{r}_\mathrm{i}$ and direction $\vec{p}_\mathrm{i}$ of an incoming ray to the position $\vec{r}_\mathrm{o}$ and direction $\vec{p}_\mathrm{o}$ of the outgoing ray:

$$\begin{bmatrix}\vec{r}_\mathrm{o}\\ \vec{p}_\mathrm{o}\end{bmatrix} = \mathbf{T}\begin{bmatrix}\vec{r}_\mathrm{i}\\ \vec{p}_\mathrm{i}\end{bmatrix} = \begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}\begin{bmatrix}\vec{r}_\mathrm{i}\\ \vec{p}_\mathrm{i}\end{bmatrix}. \tag{1}$$

The vectors $\vec{r} = [r_1, \ldots, r_D]^\mathrm{t}$ and $\vec{p} = [p_1, \ldots, p_D]^\mathrm{t}$ are D-dimensional column vectors, the ray transformation matrix $\mathbf{T}$ is $2D \times 2D$, and all block matrices are $D \times D$; as usual, the superscript $^\mathrm{t}$ denotes transposition. The ray transformation matrix of such a system is real and symplectic. Symplecticity can be expressed elegantly in the form

$$\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}^{-1} = \begin{bmatrix}\mathbf{D}^\mathrm{t} & -\mathbf{B}^\mathrm{t}\\ -\mathbf{C}^\mathrm{t} & \mathbf{A}^\mathrm{t}\end{bmatrix}, \quad \text{or} \quad \mathbf{T}^{-1} = \mathbf{J}\mathbf{T}^\mathrm{t}\mathbf{J} \tag{2}$$

with

$$\mathbf{J} = \mathrm{i}\begin{bmatrix}\mathbf{0} & -\mathbf{I}\\ \mathbf{I} & \mathbf{0}\end{bmatrix}, \quad \mathbf{J} = \mathbf{J}^{-1} = -\mathbf{J}^\mathrm{t}, \tag{3}$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{0}$ the null matrix.

We recall [1] that if λ is an eigenvalue of a real symplectic matrix $\mathbf{T}$, then $1/\lambda$, λ^*, and $1/\lambda^*$ are eigenvalues, too; as usual, complex conjugation is denoted by the superscript *. Indeed, from the realness of $\mathbf{T}$, we conclude that the characteristic equation $\det(\mathbf{T} - \lambda\mathbf{I}) = 0$ has real coefficients and that the eigenvalues are thus real or come in complex conjugated pairs: if λ is an eigenvalue, then λ^* is an eigenvalue, too. Moreover, from the symplecticity condition (2) we get

$$\begin{aligned}\det(\mathbf{T}^{-1} - \lambda\mathbf{I}) &= \det(\mathbf{J}\mathbf{T}^\mathrm{t}\mathbf{J} - \lambda\mathbf{I}) = \det[\mathbf{J}(\mathbf{T}^\mathrm{t} - \lambda\mathbf{I})\mathbf{J}]\\ &= \det(\mathbf{T}^\mathrm{t} - \lambda\mathbf{I}) = \det(\mathbf{T} - \lambda\mathbf{I})\end{aligned}$$

and we conclude that if λ is an eigenvalue, then $1/\lambda$ is an eigenvalue, too. So, for real symplectic matrices and $D \geq 2$, the eigenvalues come in complex quartets (if they are not unimodular and not real), or in complex conju-

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gated pairs (if they are unimodular, but not real), or in real pairs (in particular: double if equal to $+1$ or -1). For $D = 1$, the two eigenvalues can of course only come as a single pair, either unimodular or real. We remark that the $2D$ eigenvalues λ_n $(n = 1, \ldots, 2D)$ can be combined into a $2D$-dimensional diagonal matrix Λ such that Λ is symplectic (but possibly complex). We restrict ourselves in this paper to ray transformation matrices that can be diagonalized. This implies that the $2D$ eigenvectors $\vec{q}_n$ that are associated to the eigenvalues λ_n, $\mathbf{T}\vec{q}_n = \vec{q}_n\lambda_n$, are linearly independent and can be combined into a $2D \times 2D$ matrix $\mathbf{Q}$ such that $\mathbf{Q}$ is symplectic [1] (but possibly complex again). The similarity transformation between $\mathbf{T}$ and Λ then reads as $\mathbf{T} = \mathbf{Q}\Lambda\mathbf{Q}^{-1}$ and involves only symplectic matrices.

The case of a ray transformation matrix with D pairs of *unimodular* eigenvalues $\exp(\mathrm{i}\theta_n)$ and $\exp(-\mathrm{i}\theta_n)$ $(n = 1, \ldots, D)$ has been treated in [1]. In particular it was shown that the real symplectic $\mathbf{ABCD}$-matrix can be decomposed as

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{I} & \mathrm{i}\mathbf{w}^2 \\ \mathrm{i}\mathbf{w}^{-2} & \mathbf{I} \end{bmatrix} \times \begin{bmatrix} \exp(\mathrm{i}\Theta) & \mathbf{0} \\ \mathbf{0} & \exp(-\mathrm{i}\Theta) \end{bmatrix} \sqrt{2} \begin{bmatrix} \mathbf{I} & \mathrm{i}\mathbf{w}^2 \\ \mathrm{i}\mathbf{w}^{-2} & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix}^{-1}, \quad (4)$$

where the central matrix is a diagonal matrix containing the D complex conjugated pairs of unimodular eigenvalues $\exp(\pm\mathrm{i}\theta_n)$, where $\mathbf{w}^2$ is a diagonal scaling matrix, and where the $\mathbf{abcd}$-matrix is a *real* symplectic matrix. The three matrices in the middle correspond to a scaled, separable fractional Fourier transformer [2]:

$$\begin{bmatrix} \mathbf{w} & \mathbf{0} \\ \mathbf{0} & \mathbf{w}^{-1} \end{bmatrix} \begin{bmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{bmatrix} \begin{bmatrix} \mathbf{w}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{w} \end{bmatrix}. \quad (5)$$

We thus conclude that a first-order optical system with unimodular eigenvalues is similar to a separable fractional Fourier transformer (in the sense of similarity of their respective ray transformation matrices; and with a *real* symplectic $\mathbf{abcd}$-matrix).

We recall that a separable fractional Fourier transformer may be realized, for instance, by a combination of the one-dimensional setups suggested by Lohmann [5]: one setup consisting of a thin convex (cylindrical) lens with focal length f, preceded and followed by two identical distances d of free space, and another setup consisting of two identical thin convex (cylindrical) lenses with focal lengths f, separated by a distance d. For both setups we have $\theta = 2\arcsin(\sqrt{d/2f})$, with $0 \le \theta \le \pi$.

In the case of D pairs of *real* eigenvalues $\sigma_n \exp(\theta_n)$ and $\sigma_n \exp(-\theta_n)$ $(n = 1, \ldots, D)$, with $\sigma_n = \pm 1$ representing the sign of the eigenvalues, we can write

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{I} & -\mathbf{w}^2 \\ \mathbf{w}^{-2} & \mathbf{I} \end{bmatrix} \times \begin{bmatrix} \sigma\exp(\Theta) & \mathbf{0} \\ \mathbf{0} & \sigma\exp(-\Theta) \end{bmatrix} \sqrt{2} \begin{bmatrix} \mathbf{I} & -\mathbf{w}^2 \\ \mathbf{w}^{-2} & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix}^{-1}, \quad (6)$$

where the signs σ_n of the pairs of eigenvalues are expressed by means of the diagonal matrix σ. The three matrices in the middle now correspond to a scaled, separable hyperbolic expander, see also [6, page 183, Example: Hyperbolic expanders]:

$$\begin{bmatrix} \mathbf{w} & \mathbf{0} \\ \mathbf{0} & \mathbf{w}^{-1} \end{bmatrix} \begin{bmatrix} \sigma\cosh\Theta & \sigma\sinh\Theta \\ \sigma\sinh\Theta & \sigma\cosh\Theta \end{bmatrix} \begin{bmatrix} \mathbf{w}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{w} \end{bmatrix}. \quad (7)$$

We thus conclude that a first-order optical system with real eigenvalues is similar to a separable hyperbolic expander. We remark that a possible optical realization of a one-dimensional hyperbolic expander is similar to that of Lohmann's fractional Fourier transformer setups [5] as described above, but with a concave lens instead of a convex one, and in both setups leading to $\theta = 2\,\mathrm{arcsinh}(\sqrt{-d/2f}) \ge 0$.

The eigenfunctions of an $\mathbf{ABCD}$-system with unimodular eigenvalues, decomposed as a fractional Fourier transformer embedded in between an $\mathbf{abcd}$-system and its inverse, see (4), have been treated in [1]. They can be found by letting the eigenfunctions of a fractional Fourier transformer, i.e. the Hermite-Gauss modes, propagate through the $\mathbf{abcd}$-system. If the resulting functions are then chosen as the input functions of the cascade (4), the inverse $\mathbf{abcd}$-system will convert them into the Hermite-Gauss modes, which then act as eigenfunctions of the central fractional Fourier transformer, after which the forward $\mathbf{abcd}$-system transforms the Hermite-Gauss modes back again into the functions with which we started.

We can proceed in a similar way when we are dealing with an $\mathbf{ABCD}$-system with real eigenvalues, which system can be decomposed in the cascade (6); see also [7, Section V] for a treatment of the one-dimensional case. The eigenfunctions of such a system are now related to the eigenfunctions of the separable hyperbolic expander, and thus also to the ones of the separable magnifier with ray transformation matrix $\mathbf{A}_\mathrm{m} = \sigma\exp(\Theta)$, $\mathbf{B}_\mathrm{m} = \mathbf{C}_\mathrm{m} = \mathbf{0}$, and $\mathbf{D}_\mathrm{m} = (\mathbf{A}_\mathrm{m}^{-1})^\mathrm{t} = \sigma\exp(-\Theta)$. The input-output relationship of such a separable magnifier reads

$$\begin{aligned} f_\mathrm{o}(\vec{r}) &= f_\mathrm{i}(\mathbf{A}_\mathrm{m}^{-1}\vec{r})/\sqrt{|\det\mathbf{A}_\mathrm{m}|} \\ &= f_\mathrm{i}(\sigma\exp(-\Theta)\vec{r})\sqrt{\det\exp(-\Theta)} \end{aligned} \quad (8)$$

and it can easily be seen that the simple powers r_n^k $(n = 1, \ldots, D; k = 0, 1, \ldots)$ form a (non-orthogonal) set of eigenfunctions of the separable magnifier, and that the corresponding eigenvalues read $\sigma_n^k \exp(-k\theta_n)\sqrt{\det\exp(-\Theta)}$. Although simple powers are not orthogonal, we still might appreciate an expansion into such powers, since the resulting expansion is in fact the well-known Taylor (or Maclaurin) series. We recall that many other functions that satisfy (8) have been derived in the realm of fractals.

In determining the eigenfunctions of a more general system (but still with real eigenvalues) we may use Collins integral [8]

$$f_\mathrm{o}(\vec{r}_\mathrm{o}) = \frac{1}{\sqrt{\det \mathrm{i}\mathbf{b}}} \int_{-\infty}^{\infty} f_\mathrm{i}(\vec{r}_\mathrm{i}) \times \exp\left[i\pi\left(\vec{r}_\mathrm{i}^{\,\mathrm{t}}\mathbf{b}^{-1}\mathbf{a}\vec{r}_\mathrm{i} - 2\vec{r}_\mathrm{i}^{\,\mathrm{t}}\mathbf{b}^{-1}\vec{r}_\mathrm{o} + \vec{r}_\mathrm{o}^{\,\mathrm{t}}\mathbf{d}\mathbf{b}^{-1}\vec{r}_\mathrm{o}\right)\right] \mathrm{d}\vec{r}_\mathrm{i}, \quad (9)$$

which describes the input-output relationship of an **abcd**-system, in combination with the relationship

$$\begin{aligned}(-\mathrm{i}2\pi)^k\sqrt{p}\int_{-\infty}^{\infty} x^k \exp(-\pi p\,x^2 - \mathrm{i}2\pi ux)\,\mathrm{d}x &= \frac{\mathrm{d}^k}{\mathrm{d}u^k}\exp[-(\pi/p)\,u^2] \\ &= (-\sqrt{\pi/p})^k\exp[-(\pi/p)\,u^2]\,H_k(u\sqrt{\pi/p}). \qquad (10)\end{aligned}$$

The first equality sign in (10) follows by differentiating the Fourier transform $\exp[-(\pi/p)\,u^2]$ of $\exp(-\pi p x^2)$,

$$\sqrt{p}\int_{-\infty}^{\infty}\exp(-\pi p\,x^2 - \mathrm{i}2\pi ux)\,\mathrm{d}x = \exp[-(\pi/p)\,u^2], \qquad (11)$$

which holds for $\Re p > 0$ and for $[\Re p = 0, \Im p \neq 0]$, see, for instance, [9, (2.3.15.4)] and also [6, Section C.2, page 279, Remark: The integral of complex Gaussians]; we may also refer to [9, (2.5.22.5)] and [10, (3.691.5) and (3.691.7)] for $k=0$, and to [9, (2.5.22.3)] and [10, (3.851.1) and (3.851.3)] for $k=1$. The second equality in (10) can be found, for instance, in [11, (7.1.19)]; see also Rodrigues' formula for Hermite polynomials [11, (22.11.7)].

Let us restrict ourselves for a moment to the one-dimensional case with **a**, **b**, **c**, and **d** in Collins integral (9) chosen as $\mathbf{a} = \mathbf{d} = 1/\sqrt{2}$, $\mathbf{b} = -w^2/\sqrt{2}$, and $\mathbf{c} = w^{-2}/\sqrt{2}$, cf. the cascade (6), and determine the eigenfunctions $\mathcal{H}_k(x;\sqrt{\mathrm{i}w^2})$ of the one-dimensional hyperbolic expander (7). To do so, we choose the powers $(k!\sqrt{\mathrm{i}w^2})^{-1/2}\,(x\sqrt{2\pi/\mathrm{i}w^2})^k$ to start with and then have to determine the integral

$$\frac{2^{1/4}\,(k!\sqrt{\mathrm{i}w^2})^{-1/2}}{\sqrt{-\mathrm{i}w^2}}\exp\left(\pi\frac{x_\mathrm{o}^2}{\mathrm{i}w^2}\right) \times \int_{-\infty}^{\infty}\left(\frac{x_\mathrm{i}\sqrt{2\pi}}{\sqrt{\mathrm{i}w^2}}\right)^k \exp\left(\pi\frac{x_\mathrm{i}^2}{\mathrm{i}w^2} - 2\pi\sqrt{2}\,\frac{x_\mathrm{i}x_\mathrm{o}}{\mathrm{i}w^2}\right)\mathrm{d}x_\mathrm{i}. \quad (12)$$

We use the relationship (10) in which we first substitute $p = -(\mathrm{i}w^2)^{-1}$, leading to

$$\begin{aligned}(-\mathrm{i}2\pi)^k\frac{1}{\sqrt{-\mathrm{i}w^2}}\int_{-\infty}^{\infty} x^k\exp\left(\pi\frac{x^2}{\mathrm{i}w^2} - \mathrm{i}2\pi ux\right)\mathrm{d}x \\ = (-\mathrm{i}\sqrt{\pi\,\mathrm{i}w^2})^k\exp(\pi\,\mathrm{i}w^2\,u^2)\,H_k(\mathrm{i}u\sqrt{\pi\,\mathrm{i}w^2}), \qquad (13)\end{aligned}$$

and then $\mathrm{i}u = x_\mathrm{o}\sqrt{2}(\mathrm{i}w^2)^{-1}$, yielding

$$\begin{aligned}(-\mathrm{i}2\pi)^k\frac{1}{\sqrt{-\mathrm{i}w^2}}\int_{-\infty}^{\infty} x^k\exp\left(\pi\frac{x^2}{\mathrm{i}w^2} - 2\pi\sqrt{2}\,\frac{xx_\mathrm{o}}{\mathrm{i}w^2}\right)\mathrm{d}x \\ = (-\mathrm{i}\sqrt{\pi\,\mathrm{i}w^2})^k\exp\left(-2\pi\frac{x_\mathrm{o}^2}{\mathrm{i}w^2}\right)H_k\left(\sqrt{2\pi}\,\frac{x_\mathrm{o}}{\sqrt{\mathrm{i}w^2}}\right), \qquad (14)\end{aligned}$$

and we finally substitute the latter expression into the integral (12). We are thus led to the following expression for the eigenfunctions:

$$\mathcal{H}_k(x;\sqrt{\mathrm{i}w^2}) = 2^{1/4}\,(2^k k!\,\sqrt{\mathrm{i}w^2})^{-1/2}\exp(-\pi x^2/\mathrm{i}w^2) \times H_k(\sqrt{2\pi}\,x/\sqrt{\mathrm{i}w^2}). \qquad (15)$$

We note the remarkable resemblance between these eigenfunctions and the eigenfunctions of the fractional Fourier transformer – i.e. the Hermite-Gauss modes $\mathcal{H}_k(x;w)$ – for which we have

$$\mathcal{H}_k(x;w) = 2^{1/4}\,(2^k k!\,w)^{-1/2}\exp(-\pi x^2/w^2) \times H_k(\sqrt{2\pi}\,x/w), \qquad (16)$$

and we conclude that we can directly go from the fractional–Fourier–transformer case [equations (4) and (16)] to the hyperbolic–expander case [equations (6) and (15)] by simply replacing all occurrences of w by $\sqrt{\mathrm{i}w^2}$. We also remark that the eigenfunctions $\mathcal{H}_k$ in both cases are determined by the same generating function

$$(2/z^2)^{1/4}\exp(-s^2 + 2\sqrt{2\pi}sx/z - \pi x^2/z^2) = \sum_{k=0}^{\infty}\left(\frac{2^k}{k!}\right)^{1/2}\mathcal{H}_k(x;z)\,s^k, \qquad (17)$$

with $z = w$ for the fractional Fourier transformer and with $z = \sqrt{\mathrm{i}w^2}$ for the hyperbolic expander, which generating function is based on the one for the Hermite polynomials $H_k(x)$,

$$\exp(-s^2 + 2sx) = \sum_{k=0}^{\infty} H_k(x)\,\frac{s^k}{k!}, \qquad (18)$$

see, for instance, [11, (22.9.17)]. On the analogy of the name Hermite-Gauss modes for $\mathcal{H}_k(x;w)$, we might call the functions $\mathcal{H}_k(x;\sqrt{iw^2})$ Hermite-chirp modes.

Using the procedure outlined in [12], we readily derive the generating function at the output of an **abcd**-system, when at the input of this system a set of modes with generating function of the form [12]

$$(\det\mathbf{K}\sqrt{2})^{1/2}\exp(-\vec{s}^{\,\mathrm{t}}\mathbf{M}\vec{s} + 2\sqrt{2\pi}\vec{s}^{\,\mathrm{t}}\mathbf{K}\vec{r} - \pi\vec{r}^{\,\mathrm{t}}\mathbf{L}\vec{r}) \qquad (19)$$

is present; note that $\mathbf{L}$ and $\mathbf{M}$ are symmetric matrices. The generating function at the output has the same form as (19), with output parameters $\mathbf{K}_\mathrm{o}$, $\mathbf{L}_\mathrm{o}$, and $\mathbf{M}_\mathrm{o}$ that are related to the input parameters $\mathbf{K}_\mathrm{i}$, $\mathbf{L}_\mathrm{i}$, and $\mathbf{M}_\mathrm{i}$ by [12]

$$\begin{aligned}\mathbf{K}_\mathrm{o} &= \mathbf{K}_\mathrm{i}(\mathbf{a} + \mathbf{b}\mathrm{i}\mathbf{L}_\mathrm{i})^{-1}, \\ \mathrm{i}\mathbf{L}_\mathrm{o} &= (\mathbf{c} + \mathbf{d}\mathrm{i}\mathbf{L}_\mathrm{i})(\mathbf{a} + \mathbf{b}\mathrm{i}\mathbf{L}_\mathrm{i})^{-1}, \qquad (20) \\ \mathbf{M}_\mathrm{o} &= \mathbf{M}_\mathrm{i} - 2\mathrm{i}\mathbf{K}_\mathrm{i}(\mathbf{a} + \mathbf{b}\mathrm{i}\mathbf{L}_\mathrm{i})^{-1}\mathbf{b}\mathbf{K}_\mathrm{i}^{\mathrm{t}}.\end{aligned}$$

With $\mathbf{K}_\mathrm{i} = (\mathrm{i}\mathbf{w}^2)^{-1/2}$, $\mathbf{L}_\mathrm{i} = (\mathrm{i}\mathbf{w}^2)^{-1}$, and $\mathbf{M}_\mathrm{i} = \mathbf{I}$, cf. (17) with $z = \sqrt{\mathrm{i}w^2}$, we thus get the parameters

$$\begin{aligned} \mathbf{K} &= \mathrm{i}^{-1/2}(\mathbf{aw} + \mathbf{bw}^{-1})^{-1}, \\ \mathrm{i}\mathbf{L} &= (\mathbf{cw} + \mathbf{dw}^{-1})(\mathbf{aw} + \mathbf{bw}^{-1})^{-1}, \\ \mathbf{M} &= (\mathbf{aw} + \mathbf{bw}^{-1})^{-1}(\mathbf{aw} - \mathbf{bw}^{-1}), \end{aligned} \tag{21}$$

and the modes associated with the corresponding generating function are the eigenfunctions of the **ABCD**-system (with real eigenvalues; see (6)) under consideration. Again, we remark the resemblance to the eigenfunctions of the **ABCD**-system with unimodular eigenvalues (see (4)), for which we have [1]

$$\begin{aligned} \mathbf{K} &= (\mathbf{aw} + \mathrm{i}\mathbf{bw}^{-1})^{-1}, \\ \mathrm{i}\mathbf{L} &= (\mathbf{cw} + \mathrm{i}\mathbf{dw}^{-1})(\mathbf{aw} + \mathrm{i}\mathbf{bw}^{-1})^{-1}, \\ \mathbf{M} &= (\mathbf{aw} + \mathrm{i}\mathbf{bw}^{-1})^{-1}(\mathbf{aw} - \mathrm{i}\mathbf{bw}^{-1}), \end{aligned} \tag{22}$$

and which expressions directly lead to equations (21) by replacing $\mathbf{w}$ by $\mathrm{i}^{1/2}\mathbf{w}$.

We recall that by differentiating the generating function (19) with respect to $\vec{r}$ and $\vec{s}$, respectively, we find the derivative and recurrence relations for these eigenfunctions, and we are able to formulate a closed-form expression for them. For the case $D = 2$, with $\mathcal{H}_{m,n}(x, y) = \mathcal{H}_m(x)\mathcal{H}_n(y)$ and with the short-hand notations $\vec{r} = [x, y]^{\mathrm{t}}$ and $\partial/\partial\vec{r} = [\partial/\partial x, \partial/\partial y]^{\mathrm{t}}$, we get [13]

$$\begin{aligned} \partial\mathcal{H}_{m,n}(\vec{r})/\partial\vec{r} = &\, 2\sqrt{\pi}\,\mathbf{K}^{\mathrm{t}}\left[\sqrt{m}\,\mathcal{H}_{m-1,n}(\vec{r}), \sqrt{n}\,\mathcal{H}_{m,n-1}(\vec{r})\right]^{\mathrm{t}} \\ & -2\pi\,\mathcal{H}_{m,n}(\vec{r})\,\mathbf{L}\vec{r} \end{aligned} \tag{23}$$

and

$$\begin{aligned} & 2\sqrt{\pi}\;\mathcal{H}_{m,n}(\vec{r})\,\mathbf{K}\vec{r} \\ & \quad = \left[\sqrt{m+1}\,\mathcal{H}_{m+1,n}(\vec{r}), \sqrt{n+1}\,\mathcal{H}_{m,n+1}(\vec{r})\right]^{\mathrm{t}} \\ & \qquad +\mathbf{M}\left[\sqrt{m}\,\mathcal{H}_{m-1,n}(\vec{r}), \sqrt{n}\,\mathcal{H}_{m,n-1}(\vec{r})\right]^{\mathrm{t}}, \end{aligned} \tag{24}$$

and thus

$$\begin{aligned} 2\sqrt{\pi(m+1)}\,\mathcal{H}_{m+1,n}(\vec{r}) &= \mathcal{P}_x\mathcal{H}_{m,n}(\vec{r}), \\ 2\sqrt{\pi(n+1)}\,\mathcal{H}_{m,n+1}(\vec{r}) &= \mathcal{P}_y\mathcal{H}_{m,n}(\vec{r}), \end{aligned} \tag{25}$$

with the operators $\mathcal{P}_x$ and $\mathcal{P}_y$ defined as

$$\begin{bmatrix} \mathcal{P}_x \\ \mathcal{P}_y \end{bmatrix} = 2\pi\,(2\mathbf{K} - \mathbf{M}\mathbf{K}^{-1\,t}\mathbf{L})\,\vec{r} - (\mathbf{M}\mathbf{K}^{-1\,\mathrm{t}})\,\frac{\partial}{\partial\vec{r}}. \tag{26}$$

Note that the operators $\mathcal{P}_x$ and $\mathcal{P}_y$ commute, since the matrix product $(2\mathbf{K} - \mathbf{M}\mathbf{K}^{-1\,\mathrm{t}}\mathbf{L})(\mathbf{M}\mathbf{K}^{-1\,\mathrm{t}})^{\mathrm{t}}$ is symmetric. The closed-form expression then follows readily as [1,13]

$$\mathcal{H}_{m,n}(\vec{r}) = \frac{(\det\mathbf{K}\sqrt{2})^{1/2}\,\mathcal{P}_x^m\,\mathcal{P}_y^n\,\exp\left[-\pi\,\vec{r}^{\,\mathrm{t}}\mathbf{L}\vec{r}\right]}{2^{m+n}\sqrt{\pi^{m+n}m!n!}} \tag{27}$$

and the extension to higher dimensions is straightforward.

While for the Hermite-Gauss modes $\mathcal{H}(\vec{r}; \mathbf{w})$ we have $\mathbf{K} = \mathbf{w}^{-1}$, $\mathbf{L} = \mathbf{w}^{-2}$, and $\mathbf{M} = \mathbf{I}$, for the Hermite-chirp modes $\mathcal{H}(\vec{r}; \sqrt{\mathrm{i}\mathbf{w}^2})$ we have $\mathbf{K} = (\sqrt{\mathrm{i}\mathbf{w}^2})^{-1}$, $\mathbf{L} = (\mathrm{i}\mathbf{w}^2)^{-1}$, and $\mathbf{M} = \mathbf{I}$. But whereas the set of Hermite-Gauss modes is orthogonal, this is not the case for the set of Hermite-chirp modes; and unfortunately, since the real part of the symmetric matrix $\mathbf{K}^{\mathrm{t}}\mathbf{M}^{-1}\mathbf{K} = (\mathrm{i}\mathbf{w}^2)^{-1}$ is not positive definite, we cannot derive its bi-orthogonal partner set in the way described in [14].

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