Self-affinity in phase space

Tatiana Alieva and Martin J. Bastiaans

Technische Universiteit Eindhoven, Faculteit Elektrotechniek
Postbus 513, 5600 MB Eindhoven, Netherlands

Abstract

Based on the decomposition of coherent and partially coherent fields on the orthogonal sets of Hermite-Gauss modes, the expression for the Wigner distribution (WD) in polar coordinates has been derived. This representation allows to analyse easily the structure of the WD and to describe the field propagation through first-order optical systems including the self-imaging phenomenon.

OCIS codes: 110.1650, 110.6980, 070.1170

1 Introduction

The description of optical fields through the Wigner distribution (WD) is now a widely spread approach. It can be explained by the following facts. First of all, the WD gives us the representation of the optical field in a phase space, which is very important for the analysis of inhomogeneous structures. Moreover, the WD permits to treat by the same manner coherent as well as partially coherent fields. The evolution of the WD during optical field propagation through the widely used first-order optical systems [1] including thin lenses, spherical mirrors, graded index media, etc., is described through affine transformations. The first-order optical systems producing a rotation of the WD are used in phase-space tomography.

In this paper we derive the expression for the WD of optical fields based on the generalized Hermite-Gauss mode expansion of the complex field amplitude or the two-point correlation function for coherent or partially coherent fields, respectively. This representation of the WD permits easily to analyze its structure. In particular we consider how the modal interference reflects on the WD. We obtain the condition for self-imaging in first-order optical systems through the investigation of the self-affine structure of the WD. This approach allows to study the self-imaging of coherent as well as partially coherent optical fields. Moreover, WDs exhibiting the property of self-affinity describe a variety of optical fields with different types of symmetries. Indeed, WDs of periodic fields, Gaussian beams, Hermite-Gauss modes, fractal fields, etc., are self-affine. Then the characterisation of the self-affine structure of the WD becomes really actual.

2 Hermite-Gauss mode expansion of a complex field amplitude and a correlation function

Considering the propagation of an optical field through a linear optical system, it is often more convenient to represent the field as a superposition of the eigenmodes of the corresponding system. This approach is usually used for the case of coherent fields. Here we extend this approach to the representation of partially coherent fields in first-order optical systems.

Partially coherent, monochromatic light can be treated through the two-point correlation function $G(x_1, x_2) = \langle g(x_1) g^*(x_2) \rangle$, where the brackets indicate an ensemble average over the set of realizations of the complex field amplitude $g(x)$. The evolution of the complex field amplitude $g(x)$ during propagation through first-order optical systems, is described in the paraxial approximation of the scalar diffraction theory through the canonical integral transform, also known as the generalized Fresnel transform (GFT) [2, 3, 4] of the input field amplitude $g_i(x)$,

$$g_o(u) = R^M \left[g_i(x) \right] (u) = \int_{-\infty}^{\infty} g_i(x) K_M(x, u) dx, \tag{1}$$

with the kernel

$$K_M(x, u) = \begin{cases} \left(1/\sqrt{iB}\right) \exp\left(i\pi \left(Ax^2 + Du^2 - 2xu\right)/B\right) & B \neq 0 \\ \\ \sqrt{A} \exp\left(i\pi Cu^2/A\right) \delta(x - Au) & B = 0 \end{cases} \tag{2}$$

parametrized by a real 2×2 matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3}$$

with the determinant $AD - BC$ equal to 1. The parameters A, B, C, D depend on the concrete first-order system and the wavelength. For the sake of simplicity we will consider the one-dimensional case.

As an example, we mention that the GFT parametrized by a matrix with $A = D = \cos\theta$ and $B = -C = \sin\theta$ corresponds, except for a factor $\exp(i\theta/2)$, to the fractional Fourier transform (FT) [2, 5, 6].

According to Eqs. (1)-(2), the evolution of the correlation function is described by the two-dimensional GFT

$$G_o(u_1, u_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_i(x_1, x_2) K_M(x_1, u_1) K_M^*(x_2, u_2) dx_1 dx_2. \tag{4}$$

As it was shown in [7], the functions

$$\Phi_n(x) = \left(\sqrt{\pi} 2^n \lambda n!\right)^{-\frac{1}{2}} \exp\left(-\tfrac{1}{2}(1 + i\beta)(x/\lambda)^2\right) H_n\left(x/\lambda\right), \tag{5}$$

which we will call generalized Hermite-Gauss functions, are eigenfunctions for the GFT parametrized by the matrix (3) with eigenvalue $a = \exp(-i(n + \frac{1}{2})\theta)$, where $H_n(u)$ are the Hermite polynomials, and where the parameters θ, λ, and β are defined from the parameters of the transfer matrix by

$$\begin{aligned} \theta &= \arccos\left(\tfrac{1}{2}(A + D)\right) \\ \lambda^2 &= 2B\left(4 - (A + D)^2\right)^{-\frac{1}{2}} \\ \beta &= (A - D)\left(4 - (A + D)^2\right)^{-\frac{1}{2}}. \end{aligned} \tag{6}$$

The sets $\{\Phi_n(x)\}$ form orthonormal bases. Then every first-order optical system relates to a certain orthonormal basis of generalized Hermite-Gauss functions.

In limiting cases like the Fresnel transform ($\lambda^2 \to \infty$ and $\theta \to 0$) and the scaling transform ($\lambda^4 \to 0$ and $\beta^2 + 1 \to 0$), we cannot apply the relationships (5) and (6) for the construction of the GFT eigenfunctions. Moreover, if $|A + D| > 2$, the parameters θ, β, and λ become complex. We therefore confine ourselves to systems described by a matrix M for which $|A + D| < 2$ and $B, C \neq 0$.

We can represent a complex field amplitude $g(x)$ as a superposition of the generalized Hermite-Gauss modes,

$$g(x) = \sum_{n=0}^{\infty} g_n \Phi_n(x), \tag{7}$$

where

$$g_n = |g_n| \exp(i\varphi_n) = \int g(x)\Phi_n^*(x)dx. \tag{8}$$

This approach was used in [8, 9, 10, 11, 12] for the analysis of self-imaging of coherent fields in first-order optical systems.

The correlation function, like any other function of two variables, can also be expanded into the generalized Hermite-Gauss series in the following way:

$$G(x_1, x_2) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} g_{n,m}\Phi_n(x_1)\Phi_m^*(x_2), \tag{9}$$

where

$$g_{n,m} = |g_{n,m}| \exp(i\varphi_{n,m}) = \int\int G(x_1, x_2)\Phi_n^*(x_1)\Phi_m(x_2)dx_1dx_2. \tag{10}$$

Then the correlation function at the output plane of the first-order system associated with the given orthonormal set $\{\Phi_n(x)\}$, is given by

$$G_o(x_1, x_2) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \exp(-i(n-m)\theta) g_{n,m}\Phi_n(x_1)\Phi_m^*(x_2), \tag{11}$$

where the coefficients $g_{n,m}$ correspond to the expansion of the input correlation function. In general, the expression (11) could be used for the analysis of the self-imaging phenomenon of partially coherent light. In particular, two-dimensional eigenfunctions of the two-dimensional fractional FT were studied in [13]. A similar approach could be used for the analysis of two-dimensional functions that are self-reproducible under the GFT. Nevertheless, here we will consider the self-imaging in first-order optical systems in the framework of the WD, which allows to treat coherent and partially coherent fields in the same way and to point out the crucial difference in their Wigner distributions.

3 First-order optical systems and Wigner distribution

The Wigner distribution [14, 15] of a complex field amplitude $g(x)$ is defined by

$$W_g(x, k) = \int \langle g(x + y/2)g^*(x - y/2)\rangle \exp(-2\pi iky)dy, \tag{12}$$

where, in the case of partial coherence, the brackets indicate an ensemble average over the set of realizations of the amplitude $g(x)$. The Wigner distribution belongs to the class of bilinear distributions and gives a representation of the complex field amplitude in a phase space. The inverse transform allows to reconstruct the field amplitude (except for a constant phase factor) for coherent fields,

$$g(x) = \frac{1}{g^*(0)} \int W_g(x/2, k) \exp(2\pi ikx) dk, \tag{13}$$

or to reconstruct the two-point correlation function $G(x, y) = \langle g(x + y/2) g^*(x - y/2) \rangle$ for partially coherent ones,

$$G(x, y) = \int W_g(x, k) \exp(2\pi iky) dk. \tag{14}$$

A canonical integral transform corresponds to an affine transformation in phase space,

$$\begin{bmatrix} u \\ k_u \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ k_x \end{bmatrix}, \tag{15}$$

and therefore produces a linear transformation of the Wigner distribution: $W_{g_o}(x, k) = W_{g_i}(Dx - Bk, Ak - Cx)$. Then self-imaging in a first-order optical system is described in phase space by a Wigner distribution that is invariant under the linear transformation

$$W_{g_M}(x, k) = W_{g_M}(Dx - Bk, Ak - Cx). \tag{16}$$

This corresponds to self-reproducibility of the complex field amplitude $g(x)$ of coherent light, or of the correlation function $G(x, y)$ in the case of partially coherent light, during propagation through the first-order system. From the additivity property of the canonical integral transform, it follows that the cascade of n identical first-order optical systems defined by the matrix $M_n = (M)^n$ (with n a positive integer), also reproduces the deterministic or statistical eigenfield $g_M(x)$, and consequently $W_{g_M}(x, k) = W_{g_M}(D_n x - B_n k, A_n k - C_n x)$, where [12]

$$\begin{array}{lcl} A^{(n)} & = & \cos n\theta + \frac{1}{2}(A - D) \sin n\theta / \sin\theta \\ B^{(n)} & = & B \sin n\theta / \sin\theta \\ C^{(n)} & = & C \sin n\theta / \sin\theta \\ D^{(n)} & = & \cos n\theta - \frac{1}{2}(A - D) \sin n\theta / \sin\theta. \end{array} \tag{17}$$

The linear transformation includes rotation, shearing and scaling. Thus self-imaging during Fresnel diffraction ($A = D = 1,\ B = Z,\ C = 0$) is described by a Wigner distribution that is invariant under a shearing operation: $W_{f_Z}(x, k) = W_{f_Z}(x - Znk, k)$. The eigenfield of the scaling system ($AD = 1, B = C = 0$) is represented in phase space by the Wigner distribution $W_{f_S}(x, k) = W_{f_S}(A^{-1}x, Ak)$. Such fields are closely related to fractal structures, which are scale-invariant. The WD of coherent and partially coherent optical fields that exhibit the scaling property, was considered in [16].

Note also that the squared modulus of the GFT of $g_o(x)$ corresponds to the projection of the Wigner distribution under the related direction

$$|g_o(u)|^2 = \left| R^M \left[g_i(x) \right] (u) \right|^2 = \int W_{g_i}(Dx - Bk, Ak - Cx) dk. \tag{18}$$

4 Wigner distribution in polar coordinates

The WD is usually represented in a Cartesian system of coordinates (x, k_x), where x is a position and k_x is a spatial frequency. Meanwhile, for the description of field propagation through a quadratic refractive index medium, it is more convenient to represent the WD in polar coordinates. In this section we derive the expression for the WD in polar coordinates based on the optical field decomposition on the Hermite-Gauss modes.

Taking into account relationships (7), (9) and (12), it is easy to see that the Wigner distribution of coherent as well as of partially coherent fields can be written as

$$W_g(x, k_x) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} g_{n,m} W_{\Phi_n\Phi_m}(x, k_x) \tag{19}$$

where

$$W_{\Phi_n\Phi_m}(x, k_x) = \int \Phi_n(x + x'/2)\Phi_m^*(x - x'/2)\exp(-ik_x x')dx' \tag{20}$$

and $g_{n,m}$ for the coherent case is given by

$$g_{n,m} = |g_{n,m}|\exp(i\varphi_{n,m}) = g_n g_m^* = |g_n g_m|\exp(i(\varphi_n - \varphi_m)). \tag{21}$$

The term $W_{\Phi_n\Phi_n}$ corresponds to the Wigner distribution of a single mode, while the term $W_{\Phi_n\Phi_m}$ describes the mode interference.

Let us first consider the amplitude decomposition into the series of the Hermite-Gauss functions being eigenfunctions of the fractional Fourier transform operator. In this case we can put $\lambda = 1$, and $\beta = 0$ in Eq. (5):

$$\begin{aligned} W_{\Phi_n\Phi_m}(x, k_x) &= \int \Phi_n(x + x'/2)\Phi_m^*(x - x'/2)\exp(-ik_x x')dx' \\ &= \left(\pi 2^{n+m} n! m!\right)^{-1/2} \int \exp\left(-(x^2 + (x')^2/4)\right) \\ &\quad \times H_n\left(x + x'/2\right) H_m^*\left(x - x'/2\right)\exp(-ik_x x')dx'. \end{aligned} \tag{22}$$

In order to calculate $W_{\Phi_n\Phi_m}$ we introduce the polar coordinate system (ρ,θ), where $\rho^2 = x^2 + k_x^2$ and $\tan\theta = x/k_x$. In particular for $\theta = 0$ ($k_x = 0$) one has

$$W_{\Phi_n\Phi_m}(\rho, 0) = \int \Phi_n(\rho + \rho'/2)\Phi_m^*(\rho - \rho'/2)d\rho'. \tag{23}$$

Since a fractional FT produces a rotation in phase space,

$$\begin{array}{lcl} \Phi_n\Phi_m & \rightarrow & W_{\Phi_n\Phi_m}(\rho, \alpha) \\ \downarrow \text{fractional FT} & & \downarrow \text{rotation of WD} \\ R^\theta\left[\Phi_n\right], R^\theta\left[\Phi_m\right] & \rightarrow & W_{R^\theta[\Phi_n]R^\theta[\Phi_m]}(\rho, \alpha) = W_{\Phi_n\Phi_m}(\rho, \alpha + \theta), \end{array} \tag{24}$$

and choosing $\alpha = 0$, we obtain that $W_{\Phi_n\Phi_m}(\rho, \theta) = W_{R^\theta[\Phi_n]R^\theta[\Phi_m]}(\rho, 0)$. The determination of the Wigner distribution requires thus the calculation of the correlations between the modes in the fractional FT domains:

$$W_{\Phi_n\Phi_m}(\rho, \theta) = \int R^\theta\left[\Phi_n(x)\right](\rho + \rho'/2)R^{-\theta}\left[\Phi_m^*(x)\right](\rho - \rho'/2)d\rho'. \tag{25}$$

Since $\Phi_n(x)$ are the eigenfunctions of the fractional FT operator R^θ with eigenvalues $a = \exp(-i(n+\frac{1}{2})\theta)$, Eq. (25) reduces to

$$W_{\Phi_n\Phi_m}(\rho,\theta) = \exp\left(-i(n-m)\theta\right) \int \Phi_n(\rho+\rho'/2)\Phi_m^*(\rho-\rho'/2)d\rho'. \tag{26}$$

Using then the property [17]

$$\int \exp(-x^2)H_n(x+y)H_m(x+z)dx = 2^m\sqrt{\pi}n!z^{m-n}L_n^{m-n}(-2yz), \quad [n \leq m] \tag{27}$$

where $L_n^{m-n}(x)$ are Laguerre polynomials, we obtain

$$W_{\Phi_n\Phi_m}(\rho,\theta) = \exp\left(-i(n-m)\theta\right)(-1)^n 2^{1+(m-n)/2}\sqrt{\frac{n!}{m!}}\exp(-\rho^2)\rho^{m-n}L_n^{m-n}(2\rho^2). \tag{28}$$

The expression for the WD of the optical field in polar coordinates can then be written as

$$W_g(\rho,\theta) = 2\exp(-\rho^2)\sum_{m=0}^{\infty}\sum_{n\leq m}|g_{n,m}|\,2^{(m-n)/2}\sqrt{\frac{n!}{m!}}\rho^{m-n}L_n^{m-n}(2\rho^2)\cos\left((m-n)\theta+\varphi_{m,n}\right), \tag{29}$$

where $g_{n,m}$ are the coefficients in the Hermite-Gauss mode expansion of the complex field amplitude for the coherent case or of the correlation function for the partially coherent case. It is easy to see that the WD exhibits rotational symmetry if all coefficients $g_{n,m} = 0$ for $n \neq m$. Then the WD takes the form

$$W_g(\rho) = 2\exp(-\rho^2)\sum_{m=0}^{\infty}|g_{m,m}|\cos\left(\varphi_{m,m}\right)L_m(2\rho^2). \tag{30}$$

It follows from Eq. (21), $g_{n,m} = g_n g_m^*$, that the Wigner distribution of a coherent field is rotationally symmetric only if the complex amplitude $g(x)$ corresponds to one Hermite-Gauss mode $\Phi_n(x)$. Then we get

$$W_g(\rho) = W_{\Phi_n}(\rho) = 2\exp(-\rho^2)L_n(2\rho^2). \tag{31}$$

Note that, while $\Phi_n(x)$ is an eigenfunction of the fractional Fourier transform, $W_{\Phi_n}(\rho/\sqrt{2})$ is an eigenfunction of the fractional Hankel transform of zero order.

In the case of partial coherence, a linear superposition of Laguerre-Gauss modes with real coefficients forms a rotationally symmetric WD (30), which corresponds to the correlation function

$$G(x_1,x_2) = \sum_{m=0}^{\infty} g_{m,m}\Phi_m(x_1)\Phi_m^*(x_2). \tag{32}$$

In particular if the coefficients $g_{m,m}$ are given by [15]

$$g_{m,m} = \frac{2\sigma}{1+\sigma}\left(\frac{1-\sigma}{1+\sigma}\right)^m, \quad \text{for } 0 < \sigma \leq 1, \tag{33}$$

then the correlation function corresponds to Gauss-Schell model

$$G(x_1,x_2) = \sqrt{2\sigma}\exp\left(-\frac{\pi}{2}\left(\sigma(x_1+x_2)^2+\sigma^{-1}(x_1-x_2)^2\right)\right). \tag{34}$$

If all coefficients $g_{n,m}$ in the Hermite-Gauss expansion are real and positive, or complex but with equal phases, then the WD exhibits axial symmetry, $W_g(\rho,\theta) = W_g(\rho,-\theta)$, with respect to the axis $x = 0$.

Let us find a condition for which the WD exhibits rotational symmetry for some angle θ_0

$$W_g(\rho,\theta_0) = W_g(\rho,0). \tag{35}$$

The Wigner distribution of such fields enjoys the rotational symmetry $W_g(x,k) = W_g(x\cos n\theta_0 - k\sin n\theta_0, k\cos n\theta_0 + x\sin n\theta_0)$. Considering Eq. (29) we can conclude that for the given angle θ_0 only such coefficients $g_{n,m}$ differ from zero whose indices satisfy the equation

$$(m-n)\theta_0 = 2\pi k. \tag{36}$$

It is easy to see that if $\theta_0/2\pi$ is irrational then only $m = n$ is a solution of this equation. For partially coherent fields we then obtain the expression (30), while Eq. (31) corresponds to the coherent case.

If $\theta_0 = 2\pi p/q$ where p and q are relatively prime integers and $p < q$, then only the coefficients $g_{n,m}$ with indices n and m such that

$$m = n + ql, \tag{37}$$

where l is a nonnegative integer, differ from zero in the decomposition (29). For invariance under rotation at an angle $\theta_0 = 2\pi p/q$, the WD can be written as

$$W_g(\rho,\theta) = 2\exp(-\rho^2)\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}|g_{n,n+ql}|\,2^{ql/2}\sqrt{\frac{n!}{(n+ql)!}}\rho^{ql}L_n^{ql}(2\rho^2)\cos\left(ql\theta+\varphi_{n+ql,n}\right). \tag{38}$$

As it follows from (31), this expression reduces for the coherent case to

$$W_g(\rho,\theta) = 2\exp(-\rho^2)\sum_{l=0}^{\infty}|g_n g_{n+ql}|\,2^{ql/2}\sqrt{\frac{n!}{(n+ql)!}}\rho^{ql}L_n^{ql}(2\rho^2)\cos\left(ql\theta+\varphi_{n+ql,n}\right), \tag{39}$$

where n is a constant integer, $0 \le n < q$. Then the WD of the partially coherent light (38) can be represented as a sum of the WDs of certain coherent eigenfields of the fractional FT system (39) associated with the angle $\theta_0 = 2\pi p/q$.

The corresponding expressions for the eigenfields of the fractional FT system associated with the angle $\theta_0 = 2\pi p/q$, can be written in the following way for the coherent and the partially coherent case, respectively:

$$\begin{aligned} g(x) &= \sum_{l=0}^{\infty} g_{n+lq}\Phi_{n+lq}(x), \\ G(x_1,x_2) &= \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} g_{n,n+ql}\Phi_n(x_1)\Phi^*_{n+ql}(x_2). \end{aligned} \tag{40}$$

These results correspond to the expressions for the one- and two-dimensional eigenfunctions of the fractional FT derived in [9, 13].

5 Self-affine Wigner distribution

In the previous section the WD of the eigenfield of a fractional FT system associated with an angle θ_0 turned out to be invariant under rotation at an angle $n\theta_0$ in phase space. In this section we study the more general case and show that the WD of the eigenfield of a first-order optical system associated with the transfer matrix M is invariant under a cascade of affine transformations described by the matrix M^n. The specific properties of the eigenfield depend on the parameters of the transfer matrix M. Thus, the eigenfields of the Fresnel transform exhibit periodicity properties, while the eigenfields of the scaling transform are related to fractal structures. Note that the eigenfields of the GFT parametrized by the matrix (17), which we will consider in this section, are not fractals, because the scaling factor λ remains invariant under the cascade of the affine transformations in phase space; nevertheless, they do possess a certain symmetry in phase space.

In order to find the expression for the WD corresponding to the eigenfield of an arbitrary first-order optical system, let us find the mutual WD for two generalized Hermite-Gauss modes (5):

$$\begin{aligned} W_{\Phi_n\Phi_m}(x,k_x) &= \int \Phi_n(x+x'/2)\Phi_m^*(x-x'/2)\exp(-ik_x x')dx' \\ &= \left(\pi 2^{n+m}\lambda^2 n!m!\right)^{-1/2}\int \exp\left(-\frac{x^2+x'^2/4}{\lambda^2}\right)H_n\left(\frac{x+x'/2}{\lambda}\right) \\ &\quad \times H_m^*\left(\frac{x-x'/2}{\lambda}\right)\exp(-i(k_x+x\beta/\lambda^2)x')dx'. \end{aligned} \tag{41}$$

Introducing new variables $y=x/\lambda$, $y'=x'/\lambda$, $\omega=\lambda k_x+x\beta/\lambda$, we find out that the equation for $W_{\Phi_n\Phi_m}(y,\omega)$ is identical to Eq. (22). Then using the results from the previous section, we can derive the expression for the WD of the function $g(x)$:

$$W_g(\rho,\theta)=2\exp(-\rho^2)\sum_{m=0}^{\infty}\sum_{n\le m}|g_{n,m}|\,2^{(m-n)/2}\sqrt{\frac{n!}{m!}}\rho^{m-n}L_n^{m-n}(2\rho^2)\cos\left((m-n)\theta+\varphi_{m,n}\right), \tag{42}$$

where

$$\begin{aligned} \cot\theta &= \beta+\lambda^2 k_x/x \\ &= (A-D+2Bk_x/x)\left(4-(A+D)^2\right)^{-\frac{1}{2}} \\ \rho^2 &= (1+\beta^2)\lambda^{-2}x^2+\lambda^2k_x^2+2\beta x k_k \\ &= 2\left(4-(A+D)^2\right)^{-\frac{1}{2}}\left(Bk_x^2-Cx^2+(A-D)xk_x\right), \end{aligned} \tag{43}$$

and where the coefficients $g_{n,m}$ correspond to the decomposition of the field on the generalized Hermite-Gauss functions.

It is easy to see that ρ^2 is invariant under the affine transformations (15) and that the angle θ changes like $\theta\to\theta+\theta_0$, where $\cos\theta_0=(A+D)/2$. Then the WD in the form

$$W_g(\rho)=2\exp(-\rho^2)\sum_{m=0}^{\infty}|g_{m,m}|\cos\left(\varphi_{m,m}\right)L_m(2\rho^2) \tag{44}$$

is invariant under the affine transformation describing by a matrix M with

$$\begin{array}{rcl} A & = & \cos\theta + \beta\sin\theta \\ B & = & \lambda^2 \sin\theta \\ C & = & -\left((\beta^2+1)/\lambda^2\right)\sin\theta \\ D & = & \cos\theta - \beta\sin\theta, \end{array} \tag{45}$$

where θ takes an arbitrary real value. For the coherent case this corresponds to a certain generalized Hermite-Gauss mode, $g(x) = \Phi_n(x)$, while for partially coherent light we have a variety of suitable correlation functions.

In particular, taking the coefficients $g_{m,m}$ in the form (33), which produces the WD $W_g(\rho) = 2\sigma\exp(-\sigma\rho^2)$, we obtain the expression for the correlation function

$$G(x_1, x_2) = \frac{\sqrt{2\sigma}}{\lambda}\exp\left(-\frac{\pi}{2\lambda^2}\left(\sigma(x_1+x_2)^2 + \sigma^{-1}(x_1-x_2)^2 + i\beta(x_1-x_2)(x_1+x_2)\right)\right), \tag{46}$$

which describes self-imaging in first-order systems whose parameters are connected with λ and β by Eqs. (45).

The expansion (9) with the coefficients $g_{m,m} = (-\sigma^2/2)^m/\Gamma(1+m)$, relates to the self-affine WD

$$W_g(\rho) = 2\exp(-\rho^2)\exp(-\sigma^2/2)J_0(2\rho\sigma),$$

where $J_0(x)$ is the Bessel function of zero order.

Another example corresponds to the case of a finite number of generalized Hermite-Gauss modes

$$\begin{array}{ll} g_{m,m} = 1 & m \le n \\ g_{m,m} = 0 & m > n. \end{array}$$

Then using the relationship $\sum_{m=0}^{n} L_m(x) = L_n^1(x)$, one derives the WD in the form

$$W_g(\rho) = 2\exp(-\rho^2)L_n^1(2\rho^2).$$

In order to describe the structure of the WD that is invariant under the cascade of identical GFTs parametrized by a matrix with coefficients given by (17), where the parameter $\theta = \theta_0$ is fixed, we can use the conclusions that were made in the case of the rotationally symmetrical WD in the previous section. In particular, if $\theta_0/2\pi$ is irrational then only fields of the forms (5) and (32) for the coherent and the partially coherent case, respectively, are self-reproducible.

If $\theta_0/2\pi = p/q$ is rational, then we have the expressions (40) for the complex field amplitude and for the correlation function, where the decompositions on the generalized Hermite-Gauss functions (5) are used. As it was shown in [12], the coherent eigenfield for the GFT operator R^M can be constructed from any coherent generator field $f(x)$ through the following procedure

$$g^{(n)}(u) = \sum_{l=0}^{\infty} g_{n+lq}\Phi_{n+lq}(u) = \frac{1}{k}\sum_{l=0}^{k-1}\exp\left(i2\pi(n+\tfrac{1}{2})l/k\right)R^{M^l}\left[f(x)\right](u), \tag{47}$$

where $f(x) = \sum_{q=0}^{n-1}\sum_{l=0}^{\infty} g_{n+lq}\Phi_{n+lq}(u)$.

Finally let us consider an optical field whose Wigner distribution has the Gaussian form

$$W_f(x,k) = \exp(-(px^2 + qk^2 + tkx)). \qquad (48)$$

Self-imaging of such a field in a first-order optical system described by the matrix (3) is observed if the relationship (16) holds. This produces the following system of equations

$$\begin{cases} (D^2-1)p + C^2q - DCt & = & 0 \\ B^2p + (A^2-1)q - ABt & = & 0 \\ DBp + ACq - BCt & = & 0. \end{cases} \qquad (49)$$

For $C \neq 0$, the parameters p, q and t are connected with each other as

$$\begin{cases} q & = & -Bp/C \\ t & = & (D-A)p/C. \end{cases} \qquad (50)$$

The Gaussian Wigner distribution that is invariant under a linear transformation described by the matrix (3) with $C \neq 0$, can be written as

$$W_f(x,k) = \exp(-p(x^2 - Bk^2/C + (D-A)kx/C)). \qquad (51)$$

Thus for a fractional FT system, determined by $A = D = \cos\alpha$ and $B = -C = \sin\alpha$, we have $q = p$ and $t = 0$. In this case $W_{f_\alpha}(x,k) = \exp(-p(x^2 + k^2))$ is invariant under rotation at any angle. If $C = 0$, then $p = 0$ and $q = ABt/(A^2-1)$ for $A \neq 1$. The case $C = 0$ and $A = 1$ (and hence $D = 1$) corresponds to the Fresnel transform; we then have $p = t = 0$, and the Gaussian beam is δ-correlated.

6 Conclusions

To conclude let us make some comparisons between the self-imaging of coherent and partially coherent light in first-order optical systems. In spite of the fact that both cases are well described through the WD, which remains invariant under an affine transformation in phase space, not all structures of the WD that are suitable for partially coherent fields, are allowed for coherent light. Moreover, the WD of the partially coherent eigenfield of a first-order optical system can be represented as a superposition of the WDs of the coherent eigenfields related to this optical system.

From another point of view, only coherent light allows us to construct the eigenfield for a given optical system from a coherent generator field through a simple procedure (47), which could be realized experimentally.

The expression for the Wigner distribution in polar coordinates derived in this paper, might be very useful for the description of optical fields, because in some experiments the Wigner distribution can be measured only in polar coordinates [18].

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