On the discrete Gabor transform and the discrete Zak transform

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Abstract

Gabor's expansion of a discrete-time signal into a set of shifted and modulated versions of an elementary signal (or synthesis window) and the inverse operation – the Gabor transform – with which Gabor's expansion coefficients can be determined, are introduced. It is shown how, in the case of a finite-support analysis window and with the help of an overlap-add technique, the discrete Gabor transform can be used to determine Gabor's expansion coefficients for a signal whose support is not finite.

The discrete Zak transform is introduced and it is shown how this transform, together with the discrete Fourier transform, can be used to represent the discrete Gabor transform and the discrete Gabor expansion in sum-of-products forms. It is shown how the sum-of-products form of the Gabor transform enables us to determine Gabor's expansion coefficients in a different way, in which fast algorithms can be applied.

Using the sum-of-products forms, a relationship between the analysis window and the synthesis window is derived. It is shown how this relationship enables us to determine the optimum synthesis window in the sense that it has minimum L_2 norm, and it is shown that this optimum synthesis window resembles best the analysis window.

1 Introduction

It is sometimes convenient to describe a (discrete-time) signal $\varphi(n)$ say, not in the time domain, but in the frequency domain by means of its *frequency spectrum*, i.e., the Fourier transform of $\varphi(n)$. This frequency spectrum shows us the *global* distribution of the energy of the signal as a function of frequency. However, one is often more interested in the momentary or *local* distribution of the energy as a function of frequency. This leads to the concept of a *local frequency spectrum*, where the signal is described in time and frequency, simultaneously.

A candidate for a local frequency spectrum is *Gabor's signal expansion*. In 1946 Gabor [11] suggested the expansion of a (continuous-time) signal into a discrete set of properly shifted and modulated versions of a Gaussian elementary signal [3, 4, 5, 11, 12, 13]. Although Gabor restricted himself to a Gaussian-shaped elementary signal, his signal expansion holds for rather arbitrarily shaped elementary signals. It was shown that for an arbitrary elementary signal (or synthesis window, as it if often called) an analysis window could be found such that Gabor's expansion coefficients can be found as sampling values of a windowed Fourier transform.

In his original paper, Gabor restricted himself to a *critical* sampling of the time-frequency domain, in which case the expansion coefficients can be interpreted as independent data, i.e., degrees of freedom of a continuous-time signal. It is the aim of this paper to apply Gabor's concepts to discrete-time signals and to the case of *oversampling*; in the case of oversampling the expansion coefficients are no longer independent, of course. Moreover, we will restrict ourselves to the case of a *finite-support* analysis window, which enables us to treat all Fourier transforms as discrete Fourier transforms, for which fast algorithms exist.

In Section 2 we will first introduce the discrete Gabor expansion and its companion, the discrete Gabor transform with which the expansion coefficients can be determined. We will then demonstrate how we can determine Gabor's expansion coefficients – even for a signal that has an infinite support, in contrast to the finite-support of the analysis window – by using an overlap-add technique well-known in digital signal processing. In Section 3 we introduce the mathematical tools that we will use: the discrete Fourier transform and the discrete Zak transform. We will use these mathematical tools in Section 4, to transform the discrete Gabor expansion and the discrete Gabor transform into another, mathematically more attractive form. In particular we will demonstrate how the Gabor coefficients can be calculated by means of fast transform algorithms. In Section 5 we will present an elegant relationship between the synthesis window, which appears in the discrete Gabor expansion, and the analysis window, which appears in the discrete Gabor transform. We will show how, for an arbitrary finite-support analysis window, an optimum synthesis window can be found.

2 Discrete Gabor expansion and discrete Gabor transform

We start with *Gabor's signal expansion* [7, 9, 11] for a discrete-time signal $\varphi(n)$,

$$\varphi(n) = \sum_m \sum_{k=<K>} a_{mk} g(n - mN) e^{j2\pi kn/K}, \tag{2.1}$$

where the array a_{mk} (and also the exponential $e^{j2\pi kn/K}$, of course) is periodic in k with period K and where the expression $k = < K >$ throughout denotes a finite interval of K successive integers k; the summation over m extends over all integer values. The sequence $g(n)$ is known as the *elementary signal* or the *synthesis window*. The array of Gabor coefficients a_{mk} can be found via the *Gabor transform*

$$a_{mk} = \sum_n \varphi(n) w^*(n - mN) e^{-j2\pi kn/K}, \tag{2.2}$$

where the sequence $w(n)$ is known as the *analysis window*; the summation over n extends again over all integer values.

In Gabor's original case of critical sampling ($K = N$), it has been shown how, for a given synthesis window $g(n)$, an analysis window $w(n)$ can be found. It is the aim of this paper to show how such a window function can be found and how the array of Gabor coefficients a_{mk} can be determined in the case of oversampling $K > N$ where, moreover, the window function $w(n)$ has a finite support N_w. For convenience, we consider signals $\varphi(n)$ that have a finite support N_φ, too; in the case that we are dealing with signals of longer (or even infinite) support, we can always split the signal in parts that do have a finite support N_φ and treat all these parts separately. Under these conditions of finite support, the array a_{mk}, which is periodic in the k-variable, has a finite support M in the m-variable, where the support M satisfies the condition

$$MN \geq N_\varphi + N_w - 1. \tag{2.3}$$

We now introduce the periodized version A_{mk} of the array a_{mk} according to

$$A_{mk} = \sum_r a_{m+rM,k}. \tag{2.4}$$

Note that the periodized array A_{mk} is periodic in m (and k) with period M (and K), and that we can identify a_{mk} as one period of A_{mk}. If we introduce the periodized version $W(n)$ of the (finite support) window function $w(n)$ according to

$$W(n) = \sum_r w(n + rMN), \tag{2.5}$$

it is easy to see that the periodized array A_{mk} can be expressed in terms of the periodized window function $W(n)$ via a kind of Gabor transform [cf. Eq. (2.2)]:

$$A_{mk} = \sum_n \varphi(n) W^*(n - mN) e^{-j2\pi kn/K}. \tag{2.6}$$

Note that the periodized window function $W(n)$ is periodic with period MN and that we can identify $w(n)$ as one period of $W(n)$.

We could as well periodize the (finite support) signal $\varphi(n)$ according to

$$\Phi(n) = \sum_r \varphi(n + rMN). \tag{2.7}$$

Note that the periodized signal $\Phi(n)$ is periodic with period MN and that we can identify $\varphi(n)$ as one period of $\Phi(n)$. If we substitute from Gabor's signal expansion (2.1) into Eq. (2.7) we get

$$\Phi(n) = \sum_r \left[\sum_m \sum_{k=<K>} a_{mk} g(n + rMN - mN) e^{j2\pi k(n+rMN)/K} \right].$$

After arranging factors and requiring that K is a divisor of MN (and hence $e^{j2\pi kr(MN/K)} = 1$), we can write

$$\Phi(n) = \sum_m \sum_{k=<K>} a_{mk} \left[\sum_r g(n + rMN - mN) \right] e^{j2\pi kn/K}.$$

After introducing the periodized version $G(n)$ of the elementary signal $g(n)$ [cf. Eq. (2.5)], we can write

$$\Phi(n) = \sum_m \sum_{k=<K>} a_{mk} G(n - mN) e^{j2\pi kn/K}, \tag{2.8}$$

which relationship has the form of a Gabor expansion [cf. Eq. (2.1)].

It is not difficult to show that we also have the relationships

$$\Phi(n) = \sum_{m=<M>} \sum_{k=<K>} A_{mk} G(n - mN) e^{j2\pi kn/K} \tag{2.9}$$

and

$$A_{mk} = \sum_{n=<MN>} \Phi(n) W^*(n - mN) e^{-j2\pi kn/K}, \tag{2.10}$$

which are fully periodized versions of Gabor's signal expansion (2.1) and the Gabor transform (2.2), respectively. Equation (2.9) is known as the *discrete Gabor expansion*, while Eq. (2.10) is known as the *discrete Gabor transform* [2, 9, 16, 17].

The importance of having discrete versions of Gabor's signal expansion and the Gabor transform lies in the fact that we can formulate fast algorithms to compute them as we shall see later. However, in order to compute Gabor's expansion coefficients a_{mk} by means of the discrete Gabor transform, both the window function $w(n)$ *and* the signal $\varphi(n)$ should have a finite support. We can require that the window function has such a finite support, but we must allow the signal to have a very large (or even infinite) support. In that case we can apply overlap-add techniques by splitting the signal $\varphi(n)$ in parts and treating all parts separately. In detail we can proceed as follows. We represent the signal $\varphi(n)$ as a sequence of partial signals $\varphi_r(n)$, in which each partial signal vanishes outside an interval of length N_φ; hence

$$\varphi(n) = \sum_r \varphi_r(n), \tag{2.11}$$

with, for all values of r, $\varphi_r(n) = \varphi(n)$ for $rN_\varphi \le n \le (r+1)N_\varphi - 1$ and $\varphi_r(n) = 0$ outside that interval. On substituting the expansion (2.11) into the Gabor transform (2.2) we get

$$a_{mk} = \sum_n \left[\sum_r \varphi_r(n) \right] w^*(n - mN) e^{-j2\pi kn/K} = \sum_r \left[\sum_n \varphi_r(n) w^*(n - mN) e^{-j2\pi kn/K} \right]$$

$$= \sum_r a_{mk}(r), \tag{2.12}$$

where each partial Gabor transform

$$a_{mk}(r) = \sum_n \varphi_r(n) w^*(n - mN) e^{-j2\pi kn/K} \tag{2.13}$$

can be treated as a discrete Gabor transform [cf. Eq. (2.10)]. The summation over r in Eq. (2.12) must take into account, of course, the overlap between the partial Gabor transforms.

That such an overlap occurs, may become apparent from a simple example. Let us consider an analysis window $w(n)$ that has non-zero values for $n = -2$ through $n = 2$, only; hence we may choose $N_w = 5$. Furthermore, let us choose $N_\varphi = 4$, $N = 2$, and $M = 4$; note that the condition $MN \geq N_\varphi + N_w - 1$ is met. We now consider, for instance, the Gabor coefficient $a_{2,0}$. Due to the finite support of the analysis window, this coefficient is determined by the signal values $\varphi(2)$ through $\varphi(6)$, only [cf. Eq. (2.2)]:

$$a_{2,0} = \varphi(2)w(-2) + \varphi(3)w(-1) + \varphi(4)w(0) + \varphi(5)w(1) + \varphi(6)w(2).$$

If we introduce as before [see Eq. (2.11)] the partial signals $\varphi_r(n)$, the Gabor coefficient can as well be expressed as

$$a_{2,0} = \varphi_0(2)w(-2) + \varphi_0(3)w(-1) + \varphi_1(4)w(0) + \varphi_1(5)w(1) + \varphi_1(6)w(2),$$

which, after introducing the partial Gabor transforms $a_{mk}(r)$ [see Eq. (2.13)], immediately leads to

$$a_{2,0} = a_{2,0}(0) + a_{2,0}(1).$$

From the latter expression we conclude that, in order to calculate a particular Gabor coefficient, we may need more than one partial Gabor transform.

3 Discrete Fourier transform and discrete Zak transform

In this section we introduce the Fourier transform of a (periodized) two-dimensional array and the Zak transform [14, 15, 20, 21, 22] of a (periodized) discrete-time sequence; in Section 4 we will then use these transforms to express the discrete Gabor expansion and the discrete Gabor transform in different forms. For convenience, we introduce two integers p and q ($p \geq q \geq 1$) that do not have common factors and for which the relationship $pN = qK$ holds; note that $K/N = p/q \geq 1$ represents the degree of oversampling.

The *discrete Fourier transform* $\bar{a}(n/K, l/M)$ of the periodic array A_{mk} is defined according to

$$\bar{a}\left(\frac{n}{K}, \frac{l}{M}\right) = \sum_{m=<M>} \sum_{k=<K>} A_{mk} e^{-j2\pi(ml/M - kn/K)}; \tag{3.1}$$

we will throughout denote the discrete Fourier transform of an array by the same symbol as the array itself, but marked by a bar on top of it. Note that the discrete Fourier transform $\bar{a}(n/K, l/M)$ is periodic in the variables n and l with periods K and M, respectively. The *inverse* discrete Fourier transform reads

$$A_{mk} = \frac{1}{MK} \sum_{n=<K>} \sum_{l=<M>} \bar{a}\left(\frac{n}{K}, \frac{l}{M}\right) e^{j2\pi(ml/M - kn/K)}. \tag{3.2}$$

The *discrete Zak transform* $\tilde{w}(n, l/MN; N)$ of the periodized window function $W(n)$ is defined as a one-dimensional discrete Fourier transform of the sequence $W(n + mN)$ (with $m = < M >$) and n being a mere parameter), hence

$$\tilde{w}\left(n, \frac{l}{MN}; N\right) = \sum_{m=<M>} W(n + mN) e^{-j2\pi mN(l/MN)}; \tag{3.3}$$

we will throughout denote the discrete Zak transform of a signal by the same signal as the signal itself, but marked by a tilde on top of it. After substituting from Eq. (2.5), it can easily be shown that the discrete Zak transform can also be represented in the form

$$\tilde{w}\left(n, \frac{l}{MN}; N\right) = \sum_{m} w(n + mN)e^{-j2\pi mN(l/MN)}. \tag{3.4}$$

We remark that the discrete Zak transform $\tilde{w}(n, l/MN; N)$ is *periodic* in the frequency variable l with period M and *quasi-periodic* in the time variable n with quasi-period N:

$$\tilde{w}\left(n + mN, \frac{l + kM}{MN}; N\right) = \tilde{w}\left(n, \frac{l}{MN}; N\right)e^{j2\pi mN(l/MN)}. \tag{3.5}$$

The *inverse* relationship of the discrete Zak transform has the form

$$W(n + mN) = \frac{1}{M}\sum_{l=<M>} \tilde{w}\left(n, \frac{l}{MN}; N\right)e^{j2\pi mN(l/MN)}; \tag{3.6}$$

it will be clear that, with $m = < M >$, the variable n in the latter relationship can be restricted to a finite interval of length N. From the properties of the discrete Zak transform we mention *Parseval's energy theorem*, which leads to the relationship

$$\frac{1}{M}\sum_{n=<N>}\sum_{l=<M>}\left|\tilde{w}\left(n, \frac{l}{MN}; N\right)\right|^2 = \sum_{n=<N>} |W(n)|^2. \tag{3.7}$$

We also introduce the discrete Zak transform of $\Phi(n)$

$$\tilde{\varphi}\left(n, \frac{l}{MN}; pN\right) = \sum_{m=<M/p>} \Phi(n + mpN)e^{-j2\pi mpN(l/MN)}, \tag{3.8}$$

where the condition that p is a divisor of M should hold. If we substitute from Eq. (2.7), it is not difficult to see that we also have the relationship

$$\tilde{\varphi}\left(n, \frac{l}{MN}; pN\right) = \sum_{m} \varphi(n + mpN)e^{-j2\pi mpN(l/MN)}. \tag{3.9}$$

From the condition that p is a divisor of M, we can write $M = pL$, where L is an integer. From $pN = qK$ and $M = pL$ we conclude that $MN = qKL$, which implies that K is a divisor of MN. The latter condition is exactly the condition that should hold to be able to derive Eq. (2.8). Moreover, from $pN = qK$ and assuming that p and q do not have common factors, we also conclude that p is a divisor of K; hence we can write $K = pJ$, where J is an integer. We thus conclude that K, M, and N can be expressed in terms of the integers p, q (with $p \geq q \geq 1$, and p and q not having common factors), J and L: $K = pJ$, $M = pL$, and $N = qJ$.

4 Transformation of the discrete Gabor expansion and the discrete Gabor transform

Using the discrete Fourier transform and the discrete Zak transform defined in the previous section, it can be shown (see Appendix A) that the discrete Gabor transform can be transformed into the *sum-of-products form*:

$$\bar{a}\left(\frac{n}{K}, \frac{l + rM/p}{M}\right) = K\sum_{s=<q>} \tilde{\varphi}\left(n + sK, \frac{l}{MN}; pN\right)\tilde{w}^*\left(n + sK, \frac{l + rM/p}{MN}; N\right). \tag{4.1}$$

Note that in Gabor's original case of critical sampling ($p = q = 1$), Eq. (4.1) takes the simple *product form*

$$\bar{a}\left(\frac{n}{K}, \frac{l}{M}\right) = K\tilde{\varphi}\left(n, \frac{l}{MN}; N\right)\tilde{w}^*\left(n, \frac{l}{MN}; N\right).$$

We remark that Eq. (4.1) enables us to calculate Gabor's expansion coefficients a_{mk} in a way that is completely different from the original definition (2.2).

- We first determine the discrete Zak transform $\tilde{\varphi}(n + sK, l/MN; pN)$ of the signal $\varphi(n)$ by means of its definition (3.9). Since the discrete Zak transform is essentially a discrete Fourier transform, this can be done with a fast algorithm, especially if we choose $M/p = L$ equal to a power of 2.

- We then multiply the discrete Zak transform $\tilde{\varphi}(n + sK, l/MN; pN)$ by the discrete Zak transform $\tilde{w}^*(n + sK, [l + rM/p]/MN; N)$ of the window function $w(n)$. Note that the determination of the discrete Zak transform of the window function needs to be done only once, and that it can also be done with a fast algorithm.

- We sum over $s = <q>$. Note that this summation is not necessary if we choose $q = 1$, i.e., in the case of *integer* oversampling $K/N = p$.

- We then determine the periodized array A_{mk} by means of the inverse discrete Fourier transformation (3.2). Note that this can be done with a fast algorithm again, especially if we choose $K = pJ$ and $M = pL$ equal to powers of 2.

- We finally recognize the array of Gabor coefficients a_{mk} as one period of the periodic array A_{mk}.

By using the discrete Fourier transform and the discrete Zak transform, it can also be shown (see Appendix B) that the discrete Gabor expansion can be transformed into the *sum-of-products form*:

$$\tilde{\varphi}\left(n + sK, \frac{l}{MN}; pN\right) = \frac{1}{p}\sum_{r=<p>}\bar{a}\left(\frac{n}{K}, \frac{l + rM/p}{M}\right)\tilde{g}\left(n + sK, \frac{l + rM/p}{MN}; N\right). \tag{4.2}$$

Note that in Gabor's original case of critical sampling ($p = q = 1$, $K = N$), Eq. (4.2) takes the simple *product form*

$$\tilde{\varphi}\left(n, \frac{l}{MN}; N\right) = \bar{a}\left(\frac{n}{N}, \frac{l}{M}\right)\tilde{g}\left(n, \frac{l}{MN}; N\right).$$

We remark that, with $r = <p>$, the discrete Fourier transform $\bar{a}(n/K, l/M)$ is completely determined by the p functions

$$a_r(n, l) = \bar{a}\left(\frac{n}{K}, \frac{l + rM/p}{M}\right), \tag{4.3}$$

where the variable l extends over an interval of length M/p. Likewise, with $s = <q>$, the discrete Zak transform $\tilde{\varphi}(n, l/MN; pN)$ is completely determined by the q functions

$$\varphi_s(n, l) = \tilde{\varphi}\left(n + sK, \frac{l}{MN}; pN\right), \tag{4.4}$$

where the variable n extends over an interval of length K. Moreover, with $r = <p>$ and $s = <q>$, the discrete Zak transforms $\tilde{g}(n, l/MN; N)$ and $\tilde{w}(n, l/MN; N)$ are completely determined by the $q \times p$ functions

$$g_{sr} = \tilde{g}\left(n + sK, \frac{l + rM/p}{MN}; N\right) \tag{4.5}$$

and

$$w_{sr} = \tilde{w}\left(n + sK, \frac{l + rM/p}{MN}; N\right), \tag{4.6}$$

respectively.

Let us now, for convenience, choose $r = 0, 1, ..., p-1$; the p functions $a_r(n,l)$ can then be combined into a p-dimensional column vector of functions

$$\mathbf{a} = [a_0(n,l)\; a_1(n,l)\; ...\; a_{p-1}(n,l)]^T. \tag{4.7}$$

Likewise, with $s = 0, 1, ..., q-1$, the q functions $\varphi_s(n,l)$ can be combined into a q-dimensional column vector of functions

$$\phi = [\varphi_0(n,l)\; \varphi_1(n,l)\; ...\; \varphi_{q-1}(n,l)]^T. \tag{4.8}$$

Moreover, the $q \times p$ functions $g_{sr}(n,l)$ and $w_{sr}(n,l)$ can be combined into the $(q \times p)$-dimensional matrices of functions

$$\mathbf{G} = \begin{bmatrix} g_{00}(n,l) & g_{01}(n,l) & \cdots & g_{0,p-1}(n,l) \\ g_{10}(n,l) & g_{11}(n,l) & \cdots & g_{1,p-1}(n,l) \\ . & . & . & . \\ . & . & . & . \\ g_{q-1,0}(n,l) & g_{q-1,1}(n,l) & \cdots & g_{q-1,p-1}(n,l) \end{bmatrix} \tag{4.9}$$

and

$$\mathbf{W} = \begin{bmatrix} w_{00}(n,l) & w_{01}(n,l) & \cdots & w_{0,p-1}(n,l) \\ w_{10}(n,l) & w_{11}(n,l) & \cdots & w_{1,p-1}(n,l) \\ . & . & \cdots & . \\ . & . & \cdots & . \\ w_{q-1,0}(n,l) & w_{q-1,1}(n,l) & \cdots & w_{q-1,p-1}(n,l) \end{bmatrix}, \tag{4.10}$$

respectively. With the help of these vectors and matrices, Eqs. (4.1) and (4.2) can be expressed in the elegant matrix-vector products

$$\mathbf{a} = K\, \mathbf{W}^* \phi \tag{4.11}$$

and

$$\phi = \frac{1}{p} \mathbf{G}\mathbf{a}, \tag{4.12}$$

respectively, where, as usual, the asterisk in connection with vectors and matrices denotes complex conjugation *and* transposition. Note that Eq. (4.11) represents p equations in q unknowns, whereas Eq. (4.12) represents q equations in p unknowns. In the case of oversampling ($p > q \geq 1$) the latter set of equations is thus *underdetermined.*

5 Relationship between the analysis and the synthesis window

In this section we will prove that the discrete Gabor expansion (2.9) and the discrete Gabor transform (2.10) form a transform pair, by showing that for any analysis window $W(n)$ a synthesis window $G(n)$ can be constructed. Instead of doing this by directly combining Gabor's signal expansion and the Gabor transform, we will use the results (4.12) and (4.11) derived in the previous section.

If we substitute from Eq. (4.11) into Eq. (4.12) we get

$$\phi = \frac{K}{p} \mathbf{G}\mathbf{W}^* \phi,$$

which relation should hold for any arbitrary vector ϕ [i.e., for any arbitrary signal $\varphi(n)$]. This condition immediately leads to the relationship

$$\frac{K}{p} \mathbf{G}\mathbf{W}^* = \frac{K}{p} \mathbf{W}\mathbf{G}^* = \mathbf{I}_q, \tag{5.1}$$

where $\mathbf{I}_q$ is the $(q \times q)$-dimensional identity matrix.

We remark that in Gabor's original case of critical sampling ($p = q = 1$), the matrices **G** and **W** reduce to scalars g_{00} and w_{00}, respectively, and that Eq. (5.1) takes the simple product form

$$K w_{00} g_{00}^* = K \tilde{w} \left(n, \frac{l}{MN}; N \right) \tilde{g}^* \left(n, \frac{l}{MN}; N \right) = 1.$$

In this case of critical sampling, the discrete Zak transform of the synthesis window could thus be easily found, in principle, as the inverse of the discrete Zak transform of the analysis window, and the resulting window function would be *unique*. The possible occurrence of zeros in the discrete Zak transform of the analysis window, however, prohibits such an easy procedure. The problems caused by these zeros can be overcome by oversampling.

In the case of oversampling, the synthesis function that corresponds to a given analysis window is *not unique*. This is in accordance with the fact that in the case of oversampling the set of shifted and modulated versions of the synthesis window is overcomplete, and that Gabor's expansion coefficients are dependent and can no longer be considered as degrees of freedom. In the case of oversampling, the general condition (5.1) enables us to construct a synthesis window $G(n)$ for a given analysis window $W(n)$, by solving a set of $q \times q$ equations in $q \times p$ unknowns. Since $q < p$, this set of equations is again underdetermined.

Let us now consider Eq. (5.1) in the general case of oversampling. In that case we have $q < p$, which implies that **W** is not a square matrix and does not have a normal inverse $\mathbf{W}^{-1}$, and that Eq. (5.1) does not have a unique solution. It is well known that, under the condition that rank(**W**) = q, the *optimum solution* in the sense of the *minimum L_2 norm* can now be found with the help of the so-called *generalized (Moore-Penrose) inverse* [10] $\mathbf{W}^\dagger$, defined by

$$\mathbf{W}^\dagger = \mathbf{W}^* (\mathbf{W}\mathbf{W}^*)^{-1}; \tag{5.2}$$

note that $\mathbf{W}\mathbf{W}^\dagger = \mathbf{I}_q$ and that $\mathbf{W}^\dagger \mathbf{W}\mathbf{W}^* = \mathbf{W}^*$. The optimum solution $\mathbf{G}_{opt}$ then reads

$$\mathbf{G}_{opt} = \frac{p}{K} (\mathbf{W}^\dagger)^*. \tag{5.3}$$

Of course, if we proceed in this way, we will find, for any n and l, the minimum L_2 norm solution for the matrix **G**. It is not difficult to show, however, that the minimum L_2 norm of **G** corresponds to the minimum L_2 norm of the discrete Zak transform $\tilde{g}(n, l/MN; N)$, and thus, with the help of Parseval's energy theorem, to the minimum L_2 norm of the synthesis window $G(n)$.

The relationship (5.3) between the optimum synthesis window $G(n)$ and the analysis window $W(n)$ shows some resemblance with [19, Eqs. (35) and (83)], which are also derived using the Zak transform. Although the latter equations are derived via sums of products of Zak transforms, as well, the role of the Zak transform in the present paper is much more prominent: it is applied directly to both Gabor's expansion and the Gabor transform, to represent them in sum-of-products forms. And whereas [19, Eqs. (35) and (83)] yield only the *optimum* window function, the present approach enables us to impose *arbitrary* additional constraints on the solutions of the underdetermined sets of equations that have to be solved.

Instead of looking for the optimum solution $\mathbf{G}_{opt}$ in the sense of the minimum L_2 norm of **G**, we could as well look for the optimum solution $\mathbf{G}_F$ in the sense of the minimum L_2 norm of the difference $\mathbf{G} - \mathbf{F}$; in this way we would find the matrix **G** that resembles best the matrix **F**. To find $\mathbf{G}_F$ we proceed as follows

$$\frac{K}{p} \mathbf{W}\mathbf{G}^* = \mathbf{I}_q,$$

$$\frac{K}{p} \mathbf{W}(\mathbf{G} - \mathbf{F})^* = \mathbf{I}_q - \frac{K}{p} \mathbf{W}\mathbf{F}^*,$$

$$\frac{K}{p}(\mathbf{G}_F - \mathbf{F})^* = \mathbf{W}^\dagger \left(\mathbf{I}_q - \frac{K}{p}\mathbf{W}\mathbf{F}^* \right),$$
$$\frac{K}{p}\mathbf{G}_F^* = \mathbf{W}^\dagger + \frac{K}{p}(\mathbf{I}_p - \mathbf{W}^\dagger\mathbf{W})\mathbf{F}^*,$$

and hence

$$\mathbf{G}_F^* = \mathbf{G}_{opt}^* + (\mathbf{I}_p - \mathbf{W}^\dagger\mathbf{W})\mathbf{F}^*, \tag{5.4}$$

where $\mathbf{I}_p$ is the $(p \times p)$-dimensional identity matrix.

An obvious choice for the matrix $\mathbf{F}$ would be a matrix that is proportional to the matrix $\mathbf{W}$. From Eq. (5.4) we then have

$$\mathbf{G}_W^* = \mathbf{G}_{opt}^* + (\mathbf{I}_p - \mathbf{W}^\dagger\mathbf{W})\mathbf{W}^*,$$

but the second term in the right-hand side of this relationship vanishes, due to the fact that

$$(\mathbf{I}_p - \mathbf{W}^\dagger\mathbf{W})\mathbf{W}^* = \mathbf{W}^* - \mathbf{W}^\dagger\mathbf{W}\mathbf{W}^* = \mathbf{W}^* - \mathbf{W}^* = \mathbf{0}.$$

We thus reach the important conclusion that $\mathbf{G}_W = \mathbf{G}_{opt}$; hence, the synthesis window $G_{opt}(n)$ that has the minimum L_2 norm is the same as the synthesis window $G_W(n)$ whose difference from the analysis window $w(n)$ has the minimum L_2 norm, and resembles best this analysis window.

Let us, as an example, choose a Gaussian analysis window $w(n)$ that is symmetrical around the point $\frac{1}{2}(N-1)$:

$$w(n) = e^{-(\pi/pN^2)(n - \frac{1}{2}[N-1])^2}. \tag{5.5}$$

The discrete Zak transform $\tilde{w}(n, l/MN; N)$ of this function reads

$$\tilde{w}(n, l/MN; N) = e^{-(\pi/pN^2)(n - \frac{1}{2}[N-1])^2} \theta_3(z; e^{-\pi/p}), \tag{5.6}$$

where $\theta_3(z; e^{-\pi/p})$ is a *theta function* [1, 18] with *nome* $e^{-\pi/p}$ and where we have set

$$z = \pi\frac{l}{M} - j\pi\frac{1}{p}\left(\frac{n - \frac{1}{2}[N-1]}{N}\right).$$

We remark that the function $\theta_3(z; e^{-\pi/p})$ has zeros for $z = \pi(k + \frac{1}{2}) - j\pi(m + \frac{1}{2})/p$; hence, although a zero will not be reached for integer values of n, the value of $\tilde{w}(n, l/MN; N)$ will be very small for l in the neighbourhood of $(k + \frac{1}{2})M$ and n in the neighbourhood of $mN - \frac{1}{2}$. In Fig. 1 we have depicted the discrete Zak transform $\tilde{w}(n, l/MN; N)$ for several values of the parameter p, with $N = M = 24$.

The width of the Gaussian window (5.5) is, roughly, $N\sqrt{p}$. It will be clear that when we truncate the window function to an interval of length N_w where N_w is much larger than $N\sqrt{p}$, the discrete Zak transform of this truncated window function will almost be equal to the one of the untruncated window function.

We now apply the techniques outlined in this paper to determine the synthesis window $g_{opt}(n)$ that corresponds to a truncated Gaussian analysis window $w(n)$ [cf. Eq. (5.5)]. For convenience, we restrict ourselves to integer oversampling: $K/N = p \geq q = 1$. Moreover, we choose $N = M = 24$. In Fig. 2 we have depicted the discrete Zak transforms $\tilde{g}_{opt}(n, l/MN; N)$ of the optimum synthesis windows $g_{opt}(n)$ for different values of the oversampling parameter p, while in Fig. 3 we have depicted these window functions themselves. We remark that the resemblance between the synthesis window and the analysis window increases with increasing value of the oversampling parameter p.

6 Conclusion

We have studied Gabor's expansion of a discrete-time signal into a set of shifted and modulated versions of an elementary signal (or synthesis window). We have also considered the inverse operation – the Gabor transform – with which Gabor's expansion coefficients can be determined. In particular we have considered the discrete Gabor transform, in which the analysis window and the signal must have a finite support. And we have shown how, with the help of an overlap-add technique, the discrete Gabor transform can be used to determine Gabor's expansion coefficients for a signal whose support is not finite.

We have introduced the discrete Zak transform, and we have shown how this transform, together with the discrete Fourier transform, can be used to represent the discrete Gabor transform and the discrete Gabor expansion in mathematically more attractive sum-of-products forms. The sum-of-products form of the discrete Gabor transform enables us to determine Gabor's expansion coefficients in a different way, in which fast algorithms can be applied. This way of determining the expansion coefficients resembles the well-known procedure in which a convolution is transformed into product form by means of a Fourier transformation and which allows the determination of the convolution product by performing a normal product in the frequency domain; the use of a *fast Fourier transform* algorithm would then lead to an algorithm known as the *fast convolution*. The analogous procedure to determine Gabor's expansion coefficients might thus be called the *fast Gabor transform*.

Using the sum-of-products forms of the discrete Gabor transform and the discrete Gabor expansion enabled us to formulate a relationship between the analysis window and the synthesis window. In the general case of oversampling, this relationship leads to a set of equations that is underdetermined, which implies that the synthesis window that corresponds to a given analysis window is not unique. We have shown an easy way determine the optimum synthesis window in the sense that it has minimum L_2 norm, and we have shown that this optimum synthesis window resembles best (in the sense of minimum L_2 norm, again) the analysis window.

Appendix A. Derivation of Eq. (4.1)

In the discrete Fourier transform (3.1) of the array A_{mk}, we substitute from the Gabor transform (2.6)

$$\bar{a}\left(\frac{n}{K},\frac{l}{M}\right)=\sum_{m=<M>}\sum_{k=<K>}\left[\sum_{n'}\varphi(n')W^*(n'-mN)e^{-j2\pi kn'/K}\right]e^{-j2\pi(ml/M-kn/K)}$$

and rearrange factors

$$\bar{a}\left(\frac{n}{K},\frac{l}{M}\right)=\sum_{m=<M>}\left[\sum_{n'}\varphi(n')W^*(n'-mN)\left\{\sum_{k=<K>}e^{-j2\pi k(n'-n)/K}\right\}\right]e^{-j2\pi ml/M}.$$

We replace the sum of exponentials by a sum of Kronecker deltas

$$\bar{a}\left(\frac{n}{K},\frac{l}{M}\right)=\sum_{m=<M>}\left[\sum_{n'}\varphi(n')W^*(n'-mN)\left\{K\sum_{k}\delta[n'-n-kK]\right\}\right]e^{-j2\pi ml/M}$$

and rearrange factors again

$$\bar{a}\left(\frac{n}{K},\frac{l}{M}\right)=K\sum_{m=<M>}\left[\sum_{k}\sum_{n'}\varphi(n')W^*(n'-mN)\delta[n'-n-kK]\right]e^{-j2\pi ml/M}.$$

We evaluate the summation over n'

$$\bar{a}\left(\frac{n}{K},\frac{l}{M}\right)=K\sum_{m=<M>}\left[\sum_{k}\varphi(n+kK)W^*(n+kK-mN)\right]e^{-j2\pi ml/M}$$

and rearrange factors again

$$\bar{a}\left(\frac{n}{K},\frac{l}{M}\right) = K\sum_k \varphi(n+kK)e^{-j2\pi kK(l/MN)}$$

$$\times\left[\sum_{m=<M>} W^*\left(n-\left[m-k\frac{K}{N}\right]N\right)e^{-j2\pi(m-kK/N)N(l/MN)}\right].$$

We replace K/N by p/q

$$\bar{a}\left(\frac{n}{K},\frac{l}{M}\right) = K\sum_k \varphi(n+kK)e^{-j2\pi kK(l/MN)}$$

$$\times\left[\sum_{m=<M>} W\left(n-\left[m-k\frac{p}{q}\right]N\right)e^{j2\pi(m-kp/q)N(l/MN)}\right]^*$$

and replace the summation over k by a double summation over s and n' through the substitution $k=n'q+s$, where s extends over an interval of length q

$$\bar{a}\left(\frac{n}{K},\frac{l}{M}\right) = K\sum_{s=<q>}\sum_{n'} \varphi(n+sK+n'qK)e^{-j2\pi(n'q+s)K(l/MN)}$$

$$\times\left[\sum_{m=<M>} W\left(n-\left[m-n'p-s\frac{p}{q}\right]N\right)e^{j2\pi(m-n'p-sp/q)N(l/MN)}\right]^*.$$

We substitute $m-n'p$ by $-k$

$$\bar{a}\left(\frac{n}{K},\frac{l}{M}\right) = K\sum_{s=<q>}\sum_{n'} \varphi(n+sK+n'qK)e^{-j2\pi n'qK(l/MN)}$$

$$\times\left[\sum_{k=<M>} W(n+sK+kN)e^{-j2\pi kN(l/MN)}\right]^*$$

and recognize the definitions (3.3) and (3.9) for the discrete Zak transforms $\tilde{\varphi}(n+sK,l/MN;\,pN)$ and $\tilde{w}(n+sK,l/MN;\,N)$ of the signal $\varphi(n)$ and the window function $w(n)$, respectively, leading to

$$\bar{a}\left(\frac{n}{K},\frac{l}{M}\right) = K\sum_{s=<q>}\tilde{\varphi}\left(n+sK,\frac{l}{MN};\,pN\right)\tilde{w}^*\left(n+sK,\frac{l}{MN};\,N\right).$$

We finally replace l by $l+rM/p$ and use the periodicity property of the discrete Zak transform $\tilde{\varphi}(n+sK,l/MN;\,pN)$, which leads to the result

$$\bar{a}\left(\frac{n}{K},\frac{l+rM/p}{M}\right) = K\sum_{s=<q>}\tilde{\varphi}\left(n+sK,\frac{l}{MN};\,pN\right)\tilde{w}^*\left(n+sK,\frac{l+rM/p}{MN};\,N\right).$$

Appendix B. Derivation of Eq. (4.2)

In the discrete Gabor expansion (2.9) we substitute from the inverse discrete Fourier transform (3.2)

$$\Phi(n) = \sum_{m=<M>}\sum_{k=<K>}\left[\frac{1}{MK}\sum_{n'=<K>}\sum_{l=<M>}\bar{a}\left(\frac{n'}{K},\frac{l}{M}\right)e^{j2\pi(ml/M-kn'/K)}\right]G(n-mN)e^{j2\pi kn/K}$$

and rearrange factors

$$\Phi(n) = \frac{1}{MK} \sum_{n'=<K>} \sum_{l=<M>} \bar{a}\left(\frac{n'}{K}, \frac{l}{M}\right) \left[\sum_{m=<M>} G(n - mN) e^{j2\pi mN(l/MN)} \right]$$
$$\times \left[\sum_{k=<K>} e^{-j2\pi k(n'-n)/K)} \right].$$

We replace the sum of exponentials by a sum of Kronecker deltas and recognize the discrete Zak transform of the elementary signal $g(n)$ [cf. Eq. (3.3)]:

$$\Phi(n) = \frac{1}{MK} \sum_{n'=<K>} \sum_{l=<M>} \bar{a}\left(\frac{n'}{K}, \frac{l}{M}\right) \tilde{g}\left(n, \frac{l}{MN}; N\right) \left[K \sum_{k} \delta[n' - n - kK] \right].$$

We rearrange factors again and substitute from the periodicity property of the discrete Fourier transform $\bar{a}(n'/K, l/M)$

$$\Phi(n) = \frac{1}{M} \sum_{l=<M>} \left[\sum_{k} \sum_{n'=<K>} \bar{a}\left(\frac{n' - kK}{K}, \frac{l}{M}\right) \delta[n' - kK - n] \right] \tilde{g}\left(n, \frac{l}{MN}; N\right)$$

and we replace the summation over k together with the summation over the finite n'-interval by a summation over all n'

$$\Phi(n) = \frac{1}{M} \sum_{l=<M>} \left[\sum_{n'} \bar{a}\left(\frac{n'}{K}, \frac{l}{M}\right) \delta[n' - n] \right] \tilde{g}\left(n, \frac{l}{MN}; N\right).$$

Evaluation of the resulting summation over n' yields the intermediate result

$$\Phi(n) = \frac{1}{M} \sum_{l=<M>} \bar{a}\left(\frac{n}{K}, \frac{l}{M}\right) \tilde{g}\left(n, \frac{l}{MN}; N\right).$$

We now write down the definition of the discrete Zak transform (3.8)

$$\tilde{\varphi}\left(n, \frac{l}{MN}; pN\right) = \sum_{m=<M/p>} \Phi(n + mpN) e^{-j2\pi mpN(l/MN)}$$

and substitute from the intermediate result above

$$\tilde{\varphi}\left(n, \frac{l}{MN}; pN\right) =$$
$$= \sum_{m=<M/p>} \left[\frac{1}{M} \sum_{l'=<M>} \bar{a}\left(\frac{n + mpN}{K}, \frac{l'}{M}\right) \tilde{g}\left(n + mpN, \frac{l'}{MN}; N\right) \right] e^{-j2\pi mpN(l/MN)}.$$

We rearrange things, using the relation $pN = qK$,

$$\tilde{\varphi}\left(n, \frac{l}{MN}; pN\right) =$$
$$= \sum_{m=<M/p>} \left[\frac{1}{M} \sum_{l'=<M>} \bar{a}\left(\frac{n + mqK}{K}, \frac{l'}{M}\right) \tilde{g}\left(n + mpN, \frac{l'}{MN}; N\right) \right] e^{-j2\pi mpN(l/MN)}$$

and use the periodicity property of the discrete Fourier transform $\bar{a}(n/K, l'/M)$ and the quasi-periodicity property of the discrete Zak transform $\tilde{g}(n, l'/MN;\, N)$

$$\tilde{\varphi}\left(n, \frac{l}{MN};\, pN\right) =$$

$$= \sum_{m=<M/p>} \left[\frac{1}{M} \sum_{l'=<M>} \bar{a}\left(\frac{n}{K}, \frac{l'}{M}\right) \tilde{g}\left(n, \frac{l'}{MN};\, N\right) e^{j2\pi mpN(l'/MN)} \right] e^{-j2\pi mpN(l/MN)}.$$

We rearrange factors

$$\tilde{\varphi}\left(n, \frac{l}{MN};\, pN\right) = \frac{1}{M} \sum_{l'=<M>} \bar{a}\left(\frac{n}{K}, \frac{l'}{M}\right) \tilde{g}\left(n, \frac{l'}{MN};\, N\right) \left[\sum_{m=<M/p>} e^{j2\pi m(l'-l)(p/M)} \right]$$

and replace the sum of exponentials by a sum of Kronecker deltas

$$\tilde{\varphi}\left(n, \frac{l}{MN};\, pN\right) = \frac{1}{M} \sum_{l'=<M>} \bar{a}\left(\frac{n}{K}, \frac{l'}{M}\right) \tilde{g}\left(n, \frac{l'}{MN};\, N\right) \left[\frac{M}{p} \sum_{m} \delta\left[l' - l - m\frac{M}{p}\right] \right].$$

We replace the summation over m by a double summation over r and k through the substitution $m = kp + r$, where r extends over an interval of length p

$$\tilde{\varphi}\left(n, \frac{l}{MN};\, pN\right) = \sum_{l'=<M>} \bar{a}\left(\frac{n}{K}, \frac{l'}{M}\right) \tilde{g}\left(n, \frac{l'}{MN};\, N\right) \left[\frac{1}{p} \sum_{k} \sum_{r=<p>} \delta\left[l' - l - kM - r\frac{M}{p}\right] \right]$$

and rearrange factors

$$\tilde{\varphi}\left(n, \frac{l}{MN};\, pN\right) = \frac{1}{p} \sum_{r=<p>} \left[\sum_{k} \sum_{l'=<M>} \bar{a}\left(\frac{n}{K}, \frac{l'}{M}\right) \tilde{g}\left(n, \frac{l'}{MN};\, N\right) \delta\left[l' - kM - l - r\frac{M}{p}\right] \right].$$

We use the periodicity of the discrete Fourier transform $\bar{a}(n/K, l'/M)$ and the discrete Zak transform $\tilde{g}(n, l'/MN;\, N)$

$$\tilde{\varphi}\left(n, \frac{l}{MN};\, pN\right) = \frac{1}{p} \sum_{r=<p>} \left[\sum_{k} \sum_{l'=<M>} \bar{a}\left(\frac{n}{K}, \frac{l'-kM}{M}\right) \tilde{g}\left(n, \frac{l'-kM}{MN};\, N\right) \delta\left[l' - kM - l - r\frac{M}{p}\right] \right]$$

and replace the summation over k together with the summation over the finite l'-interval by a summation over all l'

$$\tilde{\varphi}\left(n, \frac{l}{MN};\, pN\right) = \frac{1}{p} \sum_{r=<p>} \left[\sum_{l'} \bar{a}\left(\frac{n}{K}, \frac{l'}{M}\right) \tilde{g}\left(n, \frac{l'}{MN};\, N\right) \delta\left[l' - l - r\frac{M}{p}\right] \right].$$

Evaluation of the summation over l' results in

$$\tilde{\varphi}\left(n, \frac{l}{MN};\, pN\right) = \frac{1}{p} \sum_{r=<p>} \bar{a}\left(\frac{n}{K}, \frac{l + rM/p}{M}\right) \tilde{g}\left(n, \frac{l + rM/p}{MN};\, N\right).$$

We finally replace n by $n + sK$ and use the periodicity property of the discrete Fourier transform $\bar{a}(n/K, l/M)$, which yields the result

$$\tilde{\varphi}\left(n + sK, \frac{l}{MN};\, pN\right) = \frac{1}{p} \sum_{r=<p>} \bar{a}\left(\frac{n}{K}, \frac{l + rM/p}{M}\right) \tilde{g}\left(n + sK, \frac{l + rM/p}{MN};\, N\right).$$

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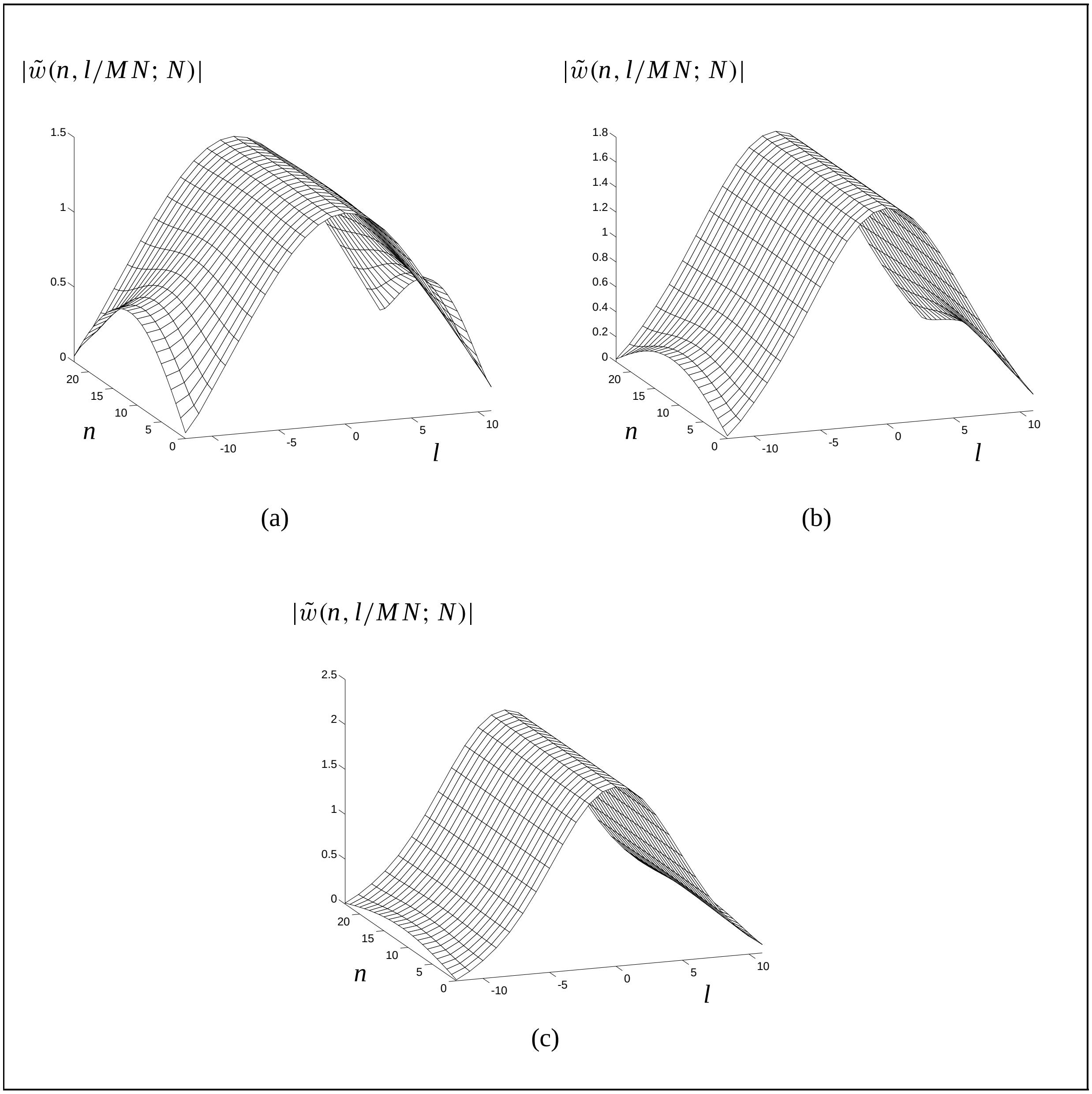

Figure 1: The discrete Zak transform $\tilde{w}(n, l/MN;\, N)$ in the case of a Gaussian window function $w(n) = e^{-(\pi/pN)^2(n-\frac{1}{2}[N-1])^2}$ (with $N = M = 24$) for different values of p: (a) $p = 2$, (b) $p = 3$, (c) $p = 4$.

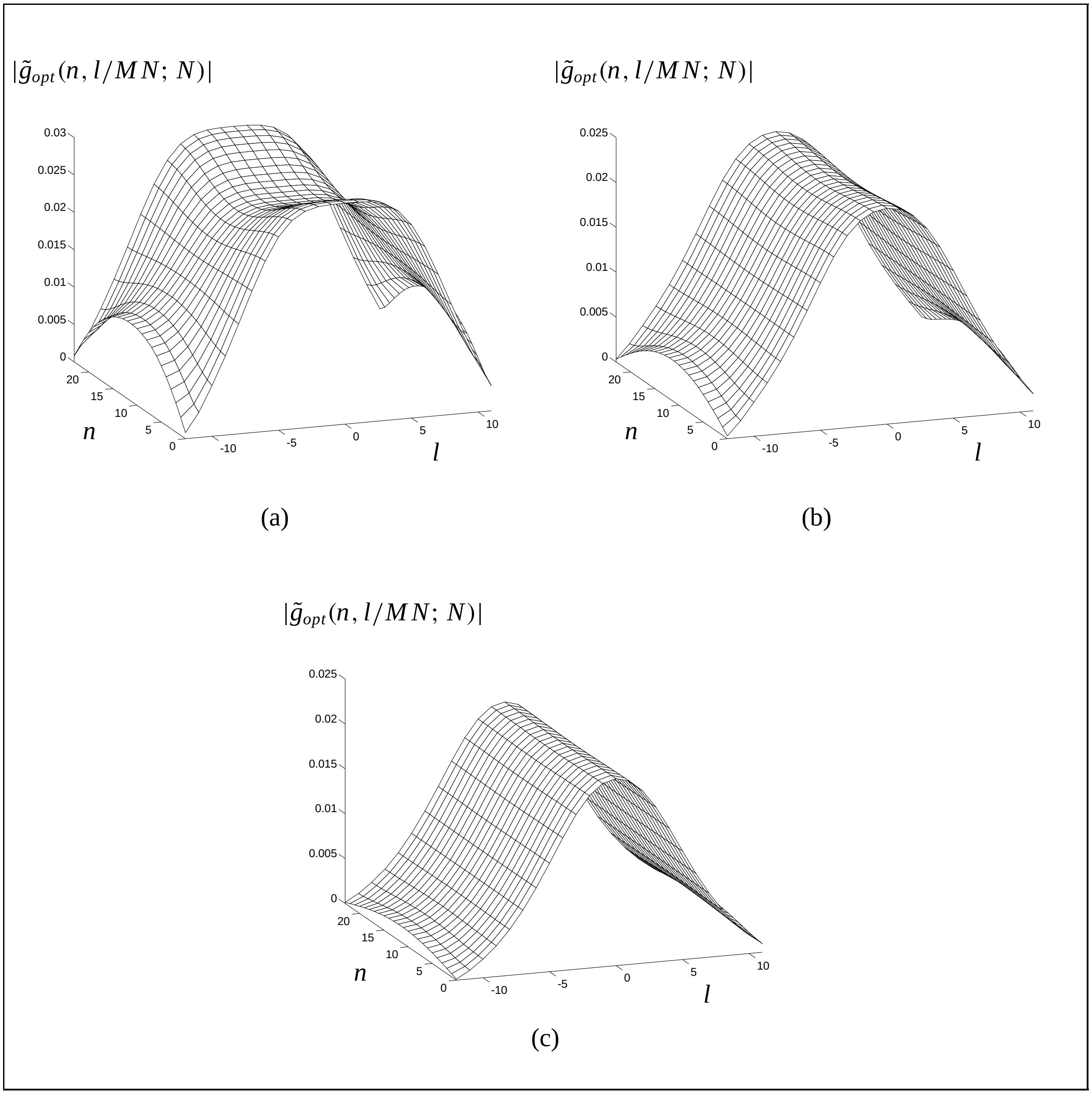

Figure 2: The discrete Zak transform $\tilde{g}_{opt}(n, l/MN; N)$ in the case of a Gaussian window function $w(n) = e^{-(\pi/pN)^2(n-\frac{1}{2}[N-1])^2}$ (with $N = M = 24$) for different values of oversampling $p = K/N$: (a) $p = 2$, (b) $p = 3$, (c) $p = 4$.

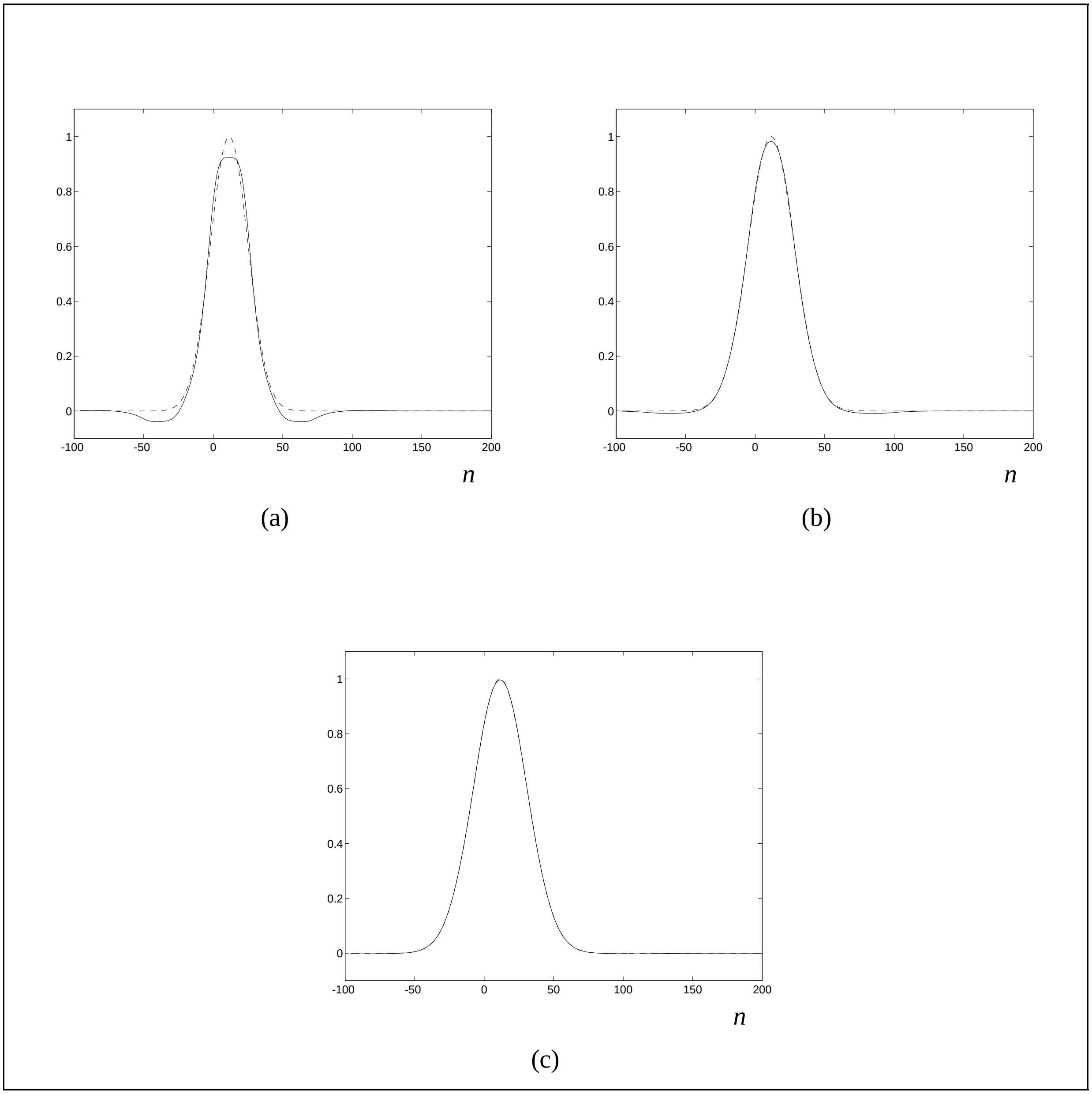

Figure 3: A Gaussian analysis window $w(n) = e^{-(\pi/pN)^2(n-\frac{1}{2}[N-1])^2}$ (with $N = 24$, dashed line) and its corresponding optimum synthesis window $g_{opt}(n)$ (solid line) for different values of oversampling $p = K/N$: (a) $p = 2$, (b) $p = 3$, (c) $p = 4$.