Multidimensional Reassignment Method

Igor Djurović,[1] LJubiša Stanković,[1] Martin J. Bastiaans[2]
[1]Elektrotehnički fakultet, University of Montenegro, Podgorica, Montenegro/Yu.
Tel./Fax +381 81 244 921. Email: igordj@cg.ac.yu.
[2]Technische Universiteit Eindhoven, Eindhoven, Netherlands. Email: M.J.Bastiaans@ele.tue.nl

Abstract— **Application of the reassignment method to time-frequency analysis of multidimensional signals is considered. Basic properties of reassigned multidimensional distributions are presented. Generally, multidimensional reassignment distributions are numerically very complex. The multidimensional S-method based reassignment distribution, resulting in a quite simple realization scheme, is proposed.**

Keywords— **Time-frequency analysis, Wigner distribution, spectrogram, reassignment method.**

I. Introduction

The most important time-frequency representations, such as the spectrogram and the Wigner distribution, belong to the Cohen class of distributions [1]. In the realizations and applications of distributions from this class, there are several well-known undesired effects, such as cross-terms, inner interferences, resolution, and noise influence [2], [3]. The reassignment method was introduced in order to improve readability of time-frequency distributions [4]. It can be helpful for signal parametric identification in a high-noise environment [5]. The main problem that exists in the realization of reassigned distributions is their numerical complexity. Namely, in order to get the reassigned form of a distribution, it is necessary to calculate three time-frequency distributions instead of one. Several attempts have been made in the direction of reducing the calculation complexity. A simplified form of reassigned distributions is proposed in [6]. Recursive realizations of reassigned distributions have been considered in [7]. The aim of this paper is to present multidimensional reassigned distributions, since multidimensional space/spatial-frequency analysis is an interesting research area [8], [9], [10]. In general, the multidimensional signal case will further increase the computational cost of implementing reassigned distributions, beyond an acceptable level. This was the reason for considering a simplified reassignment form, based on the multidimensional S-method [11], which can easily be realized.

The paper is organized as follows. A review of multidimensional space/spatial-frequency transformations is given in Section II. The multidimensional reassignment method is derived in Section III. The reassigned form of the S-method is introduced in Section IV. In this section a modified form of the reassigned S-method is introduced, which uses the displacement along space coordinates only. The theory is illustrated on a numerical example in Section V.

II. Review

Multidimensional signals appear in optics, image and video processing, and in numerous other applications. We denote a multidimensional signal as $x(\vec{t})$, where $\vec{t}$ is a vector of space components $\vec{t}=(t_1,t_2,...,t_Q)$, i.e., $x(\vec{t})=x(t_1,...,t_Q)$. The Fourier transform of a multidimensional signal is defined as

$$X(\omega_1,\omega_2,...,\omega_Q)=$$

$$=\int_{t_1}\int_{t_2}...\int_{t_Q} x(\vec{t})e^{-j(\omega_1 t_1+...+\omega_Q t_Q)}dt_1...dt_Q. \quad (1)$$

It can be written as

$$X(\vec{\omega})=\int_{\vec{t}} x(\vec{t})e^{-j\vec{\omega}\vec{t}}d\vec{t}, \quad (2)$$

where $\vec{\omega}$ is the frequency vector $\vec{\omega}=(\omega_1,\omega_2,...,\omega_Q)$, and where $\vec{\omega}\vec{t}$ represents a scalar product:

$$\vec{\omega}\vec{t}=\omega_1 t_1+\omega_2 t_2+...+\omega_Q t_Q. \quad (3)$$

The inverse Fourier transform can be written in the following form:

$$x(\vec{t})=\frac{1}{(2\pi)^Q}\int_{\vec{\omega}} X(\vec{\omega})e^{j\vec{\omega}\vec{t}}d\vec{\omega}. \quad (4)$$

Space/spatial-frequency distributions are introduced for the analysis of multidimensional signals with space-varying spectral content. The simplest space/spatial-frequency transform is the short-time Fourier transform defined as

$$STFT_w(\vec{t},\vec{\omega})=\int_{\vec{\tau}} x(\vec{t}+\vec{\tau})w^*(\vec{\tau})e^{-j\vec{\omega}\vec{\tau}}d\vec{\tau}, \quad (5)$$

where $w(\vec{\tau})=w(\tau_1,...,\tau_Q)$ represents the multidimensional window function. The window function will play an important role in the reassigned distribution, thus it is used as an index in $STFT_w(\vec{t},\vec{\omega})$. The spectrogram (squared modulus of the short-time Fourier transform) is used in practice:

$$SPEC_w(\vec{t},\vec{\omega})=\left|STFT_w(\vec{t},\vec{\omega})\right|^2. \quad (6)$$

The spectrogram possesses several drawbacks, such as low concentration and low space/spatial-frequency resolution. These are the reasons for introducing the multidimensional Wigner distribution:

$$WD(\vec{t},\vec{\omega})=\int_{\vec{\tau}} x(\vec{t}+\vec{\tau}/2)x(\vec{t}-\vec{\tau}/2)e^{-j\vec{\omega}\vec{\tau}}d\vec{\tau}. \quad (7)$$

For a linear frequency-modulated signal

$$x(\vec{t}) = Ae^{j\Phi(\vec{t})}, \tag{8}$$

where the phase function is equal to

$$\Phi(\vec{t}) = \sum_{i=1}^{Q}\sum_{j=1}^{Q} a_{ij}t_i t_j + \sum_{i=1}^{Q} b_i t_i + c, \tag{9}$$

the Wigner distribution is ideally concentrated along the local frequency:

$$\begin{aligned} WD(\vec{t},\vec{\omega}) &= (2\pi)^Q \prod_{i=1}^{Q} \delta\left(\omega_i - 2a_{ii}t_i - \sum_{j=1, j\neq i}^{Q} a_{ij}t_j - b_i\right) \\ &= (2\pi)^Q \prod_{i=1}^{Q} \delta\left(\omega_i - \partial\Phi(\vec{t})/\partial t_i\right). \end{aligned}$$

Unfortunately, the Wigner distribution has a serious drawback. Namely, for a multicomponent signal

$$x(\vec{t}) = \sum_{m=1}^{M} x_m(\vec{t}), \tag{10}$$

it exhibits very emphatic cross-terms:

$$\begin{aligned} WD(\vec{t},\vec{\omega}) &= \sum_{m=1}^{M} WD_{mm}(\vec{t},\vec{\omega}) \\ &+ 2\,\mathrm{Re}\sum_{m=1}^{M}\sum_{m>n}^{M} WD_{mn}(\vec{t},\vec{\omega}), \end{aligned} \tag{11}$$

where

$$WD_{mn}(\vec{t},\vec{\omega}) = \int_{\vec{\tau}} x_m(\vec{t}+\vec{\tau}/2)x_n^*(\vec{t}-\vec{\tau}/2)e^{-j\vec{\omega}\vec{\tau}}d\vec{\tau}. \tag{12}$$

The signal components $WD_{mm}(\vec{t},\vec{\omega})$, $m = 1, ..., M$, are known as auto-terms, while the undesired components $WD_{mn}(\vec{t},\vec{\omega})$, for $m \neq n$, are the cross-terms. The cross-terms can mask the auto-terms and may make detection of auto-terms and estimation of signal parameters very difficult. Numerous reduced interference distributions have been developed in order to get a good trade-off between the concentration of the signal's auto-terms and reduction of the cross-terms. The most important reduced interference distributions belong to the Cohen class of distributions. For multidimensional signals this class reads

$$TF(\vec{t},\vec{\omega}) = \frac{1}{(2\pi)^Q}\int_{\vec{u}}\int_{\vec{v}} \Pi(\vec{u},\vec{v})WD(\vec{t}-\vec{u},\vec{\omega}-\vec{v})d\vec{u}d\vec{v}, \tag{13}$$

where $\Pi(\vec{t},\vec{\omega})$ is the kernel function in the space/spatial-frequency domain. The kernel function specifies the distribution from the Cohen class. For example, a kernel function of the form

$$\Pi(\vec{t},\vec{\omega}) = WD_w(\vec{t},\vec{\omega}), \tag{14}$$

where $WD_w(\vec{t},\vec{\omega})$ is the Wigner distribution of the window function,

$$WD_w(\vec{t},\vec{\omega}) = \int_{\vec{\tau}} w(\vec{t}+\vec{\tau}/2)w^*(\vec{t}-\vec{\tau}/2)e^{-j\vec{\omega}\vec{\tau}}d\vec{\tau}, \tag{15}$$

produces the spectrogram. The kernel function

$$\Pi(\vec{t},\vec{\omega}) = (2\pi)^Q\delta(\vec{\omega})\delta(\vec{t}) \tag{16}$$

gives the Wigner distribution. Here, we will use the S-method [11], [12] as a multidimensional distribution that can give highly concentrated auto-terms, with reduced interferences, and which will be very convenient for the reassignment method. The S-method is defined as

$$\begin{aligned} &SM_{w,w}(\vec{t},\vec{\omega}) \\ &= \frac{1}{\pi^Q}\int_{\vec{\theta}} P(\vec{\theta})STFT_w(\vec{t},\vec{\omega}+\vec{\theta})STFT_w^*(\vec{t},\vec{\omega}-\vec{\theta})d\vec{\theta}. \end{aligned} \tag{17}$$

For a very narrow frequency window, $P(\vec{\theta}) = \pi^Q\delta(\vec{\theta})$, the S-method reduces to the spectrogram, $SM_{ww}(\vec{t},\vec{\omega}) = SPEC_w(\vec{t},\vec{\omega})$, while for a very wide frequency window, $P(\vec{\theta}) = 1$, it is equal to the pseudo Wigner distribution:

$$\begin{aligned} &PWD(\vec{t},\vec{\omega}) \\ &= \int_{\vec{\tau}} w(-\vec{\tau}/2)w^*(\vec{\tau}/2)x(\vec{t}+\vec{\tau}/2)x^*(\vec{t}-\vec{\tau}/2)e^{-j\vec{\omega}\vec{\tau}}d\vec{\tau}. \end{aligned}$$

For a frequency window of the form

$$P(\vec{\theta}) = \begin{cases} 1 & |\theta_1| \leq \Delta_1, |\theta_2| \leq \Delta_2, ..., |\theta_Q| \leq \Delta_Q, \\ 0 & \text{elsewhere,} \end{cases} \tag{18}$$

where Δ_1,Δ_2,...,Δ_Q are chosen appropriately [13], the S-method would give auto-terms close to those in the Wigner distribution, almost avoiding cross-terms:

$$SM_{ww}(\vec{t},\vec{\omega}) \simeq \sum_{m=1}^{M} PWD_{mm}(\vec{t},\vec{\omega}). \tag{19}$$

The kernel function of the S-method is given as

$$\Pi(\vec{t},\vec{\omega}) = 2^Q p(2\vec{t})WD_w(\vec{t},\vec{\omega}), \tag{20}$$

where $p(\vec{t})$ is the inverse Fourier transform of the frequency window: $p(\vec{t}) = IFT\{P(\vec{\theta})\}$.

III. Multidimensional Reassignment Method

The reassignment method has been introduced in time-frequency analysis in order to produce an accurate time-frequency representation of signals embedded in a high-noise environment. Here, the reassignment method is applied to multidimensional signals. By analogy with the one-dimensional case, the reassigned distributions for Q-dimensional signals can be defined as

$$RTF(\vec{t},\vec{\omega}) = \frac{1}{(2\pi)^Q}\int_{\vec{\tau}}\int_{\vec{\theta}} TF(\vec{\tau},\vec{\theta})$$

$$\times\,\delta\left(\vec{t}-\vec{t}_r(\vec{\tau},\vec{\theta})\right)\delta\left(\vec{\omega}-\vec{\omega}_r(\vec{\tau},\vec{\theta})\right)d\vec{\tau}d\vec{\theta}. \quad (21)$$

The displacements along the space and frequency coordinates are defined as

$$\vec{t}_r(\vec{\tau},\vec{\theta}) = (\tau^{(1)},\tau^{(2)},...,\tau^{(Q)}), \quad (22)$$
$$\vec{\omega}_r(\vec{\tau},\vec{\theta}) = (\theta^{(1)},\theta^{(2)},...,\theta^{(Q)}), \quad (23)$$

where

$$\tau^{(i)} = \tau_i - \frac{\int_{\vec{u}}\int_{\vec{v}} u_i \Pi(\vec{u},\vec{v})WD(\vec{\tau}-\vec{u},\vec{\theta}-\vec{v})d\vec{u}d\vec{v}}{\int_{\vec{u}}\int_{\vec{v}} \Pi(\vec{u},\vec{v})WD(\vec{\tau}-\vec{u},\vec{\theta}-\vec{v})d\vec{u}d\vec{v}}, \quad (24)$$
$$\theta^{(i)} = \theta_i - \frac{\int_{\vec{u}}\int_{\vec{v}} v_i \Pi(\vec{u},\vec{v})WD(\vec{\tau}-\vec{u},\vec{\theta}-\vec{v})d\vec{u}d\vec{v}}{\int_{\vec{u}}\int_{\vec{v}} \Pi(\vec{u},\vec{v})WD(\vec{\tau}-\vec{u},\vec{\theta}-\vec{v})d\vec{u}d\vec{v}}, \quad (25)$$

with $i=1,...,Q$. In (21), $TF(\vec{t},\vec{\omega})$ denotes a distribution from the Cohen class (13) with the kernel function $\Pi(\vec{t},\vec{\omega})$.

A. Reassigned Spectrogram

For the spectrogram with kernel function (14), the displacements are given as

$$\tau^{(i)} = \tau_i + \mathrm{Re}\left\{\frac{STFT_{Twi}(\vec{\tau},\vec{\theta})}{STFT_w(\vec{\tau},\vec{\theta})}\right\},\ i=1,...,Q, \quad (26)$$
$$\theta^{(i)} = \theta_i - \mathrm{Im}\left\{\frac{STFT_{Dwi}(\vec{\tau},\vec{\theta})}{STFT_w(\vec{\tau},\vec{\theta})}\right\},\ i=1,...,Q, \quad (27)$$

where indexes denote windows applied to the short-time Fourier transform [$Twi \leftrightarrow t_i w(\vec{t})$, $Dwi \leftrightarrow \partial w(\vec{t})/\partial t_i$]:

$$STFT_{Twi}(\vec{t},\vec{\omega}) = \int_{\vec{\tau}} x(\vec{t}+\vec{\tau})\tau_i w^*(\vec{\tau})e^{-j\vec{\omega}\vec{\tau}}d\vec{\tau}, \quad (28)$$
$$STFT_{Dwi}(\vec{t},\vec{\omega}) = \int_{\vec{\tau}} x(\vec{t}+\vec{\tau})\frac{\partial w^*(\vec{\tau})}{\partial \tau_i}e^{-j\vec{\omega}\vec{\tau}}d\vec{\tau}. \quad (29)$$

Thus, for the determination of the reassigned spectrogram, it is necessary to calculate $2Q+1$ different short-time Fourier transforms. It is easy to conclude that the reassigned spectrogram is numerically very inefficient. A similar conclusion can be drawn for reassigned versions of almost all other distributions from the Cohen class.

B. Properties of reassigned representations

The main properties of multidimensional reassignment distributions can be derived by analogy with one-dimensional signals [4]:

1. Reassigned distributions are ideally concentrated along the local frequency for multidimensional linear FM signals (8), (9), and delta pulses.
2. Reassigned distributions are not bilinear.
3. Reassigned distributions are space/spatial-frequency invariant. If two signals $y(\vec{t})$ and $x(\vec{t})$ are related by

$$y(\vec{t}) = x(\vec{t}-\vec{t}_0)e^{-j\vec{\omega}_0\vec{t}}, \quad (30)$$

their space/spatial-frequency distributions from the Cohen class are related by

$$TF_y(\vec{t},\vec{\omega}) = TF_x(\vec{t}-\vec{t}_0,\vec{\omega}-\vec{\omega}_0). \quad (31)$$

The same property holds for the reassigned distributions:

$$RTF_y(\vec{t},\vec{\omega}) = RTF_x(\vec{t}-\vec{t}_0,\vec{\omega}-\vec{\omega}_0). \quad (32)$$

4. In the case of the Wigner distribution, the reassigned representation is the Wigner distribution itself. Note that there is no displacement for the Wigner distribution, i.e.,

$$\tau^{(i)} = \tau_i,\ i=1,...,Q,\quad \vec{t}_r(\vec{\tau},\vec{\theta}) = \vec{\tau} \quad \text{and}$$
$$\theta^{(i)} = \theta_i,\ i=1,...,Q,\quad \vec{\omega}_r(\vec{\tau},\vec{\theta}) = \vec{\theta}.$$

5. Reassigned distributions satisfy the unbiased energy condition, if this condition is satisfied by the original distributions $TF(\vec{t},\vec{\omega})$. Thus, if for a nonreassigned distribution the relation

$$\frac{1}{(2\pi)^Q}\int_{\vec{t}}\int_{\vec{\omega}} TF(\vec{t},\vec{\omega})d\vec{t}d\vec{\omega} = \int_{\vec{t}}|x(\vec{t})|^2 d\vec{t} = E_x \quad (33)$$

holds, then

$$\frac{1}{(2\pi)^Q}\int_{\vec{t}}\int_{\vec{\omega}} RTF(\vec{t},\vec{\omega})d\vec{t}d\vec{\omega} = E_x. \quad (34)$$

Properties 1-5 can easily be proven along the lines presented for the one-dimensional case in [4].

IV. Reassigned S-method

The reassigned form of the S-method, with the kernel function (20), is produced by using displacements:

$$\tau^{(i)} = \tau_i + \mathrm{Re}\left\{\frac{SM_{Twi,w}(\vec{\tau},\vec{\theta})}{SM_{w,w}(\vec{\tau},\vec{\theta})}\right\},\ i=1,...,Q, \quad (35)$$
$$\theta^{(i)} = \theta_i - \mathrm{Im}\left\{\frac{SM_{Dwi,w}(\vec{\tau},\vec{\theta})}{SM_{w,w}(\vec{\tau},\vec{\theta})}\right\},\ i=1,...,Q, \quad (36)$$

where $SM_{\alpha,\beta}(\vec{t},\vec{\omega})$ denotes the S-method calculated by using two different short-time Fourier transforms, $STFT_\alpha(\vec{t},\vec{\omega}+\vec{\theta})$ and $STFT_\beta(\vec{t},\vec{\omega}-\vec{\theta})$, in (17). It can easily be seen that for $P(\vec{\theta}) = \pi^Q\delta(\vec{\theta})$, when the S-method reduces to the spectrogram, the displacements of the S-method are equal to the displacements of the spectrogram.

It is well known that the auto-terms of the S-method are close to the auto-terms of the Wigner distribution [11]. From this fact we can conclude that both (space *and* frequency) displacements are small. Thus, by using the space or frequency displacement only, a significant improvement of the space/spatial-frequency representation can be achieved.

For the rectangular multidimensional frequency window $P(\vec{\theta})$ given by (18), the spatial displacement can be written in the form:

$$\tau^{(i)} = \tau_i + \frac{1}{SM_{w,w}(\vec{\tau},\vec{\theta})}$$
$$\times\ \mathrm{Im}\left\{STFT_w(\vec{t},\omega_1,...,\omega_i+\Delta_i,...,\omega_Q)\right.$$
$$\left.\times\ STFT_w^*(\vec{t},\omega_1,...,\omega_i-\Delta_i,...,\omega_Q)\right\}. \quad (37)$$

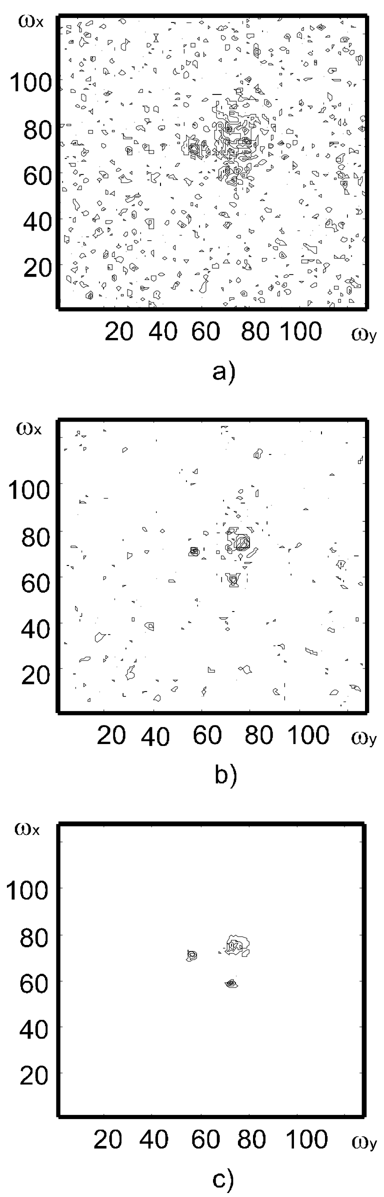

Fig. 1. Space/spatial-frequency representations of a 2D signal: a) Spectrogram, b) S-Method, c) Reassigned S-method.

The spatial displacements (37) can be calculated by using only one single short-time Fourier transform with the window $w(\vec{t})$, instead of $2Q+1$ different transforms in the general case. Both the S-method and the spatial displacements can be calculated by using only $STFT_w(\vec{t},\vec{\omega})$. This is the reason for using spatial displacements in our numerical calculation.

V. Example

Consider a two-dimensional multicomponent signal

$$f(\vec{t}) = \exp\left[j42\pi(t_1^3+t_2^3)\right] + \cos\left(20\pi t_1^2 - 16\pi t_2^2\right), \quad (38)$$

corrupted with complex Gaussian, white noise, with variance $\sigma_{\nu\nu}^2 = 5$. The sampling intervals along both coordinates are $\Delta t_i = 1/128$, $i = 1,2$. The spectrogram, the S-method with frequency window width $2\Delta_1 = 2\Delta_2 = 8\pi$, and the corresponding reassigned S-method for $(t_1,t_2) = (0.4, 0.4)$ are shown in Fig. 1. A separable Hanning window $w(\vec{t}) = w(t_1)w(t_2)$ with a width of $N = 128$ samples is used. We can easily conclude that the reassigned S-method produces an almost ideally concentrated representation along the local frequencies. Computation is also very simple and requires only one short-time Fourier transform. It can be done in a recursive manner [11], including a possibility of VLSI implementation [14].

VI. Conclusion

The reassignment method for multicomponent signals is introduced. Properties of reassigned space/spatial-frequency distributions are presented. The significant calculation complexity of this method is reduced by using the multidimensional S-method and spatial displacement only.

VII. Acknowledgment

This work is supported by the Volkswagen Stiftung, Federal Republic of Germany.

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