On the moments of the Wigner distribution of rotationally symmetric partially coherent light

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The Wigner distribution of rotationally symmetric partially coherent light is considered and the constraints for its moments are derived. While all odd-order moments vanish, these constraints lead to a drastic reduction in the number of parameters that we need to describe all even-order moments: whereas in general we have $(N+1)(N+2)(N+3)/6$ different moments of order N, this number reduces to $(1+N/2)^2$ in the case of rotational symmetry. A way to measure the moments as intensity moments in the output planes of (generally anamorphic) fractional Fourier transform systems is presented. © 2003 Optical Society of America

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The Wigner distribution[1] of partially coherent light is defined in terms of the cross-spectral density[2,3] $\Gamma(x_1, y_1, x_2, y_2)$ by

$$W(x, y, u, v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Gamma(x + \tfrac{1}{2}x', y + \tfrac{1}{2}y', x - \tfrac{1}{2}x', y - \tfrac{1}{2}y')\, e^{-j2\pi(ux' + vy')}\, dx'\, dy'. \quad (1)$$

The (real-valued) Wigner distribution $W(x, y, u, v)$ represents partially coherent light in a combined space/spatial-frequency domain, the so-called phase space, where u and v are the spatial-frequency variables associated to the space variables x and y, respectively. In previous papers, the special but important case of rotational symmetry has been studied extensively; we mention twisted Gaussian-Schell model light[4] and the characterization of rotationally symmetric light in terms of second-order moments.[5,6] In this letter we present an extension to higher-order moments.

To formulate rotational symmetry for the Wigner distribution $W(x, y, u, v)$, we express the space variables x and y in polar coordinates, $x = \rho\cos\phi$ and $y = \rho\sin\phi$, and, with the angle ϕ as an offset, we do the same with the spatial-frequency variables u and v, $u = \zeta\cos(\phi+\theta)$ and $v = \zeta\sin(\phi+\theta)$. We can then formulate an expression for $W(x, y, u, v)$ in terms of the four variables ρ, ϕ, ζ, and θ; for rotational symmetry we require that this expression does not depend on the angle ϕ:

$$W(\rho\cos\phi, \rho\sin\phi, \zeta\cos(\phi+\theta), \zeta\sin(\phi+\theta)) = W_\circ(\rho, \zeta, \theta). \quad (2)$$

The (normalized) moments μ_{pqrs} of the Wigner distribution are defined as

$$\mu_{pqrs}\, E = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(x, y, u, v) x^p u^q y^r v^s dx dy du dv, \quad (3)$$

where $\mu_{0000} = 1$ and the normalization constant E represents the intensity of the light. In general there are $(N+1)(N+2)(N+3)/6$ moments[7] of order $N = p+q+r+s$. In the case of rotational symmetry, however, the number of parameters that we need to describe all even-order moments is reduced drastically to $(1+N/2)^2$. This can easily be seen from Eq. (2), from which we conclude that the four-dimensional Wigner distribution $W(x, y, u, v)$ is completely determined by the three-dimensional function $W(x, 0, u, v)$, where, moreover, $W(x, 0, u, v)$ is an even function of x; and this three-dimensional function has $(1 + N/2)^2$ different nonvanishing moments of even order N.

Using the symmetry condition (2), we write

$$\begin{aligned} \mu_{pqrs}\, E &= \int_0^\infty \int_0^\infty \int_0^{2\pi} W_\circ(\rho, \zeta, \theta)\rho^{p+r+1}\zeta^{q+s+1} d\rho d\zeta d\theta \\ &\quad \times \int_0^{2\pi} (\cos\phi)^p (\cos(\phi+\theta))^q (\sin\phi)^r (\sin(\phi+\theta))^s d\phi. \end{aligned} \tag{4}$$

From the special form of the integral over ϕ, we conclude that all odd-order moments (i.e., $N = p + q + r + s$ is odd) are zero. Moreover, using the definition of the beta function $B(x, y) = 2\int_0^{\pi/2} (\cos\varphi)^{2x-1}(\sin\varphi)^{2y-1}\, d\varphi = \Gamma(x)\Gamma(y)/\Gamma(x + y)$, this integral can be expressed as

$$\begin{aligned} I_{pqrs}(\theta) &= \frac{1}{\pi}\int_0^{2\pi} (\cos\phi)^p (\cos(\phi+\theta))^q (\sin\phi)^r (\sin(\phi+\theta))^s d\phi \\ &= \frac{1}{2\pi}\sum_{k=0}^{q}\sum_{l=0}^{s} \begin{pmatrix} q \\ k \end{pmatrix} \begin{pmatrix} s \\ l \end{pmatrix} (-1)^k \left[1 + (-1)^{p+q-k+l}\right]\left[1 + (-1)^{r+s+k-l}\right] \\ &\quad \times B\left(\frac{p+q-k+l+1}{2}, \frac{r+s+k-l+1}{2}\right) (\cos\theta)^{q+s-k-l}(\sin\theta)^{k+l}. \end{aligned} \tag{5}$$

Since nonvanishing values under the summation appear only if both $p + q - k + l$ and $r + s + k - l$ are even, we can use the property $\Gamma(n + \frac{1}{2})2^n/\sqrt{\pi} = (2n - 1)!! = 1 \cdot 3 \cdot 5 \cdots (2n - 1)$.

It is advantageous to write the moments as $\mu_{p,q,m-p,n-q}$ where $p + r = m$ and $q + s = n$, and to group those moments together that have identical m and n. For easy reference, $I_{pqrs}(\theta)$ has been tabulated below for second-order moments [(m, n) = (2,0), (1,1), and (0,2)] and fourth-order moments [(m, n) = (4,0), (3,1), (2,2), (1,3), and (0,4)] in such a way that equal m and n (with $m + n = 2$ and $m + n = 4$, respectively) are grouped together. Identical values of $I_{pqrs}(\theta)$ in the same block then lead to companion moments μ_{pqrs}. For different choices of p and q (but with constant $m = p + r$ and $n = q + s$) we can easily find relations between the different moments μ_{pqrs}. In particular, we find that in any (m, n) block, the number of nonvanishing parameters equals $1 + \min(m, n)$, leading to a total of $(1 + N/2)^2$ parameters to describe the moments of order $N = m + n$.

Let us consider the second-order moments, which can be represented elegantly in the usual form of a real, symmetric 4×4 matrix. As a consequence from the moment relations, this matrix now takes a special form[4] and is determined by four parameters instead of the ten parameters in the general case. In particular we observe that three moments appear in pairs with a positive companion (μ_{2000}, μ_{1100}, and μ_{0200}) and one moment forms a pair with a negative companion (μ_{1001}); moreover, two moments vanish (μ_{1010} and μ_{0101}).

Let us now consider the fourth-order moments. From the moment relations we conclude that the moments are determined by nine parameters, whereas in the general case we would need 35 parameters. In particular we observe that two moments appear in pairs with a positive companion (μ_{2200} and μ_{2002}); two moments appear in triples with positive companions (μ_{4000} and μ_{0400}); five moments appear in quadruples: two moments with positive companions (μ_{3100} and μ_{1300}) and three moments with one positive and two negative companions (μ_{3001}, μ_{2101}, and μ_{1201}); moreover, four moments vanish (μ_{3010}, μ_{1030}, μ_{0301}, and μ_{0103}) and the moment μ_{1111} follows from the relation $\mu_{1111} = (\mu_{2200} - \mu_{2002})/2$.

Following the procedure described in Ref. 7, the moments can be determined from measurement of the intensity distribution $\Gamma(x, y, x, y)$ in the output plane of some (possibly

m	n	$I_{pqrs}(\theta)$	μ_{pqrs}	companion
2	0	1	μ_{2000}	
2	0	0	μ_{1010}	–
2	0	1	μ_{0020}	μ_{2000}
1	1	$\cos\theta$	μ_{1100}	
1	1	$\sin\theta$	μ_{1001}	
1	1	$-\sin\theta$	μ_{0110}	$-\mu_{1001}$
1	1	$\cos\theta$	μ_{0011}	μ_{1100}
0	2	1	μ_{0200}	
0	2	0	μ_{0101}	–
0	2	1	μ_{0002}	μ_{0200}
4	0	$3/4$	μ_{4000}	
4	0	0	μ_{3010}	–
4	0	$1/4$	μ_{2020}	$\mu_{4000}/3$
4	0	0	μ_{1030}	–
4	0	$3/4$	μ_{0040}	μ_{4000}
3	1	$3\cos\theta/4$	μ_{3100}	
3	1	$3\sin\theta/4$	μ_{3001}	
3	1	$-\sin\theta/4$	μ_{2110}	$-\mu_{3001}/3$
3	1	$\cos\theta/4$	μ_{2011}	$\mu_{3100}/3$
3	1	$\cos\theta/4$	μ_{1120}	$\mu_{3100}/3$
3	1	$\sin\theta/4$	μ_{1021}	$\mu_{3001}/3$
3	1	$-3\sin\theta/4$	μ_{0130}	$-\mu_{3001}$
3	1	$3\cos\theta/4$	μ_{0031}	μ_{3100}

m	n	$I_{pqrs}(\theta)$	μ_{pqrs}	companion
2	2	$(2+\cos 2\theta)/4$	μ_{2200}	
2	2	$\sin 2\theta/4$	μ_{2101}	
2	2	$(2-\cos 2\theta)/4$	μ_{2002}	
2	2	$-\sin 2\theta/4$	μ_{1210}	$-\mu_{2101}$
2	2	$\cos 2\theta/4$	μ_{1111}	$(\mu_{2200}-\mu_{2002})/2$
2	2	$\sin 2\theta/4$	μ_{1012}	μ_{2101}
2	2	$(2-\cos 2\theta)/4$	μ_{0220}	μ_{2002}
2	2	$-\sin 2\theta/4$	μ_{0121}	$-\mu_{2101}$
2	2	$(2+\cos 2\theta)/4$	μ_{0022}	μ_{2200}
1	3	$3\cos\theta/4$	μ_{1300}	
1	3	$\sin\theta/4$	μ_{1201}	
1	3	$\cos\theta/4$	μ_{1102}	$\mu_{1300}/3$
1	3	$3\sin\theta/4$	μ_{1003}	$3\mu_{1201}$
1	3	$-3\sin\theta/4$	μ_{0310}	$-3\mu_{1201}$
1	3	$\cos\theta/4$	μ_{0211}	$\mu_{1300}/3$
1	3	$-\sin\theta/4$	μ_{0112}	$-\mu_{1201}$
1	3	$3\cos\theta/4$	μ_{0013}	μ_{1300}
0	4	$3/4$	μ_{0400}	
0	4	0	μ_{0301}	–
0	4	$1/4$	μ_{0202}	$\mu_{0400}/3$
0	4	0	μ_{0103}	–
0	4	$3/4$	μ_{0004}	μ_{0400}

anamorphic) fractional Fourier transform systems – with a fractional angle α in the x-direction and a fractional angle β in the y-direction, say – for appropriately chosen values of α and β. In the output plane we then measure the intensity moments $\mu^{\text{out}}_{p0r0}(\alpha,\beta)$, cf. Eq. (3) with $q=s=0$, which are completely determined by the output intensity distribution. The general relationship between the output intensity moments and the moments in the input plane reads[7]

$$\mu^{\text{out}}_{p0r0}(\alpha,\beta) \;=\; \sum_{k=0}^{p}\sum_{m=0}^{r}\binom{p}{k}\binom{r}{m}\mu_{p-k,k,r-m,m}\cos^{p-k}\alpha\,\sin^{k}\alpha\,\cos^{r-m}\beta\,\sin^{m}\beta. \tag{6}$$

In the case of second-order moments, the set of relevant equations in which the intensity moments $\mu^{\text{out}}_{2000}(\alpha,\beta)$, $\mu^{\text{out}}_{1010}(\alpha,\beta)$, and $\mu^{\text{out}}_{0020}(\alpha,\beta)$ at the output of a fractional Fourier transform system with fractional angles α and β are expressed in terms of the input moments, reduces to

$$\mu^{\text{out}}_{2000}(\alpha,\beta) \;=\; \mu_{2000}\cos^2\alpha + 2\mu_{1100}\cos\alpha\sin\alpha + \mu_{0200}\sin^2\alpha, \tag{7}$$
$$\mu^{\text{out}}_{1010}(\alpha,\beta) \;=\; \mu_{1001}\sin(\beta-\alpha). \tag{8}$$

To measure the moment μ_{1001} from the intensity moment $\mu^{\text{out}}_{1010}(\alpha,\beta)$, we clearly need an anamorphic system, $\alpha\neq\beta$. Together with two additional isotropic systems, $\alpha=\beta$, we can then construct four equations from measurements of the intensity distributions in the three output planes, and we conclude that the four second-order moments can be determined from the knowledge of the intensity distributions in the output plane of three fractional Fourier transform systems, where one of them has to be anamorphic, see also Ref. 6. We would not need the anamorphic system if the rotationally symmetric light satisfies the additional condition that $W_\circ(\rho, \zeta, \theta)$ is an even function of θ.

In the case of fourth-order moments, the set of relevant equations for the output intensity moments[7] reduces to

$$\begin{aligned} \mu_{4000}^{\text{out}}(\alpha, \beta) &= \mu_{4000} \cos^4 \alpha + 4\mu_{3100} \cos^3 \alpha \sin \alpha + 6\mu_{2200} \cos^2 \alpha \sin^2 \alpha \\ &\quad + 4\mu_{1300} \cos \alpha \sin^3 \alpha + \mu_{0400} \sin^4 \alpha, \end{aligned} \tag{9}$$

$$\mu_{3010}^{\text{out}}(\alpha, \beta) = \left(\mu_{3001} \cos^2 \alpha + 3\mu_{2101} \cos \alpha \sin \alpha + 3\mu_{1201} \sin^2 \alpha\right) \sin(\beta - \alpha), \tag{10}$$

$$\begin{aligned} 3\mu_{2020}^{\text{out}}(\alpha, \beta) &= \mu_{4000} \cos^2 \alpha \cos^2 \beta + 2\mu_{3100} \cos \alpha \cos \beta \sin(\alpha + \beta) \\ &\quad + 6\mu_{2200} \cos \alpha \sin \alpha \cos \beta \sin \beta + 3\mu_{2002} \sin^2(\beta - \alpha) \\ &\quad + 2\mu_{1300} \sin \alpha \sin \beta \sin(\alpha + \beta) + \mu_{0400} \sin^2 \alpha \sin^2 \beta. \end{aligned} \tag{11}$$

To determine the moments μ_{3001}, μ_{2101}, and μ_{1201} from the intensity moment $\mu_{3010}^{\text{out}}(\alpha, \beta)$, and the moment μ_{2002} from the intensity moment $\mu_{2020}^{\text{out}}(\alpha, \beta)$, we need anamorphic systems; and obviously we need three of them. Together with two additional isotropic systems, we can then construct nine equations from measurements of the intensity distributions in the five output planes, with which the nine moments can be determined. We note that even in the highly symmetric case that $W_\circ(\rho, \zeta, \theta)$ is an even function of θ, we still need an anamorphic system. Such a system would not be necessary if $W_\circ(\rho, \zeta, \theta)$ does not depend on θ at all, in which case only the strictly even-order moments (i.e., p, q, r, and s are even) remain and all other moments vanish.

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