Gabor's signal expansion based on a non-orthogonal sampling geometry

Martin J. Bastiaans

Technische Universiteit Eindhoven, Faculteit Elektrotechniek
Postbus 513, 5600 MB Eindhoven, Netherlands
tel: +31 40 247 3319 – fax: +31 40 2448375
email: M.J.Bastiaans@ele.tue.nl

ABSTRACT

Gabor's signal expansion and the Gabor transform are formulated on a non-orthogonal time-frequency lattice instead of on the traditional rectangular lattice. The reason for doing so is that a non-orthogonal sampling geometry might be better adapted to the form of the window functions (in the time-frequency domain) than an orthogonal one: the set of shifted and modulated versions of the usual Gaussian synthesis window, for instance, corresponding to circular contour lines in the time-frequency domain, can be arranged more tightly in a hexagonal geometry than in a rectangular one. Oversampling in the Gabor scheme, which is required to have mathematically more attractive properties for the analysis window, then leads to better results in combination with less oversampling. The procedure presented in this paper is based on considering the non-orthogonal lattice as a sub-lattice of a denser orthogonal lattice that is oversampled by a rational factor. In doing so, Gabor's signal expansion on a non-orthogonal lattice can be related to the expansion on an orthogonal lattice (restricting ourselves, of course, to only those sampling points that are part of the non-orthogonal sub-lattice), and all the techniques that have been derived for rectangular sampling – including an optical means of generating Gabor's expansion coefficients via the Zak transform in the case of integer oversampling – can be used, albeit in a slightly modified form.

Keywords: Gabor's signal expansion, Gabor transform, Zak transform, time-frequency signal analysis

1. INTRODUCTION

In 1946,[1] Gabor suggested the representation of a time signal in a combined time-frequency domain; in particular he proposed to represent the signal as a superposition of shifted and modulated versions of a so-called elementary signal or synthesis window $g(t)$. Moreover, as a synthesis window $g(t)$ he chose a Gaussian signal,

$$g(t) = 2^{1/4}\, e^{-\pi(t/\sigma_t)^2} \tag{1}$$

with Fourier transform

$$\bar{g}(\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt = 2^{1/4}\, \sigma_t\, e^{-\pi(\omega/\sigma_\omega)^2} \qquad (\sigma_t \sigma_\omega = 2\pi), \tag{2}$$

because such a signal has a good localization, both in the time domain and in the frequency domain. The other choices that Gabor made were (i) that his signal expansion was formulated on a rectangular lattice in the time-frequency domain, $(mT, k\Omega)$, (ii) that the sampling distances T and Ω satisfied the relation $\Omega T = 2\pi$, and (iii) that they were in like manners proportional to the widths σ_t and σ_ω of the synthesis window and its Fourier transform, respectively: $T/\sigma_t = \Omega/\sigma_\omega$.

The coefficients in Gabor's signal expansion can be determined by using an analysis window $w(t)$. In the case of critical sampling, i.e., $\Omega T = 2\pi$, the analysis window $w(t)$ follows uniquely from the given synthesis window $g(t)$. However, such a unique analysis window appears to have some mathematically very unattractive properties. For this reason, the expansion should be formulated on a denser lattice, $\Omega T < 2\pi$. This makes the analysis window no longer unique and thus allows for finding an analysis window that is optimal in some way. We can, for instance, look for the 'optimum' analysis window that resembles best the synthesis window; a better resemblance can then be reached for a higher degree of oversampling. Some examples of optimum analysis windows $w(t)$ that correspond to a Gaussian synthesis window $g(t)$ for different degrees of oversampling have been presented in Fig. 1.

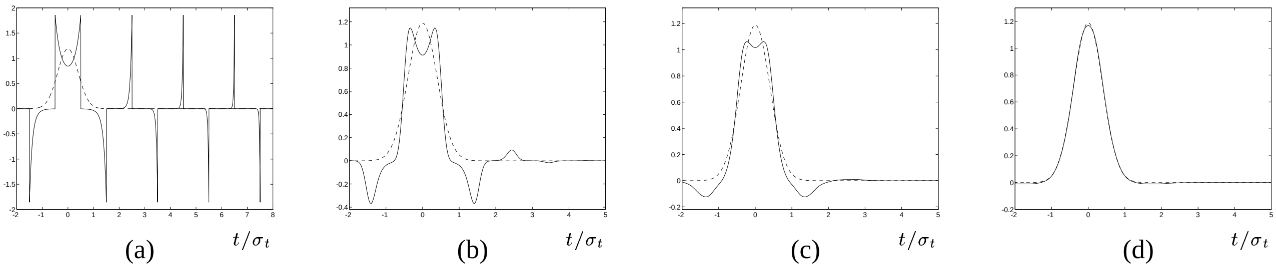


Figure 1. A Gaussian synthesis window $g(t) = 2^{1/4}\exp[-\pi(t/\sigma_t)^2]$ (dashed line) and its corresponding optimum analysis window $(T/q)w(t)$ (solid line), for different values of rational oversampling $2\pi/\Omega T = p/q$, while maintaining the proportionality condition $\sigma_t/T = \sigma_\omega/\Omega = \sqrt{2\pi/\Omega T}$: (a) no oversampling $2\pi/\Omega T = 1$, (b) $2\pi/\Omega T = 7/6$, (c) $2\pi/\Omega T = 3/2$, and (d) $2\pi/\Omega T = 3$.

A better resemblance can also be reached if we adapt the structure of the lattice to the form of the window as it is represented in the time-frequency domain. The time-frequency representation of a Gaussian window, for instance, for which its Wigner distribution[2,3] reads

$$F_g(t,\omega) = \int_{-\infty}^{\infty} g(t+\tfrac{1}{2}t')g^*(t-\tfrac{1}{2}t')e^{-j\omega t'}dt' = 2\,\sigma_t\,e^{-2\pi[(t/\sigma_t)^2+(\omega/\sigma_\omega)^2]}, \tag{3}$$

has circular contour lines in the time-frequency domain; and it is well known that circles are better packed on a hexagonal lattice than on a rectangular lattice, see Fig. 2. Gabor's signal expansion on such a hexagonal, non-orthogonal lattice then leads to a better resemblance between the window functions $g(t)$ and $w(t)$ than the expansion on a rectangular, orthogonal lattice does.

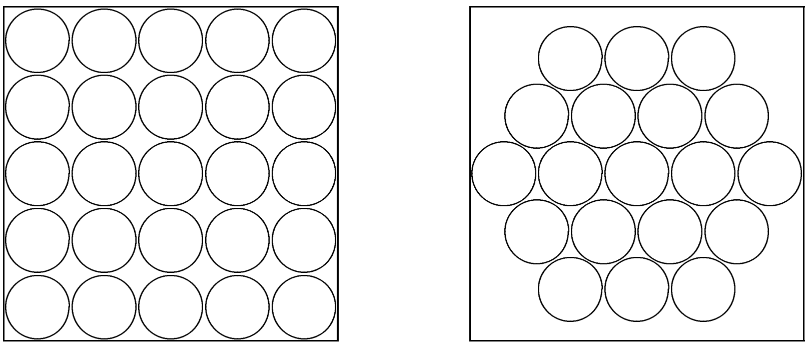


Figure 2. A rectangular and a hexagonal packing of circles, with filling factor $\pi/4 = 0.7854$ and $\pi/2\sqrt{3} = 0.9069$, respectively.

In this paper we consider the case of a non-orthogonal sampling geometry. After a short introduction of Gabor's signal expansion and the Gabor transform on a rectangular lattice, we will introduce the Fourier transform of the array of expansion coefficients and the Zak transforms of the signal and the window functions to formulate the Gabor expansion and the Gabor transform in product forms.[4,5] These product forms hold in the case of critical sampling on a rectangular lattice ($2\pi/\Omega T = 1$), but – in a modified form[6,7] – product forms can also be formulated in the case of rational oversampling ($2\pi/\Omega T = p/q > 1$); the use of an orthogonal lattice, however, is crucial!

We will then introduce a non-orthogonal sampling geometry, and we will consider the non-orthogonal sampling lattice as a sub-lattice of a denser lattice that is orthogonal. In doing so, Gabor's signal expansion on a non-orthogonal lattice can be related to the expansion on an orthogonal lattice, and all the techniques that have been developed for rectangular sampling – not only for continuous-time signals as described here, but for discrete-time signals, as well – can be used,[5–8] albeit in a slightly modified form.

2. GABOR'S SIGNAL EXPANSION ON A RECTANGULAR LATTICE

We start with the usual Gabor expansion[1,4–6,8] on a rectangular time-frequency lattice, in which case a signal $\varphi(t)$ can be expressed as a linear combination of properly shifted and modulated versions $g_{mk}(t) = g(t - mT)\exp(jk\Omega t)$ of a synthesis window $g(t)$:

$$\varphi(t) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_{mk} g_{mk}(t) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_{mk} g(t - mT) e^{jk\Omega t}. \tag{4}$$

The time step T and the frequency step Ω satisfy the relationship $\Omega T \leq 2\pi$; note that the factor $2\pi/\Omega T$ represents the degree of oversampling, and that in his original paper[1] Gabor considered the case of critical sampling, i.e. $\Omega T = 2\pi$. The expansion coefficients a_{mk} follow from sampling the windowed Fourier transform with analysis window $w(t)$, $\int_{-\infty}^{\infty} \varphi(t) w^*(t - \tau) \exp(-j\omega t) dt$, on the rectangular lattice $(\tau = mT, \omega = k\Omega)$:

$$a_{mk} = \int_{-\infty}^{\infty} \varphi(t) w_{mk}^*(t) dt = \int_{-\infty}^{\infty} \varphi(t) w^*(t - mT) e^{-jk\Omega t} dt. \tag{5}$$

This relationship is known as the Gabor transform.

The synthesis window $g(t)$ and the analysis window $w(t)$ are related to each other in such a way that their shifted and modulated versions constitute two sets of functions that are biorthogonal:

$$\sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} g_{mk}(t_1) w_{mk}^*(t_2) = \delta(t_1 - t_2), \tag{6}$$

with $\delta(t)$ the Dirac delta function. If the biorthogonality condition (6) is satisfied, the Gabor transform (5) and Gabor's signal expansion (4) form a transform pair in the following sense: if we start with an arbitrary signal $\varphi(t)$ and determine its expansion coefficients a_{mk} via the Gabor transform (5), the signal can be reconstructed via the Gabor expansion (4).

The biorthogonality relation (6) leads immediately to the equivalent but simpler expression

$$\frac{2\pi}{\Omega} \sum_{m=-\infty}^{\infty} g(t - mT) w^* \left(t - \left[mT + n\frac{2\pi}{\Omega} \right] \right) = \delta_n, \tag{7}$$

where δ_n is the Kronecker delta. In the case of critical sampling, i.e., $2\pi/\Omega = T$, the biorthogonality relation (7) reduces to

$$T \sum_{m=-\infty}^{\infty} g(t - mT) w^*(t - [m + n]T) = \delta_n \tag{8}$$

and the analysis window $w(t)$ follows uniquely from a given synthesis window $g(t)$, or vice versa. An elegant way to find the analysis window if the synthesis window is given, is presented in the next section.

3. FOURIER TRANSFORM AND ZAK TRANSFORM

It is well known,[4–6,8] that in the case of critical sampling, $\Omega T = 2\pi$, Gabor's signal expansion (4) and the Gabor transform (5) can be transformed into product form. We therefore need the Fourier transform $\bar{a}(\xi, \eta)$ of the two-dimensional array of Gabor coefficients a_{mk}, defined by

$$\bar{a}(\xi, \eta) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_{mk} e^{-j2\pi(m\eta - k\xi)}, \tag{9}$$

and the Zak transforms $\tilde{\varphi}(\xi T, \eta 2\pi/T; T)$, $\tilde{g}(\xi T, \eta 2\pi/T; T)$, and $\tilde{w}(\xi T, \eta 2\pi/T; T)$ of the signal $\varphi(t)$ and the window functions $g(t)$ and $w(t)$, respectively, where the Zak transform $\tilde{f}(t, \omega; \tau)$ of a function $f(t)$ is defined as,[5,6,8–11]

$$\tilde{f}(t, \omega; \tau) = \sum_{n=-\infty}^{\infty} f(t + n\tau) e^{-jn\tau\omega}. \tag{10}$$

Note that the Fourier transform $\bar{a}(\xi,\eta)$ is periodic in ξ and η with period 1, and that the Zak transform $\tilde{f}(t,\omega;\tau)$ is periodic in ω with period $2\pi/\tau$ and quasi-periodic in t with period τ: $\tilde{f}(t+m\tau,\omega+k2\pi/\tau;\tau)=\tilde{f}(t,\omega;\tau)\exp(jm\tau\omega)$.

Upon substituting from the Fourier transform (9) and the Zak transforms [cf. Eq. (10)] into Eqs. (4) and (5), it is not too difficult to show that Gabor's signal expansion (4) can be transformed into the product form

$$\tilde{\varphi}\left(\xi T,\eta\frac{2\pi}{T};T\right)=\bar{a}(\xi,\eta)\tilde{g}\left(\xi T,\eta\frac{2\pi}{T};T\right), \tag{11}$$

while the Gabor transform (5) can be transformed into the product form

$$\bar{a}(\xi,\eta)=T\tilde{\varphi}\left(\xi T,\eta\frac{2\pi}{T};T\right)\tilde{w}^*\left(\xi T,\eta\frac{2\pi}{T};T\right). \tag{12}$$

In particular the product form (12) is useful for determining Gabor's expansion coefficients. Since a Zak transform is merely a Fourier transform [cf. Eq. (10)], the expansion coefficients can be determined by Fourier transformations and multiplications; and if things are formulated for discrete-time signals, we can use the *fast* Fourier transform to formulate a fast algorithm for the Gabor transform.[6,7]

The relationship between the Zak transforms of the analysis window $w(t)$ and the synthesis window $g(t)$ now follows from substituting from Eq. (12) into Eq. (11) and reads

$$T\tilde{g}\left(\xi T,\eta\frac{2\pi}{T};T\right)\tilde{w}^*\left(\xi T,\eta\frac{2\pi}{T};T\right)=1. \tag{13}$$

From the latter relationship we conclude that (the Zak transform of) the analysis window $w(t)$ follows uniquely from (the Zak transform of) the given synthesis window $g(t)$; see Fig. 1a for the analysis window that corresponds to the Gaussian synthesis window and Fig. 3a for the Zak transform of that Gaussian window. Unfortunately, in general, the unique analysis window $w(t)$ has some very unattractive mathematical properties. We are therefore urged to consider Gabor's signal expansion on a denser lattice, in which case the analysis window is no longer unique. This enables us to choose an analysis window that is better suited to our purpose of determining Gabor's expansion coefficients.

4. RATIONAL OVERSAMPLING

In the case of oversampling by a rational factor, $2\pi/\Omega T=p/q\geq 1$, with p and q relatively prime, positive integers, $p>q\geq 1$, Gabor's expansion (4) and the Gabor transform (5) can be transformed (see Appendices A and B) into the sum-of-products forms,[6,7] cf. Eqs. (11) and (12),

$$\varphi_s(\xi,\eta) = \frac{1}{p}\sum_{r=0}^{p-1} g_{s,r}(\xi,\eta)a_r(\xi,\eta) \quad (s=0,1,\ldots,q-1), \tag{14}$$

$$a_r(\xi,\eta) = \frac{pT}{q}\sum_{s=0}^{q-1} w^*_{s,r}(\xi,\eta)\varphi_s(\xi,\eta) \quad (r=0,1,\ldots,p-1), \tag{15}$$

respectively, where we have introduced the shorthand notations

$$\begin{aligned}
a_r(\xi,\eta) &= \bar{a}\left(\xi,\eta+\frac{r}{p}\right),\\
\varphi_s(\xi,\eta) &= \tilde{\varphi}\left([\xi+s]\frac{pT}{q},\eta\frac{2\pi}{T};pT\right),\\
g_{s,r}(\xi,\eta) &= \tilde{g}\left([\xi+s]\frac{pT}{q},\left[\eta+\frac{r}{p}\right]\frac{2\pi}{T};T\right),\\
w_{s,r}(\xi,\eta) &= \tilde{w}\left([\xi+s]\frac{pT}{q},\left[\eta+\frac{r}{p}\right]\frac{2\pi}{T};T\right),
\end{aligned}$$

with $0 \leq \xi < 1$ and $s = 0, 1, \ldots, q-1$ [and hence $0 \leq (\xi+s)/q < 1$], and $0 \leq \eta < 1/p$ and $r = 0, 1, \ldots, p-1$ [and hence $0 \leq \eta + r/p < 1$]. The relationship between the Zak transforms of the analysis window $w(t)$ and the synthesis window $g(t)$ then follows from substituting from Eq. (15) into Eq. (14) and reads [cf. Eq. (13)]

$$\frac{T}{q} \sum_{r=0}^{p-1} g_{s_1,r}(\xi,\eta) w^*_{s_2,r}(\xi,\eta) = \delta_{s_1-s_2}, \tag{16}$$

with $s_1, s_2 = 0, 1, \ldots, q-1$. For each (ξ,η) and given the synthesis window $g(t)$, the latter relationship represents a set of q^2 equations for the pq unknowns $w^*_{s,r}(\xi,\eta)$, which set of equations is underdetermined since $p > q$, and we conclude that the analysis window does not follow uniquely from the synthesis window. Note that for $q = 1$, i.e., for integer oversampling by a factor p, the sum-of-products form (15) reduces to a product again [cf. Eq. (12)]:

$$\bar{a}(\xi,\eta) = pT\tilde{\varphi}\left(\xi pT, \eta\frac{2\pi}{T}; pT\right) \tilde{w}^*\left(\xi pT, \eta\frac{2\pi}{T}; T\right). \tag{17}$$

After combining the p functions $a_r(\xi,\eta)$ into a p-dimensional column vector $\mathbf{a}(\xi,\eta)$, the q functions $\varphi_s(\xi,\eta)$ into a q-dimensional column vector $\boldsymbol{\phi}(\xi,\eta)$, and the $q \times p$ functions $g_{s,r}(\xi,\eta)$ and $w_{s,r}(\xi,\eta)$ into the $q \times p$-dimensional matrices $\mathbf{G}(\xi,\eta)$ and $\mathbf{W}(\xi,\eta)$, respectively, the sum of products forms can be expressed as matrix-vector and matrix-matrix multiplications:

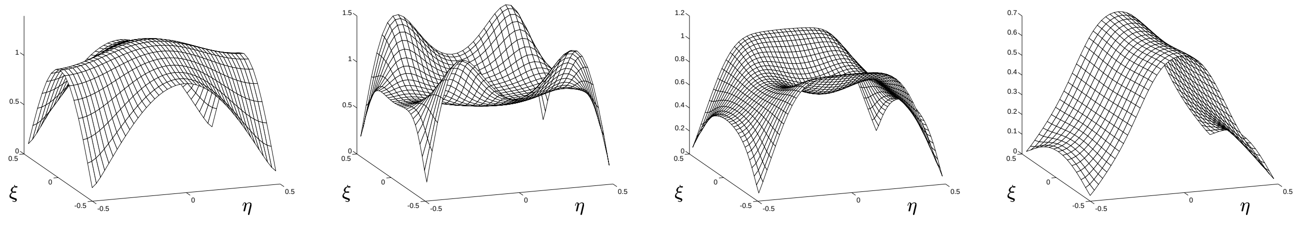
$$\boldsymbol{\phi}(\xi,\eta) = \frac{1}{p}\,\mathbf{G}(\xi,\eta)\mathbf{a}(\xi,\eta), \tag{18}$$

$$\mathbf{a}(\xi,\eta) = \frac{pT}{q}\,\mathbf{W}^*(\xi,\eta)\boldsymbol{\phi}(\xi,\eta), \tag{19}$$

$$\mathbf{I}_q = \frac{T}{q}\,\mathbf{G}(\xi,\eta)\mathbf{W}^*(\xi,\eta), \tag{20}$$

where $\mathbf{I}_q$ denotes the $q \times q$-dimensional identity matrix and where, as usual, the asterisk in connection with vectors and matrices denotes complex conjugation *and* transposition.

The latter relationship again represents q^2 equations for pq unknowns, and the $p \times q$ matrix $\mathbf{W}^*(\xi,\eta)$ cannot be found by a simple inversion of the $q \times p$ matrix $\mathbf{G}(\xi,\eta)$. An optimum solution that is often used, is based on the generalized inverse and reads $\mathbf{W}^*_{opt}(\xi,\eta) = (q/T)\,\mathbf{G}^*(\xi,\eta)[\mathbf{G}(\xi,\eta)\mathbf{G}^*(\xi,\eta)]^{-1}$. This solution for $\mathbf{W}(\xi,\eta)$ is optimal in the sense that (i) it yields the analysis window $w(t)$ with the lowest L^2 norm, (ii) it yields the Gabor coefficients a_{mk} with the lowest L^2 norm, and (iii) it yields the analysis window that – in an L^2 sense, again – best resembles the synthesis window. As examples we have represented in Fig. 3 the absolute values of the Zak transforms that correspond to the Gaussian synthesis window $g(t)$ and to the optimum analysis windows $w(t)$ depicted in Figs. 1b, c, and d.



(a) (b) (c) (d)

Figure 3. (a) The absolute value $|\tilde{g}(\xi T, \eta 2\pi/T; T)|$ of the Zak transform that corresponds to the Gaussian synthesis window $g(t) = 2^{1/4}\exp[-\pi(t/T)^2]$; and the absolute values $|\tilde{w}(\xi T, \eta 2\pi/T; T)|$ of the Zak transforms that correspond to the optimum analysis windows $w(t)$ in Figs. 1b, c, and d, respectively, i.e., for different values of oversampling: (b) $2\pi/\Omega T = 7/6$, (c) $2\pi/\Omega T = 3/2$, and (d) $2\pi/\Omega T = 3$.

The optimum solution gets better if the degree of oversampling p/q becomes higher. However, there is another way of finding a better solution, based on the structure of the lattice. If the lattice structure is adapted to the form of the window function as it is represented in the time-frequency domain, the optimum solution will be better, even for a lower degree of oversampling. In Section 6 we will therefore consider the case of a non-orthogonal sampling geometry, but we will do that in such a way that we can relate this non-orthogonal sampling to orthogonal sampling. In that case we will still be able to use product forms of Gabor's expansion and the Gabor transform, and benefit from all the techniques that have been developed for them.

5. COHERENT-OPTICAL GENERATION OF THE GABOR COEFFICIENTS VIA THE ZAK TRANSFORM

The product form (17) of the Gabor transform in the case of integer oversampling, suggests a generation of this transform by coherent-optical means.[5,6,12,13] Apart from being able to realize ideal imaging, coherent optics is well suited for two specific signal operations, viz., multiplication by a constant function (like in a slide projector, for instance) and Fourier transformation (between the back and the front focal plane of a lens, for instance). These two operations are exactly the ones that are needed for generation of the Gabor transform.

Let a plane wave of monochromatic laser light be normally incident upon a transparency situated in the input plane of a coherent-optical system, see Fig. 4. The transparency contains the time signal $\varphi(t)$ in a rastered format. With $X_o \sim pT$ being the width of this raster and $p\mu X_o$ (with $\mu > 0$) being the spacing between the raster lines, the light amplitude $\varphi_i(x_i, y_i)$ just behind the transparency reads

$$\varphi_i(x_i, y_i) = \text{rect}\left(\frac{x_i}{X_o}\right) \sum_{n=-\infty}^{\infty} \varphi\left(\frac{x_i + nX_o}{X_o} pT\right) \delta(y_i - np\mu X_o), \tag{21}$$

where rect(z) represents a rectangular window function: rect$(z) = 1$ for $-\frac{1}{2} < z \leq \frac{1}{2}$ and rect$(z) = 0$ for z outside this range.

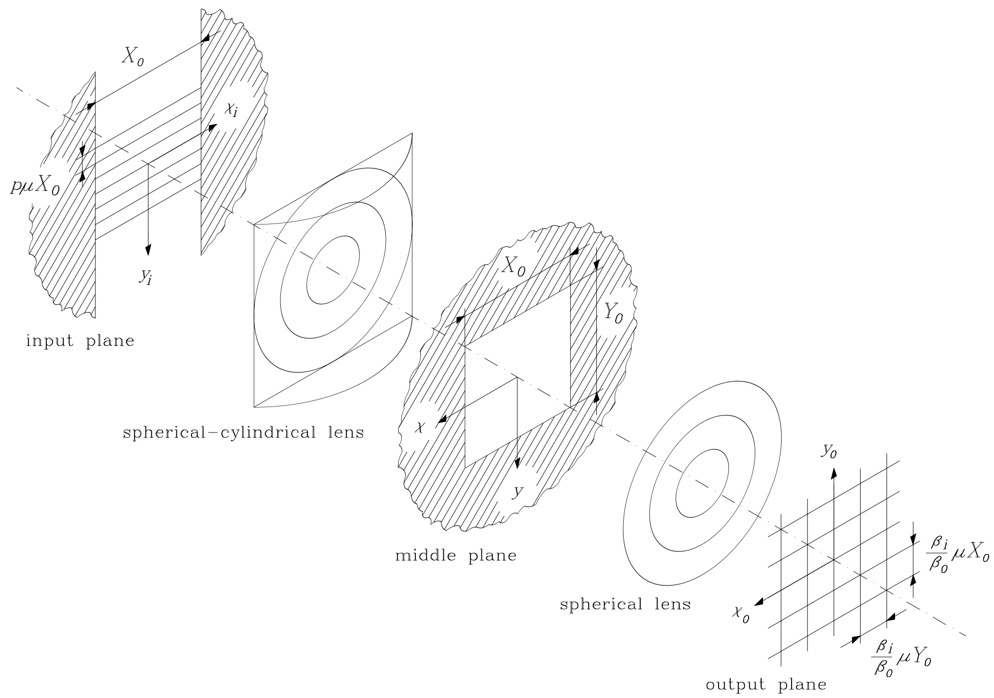


Figure 4. Coherent-optical setup for generation of the Gabor transform.

An anamorphic optical system between the input plane and an intermediate middle plane performs a Fourier transformation in the y-direction and an ideal imaging (possibly with inversion) in the x-direction. Such an anamorphic system can be realized, for instance, using a combination of a spherical and a cylindrical lens. The anamorphic operation results in the light amplitude

$$\varphi_1(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi_i(x_i,y_i) e^{-j\beta_i y y_i} \delta(x-x_i) dx_i dy_i = \text{rect}\left(\frac{x}{X_o}\right) \tilde{\varphi}\left(\frac{x}{X_o}pT, \frac{y}{Y_o}\frac{2\pi}{T}; pT\right) \tag{22}$$

just in front of the intermediate plane. The parameter β_i contains the effect of the wavelength λ of the laser light and the focal length f_i of the spherical and the cylindrical lens: $\beta_i = 2\pi/\lambda f_i$; moreover, the vertical distance Y_o is defined through $\beta_i \mu X_o Y_o = 2\pi$.

A transparency with amplitude transmittance

$$m(x,y) = \text{rect}\left(\frac{x}{X_o}\right) \text{rect}\left(\frac{y}{Y_o}\right) pT \tilde{w}^* \left(\frac{x}{X_o}pT, \frac{y}{Y_o}\frac{2\pi}{T}; T\right) \tag{23}$$

is situated in the intermediate plane. Just behind this transparency, the light amplitude takes the form

$$\varphi_2(x,y) = m(x,y)\varphi_1(x,y) = \text{rect}\left(\frac{x}{X_o}\right) \text{rect}\left(\frac{y}{Y_o}\right) \bar{a}\left(\frac{x}{X_o}, \frac{y}{Y_o}\right), \tag{24}$$

where use has been made of the product form (17). Note that, with $\xi = x/X_o$ and $\eta = y/Y_o$, the aperture rect(ξ)rect(η) contains *one* period of the periodic Fourier transform $\bar{a}(\xi,\eta)$, p horizontal periods of the (periodic) Zak transform $\tilde{\varphi}\,(\xi pT, \eta 2\pi/T; pT)$, and p vertical quasi-periods of the (quasi-periodic) Zak transform $\tilde{w}^*\,(\xi pT, \eta 2\pi/T; T)$.

Finally, a two-dimensional Fourier transformation is performed between the intermediate plane and the output plane. Such a Fourier transformation can be realized, for instance, using a spherical lens. The light amplitude in the output plane then takes the form

$$\begin{aligned} \varphi_o(x_o,y_o) &= \frac{1}{X_o Y_o}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi_2(x,y) e^{-j\beta_o(x_o x - y_o y)} dx dy \\ &= \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} a_{mk}\, \text{sinc}\left(\frac{\beta_o}{\beta_i}\frac{y_o}{\mu X_o} - m\right) \text{sinc}\left(\frac{\beta_o}{\beta_i}\frac{x_o}{\mu Y_o} - k\right), \end{aligned} \tag{25}$$

where the sinc-function $\text{sinc}(z) = \sin(\pi z)/(\pi z)$ has been introduced; the parameter β_o, again, contains the effects of the wave length λ of the laser light and the focal length f_o of the spherical lens: $\beta_o = 2\pi/\lambda f_o$. We conclude that the Gabor transform appears on a rectangular lattice of points

$$a_{mk} = \varphi_o\left(k\frac{\beta_i}{\beta_o}\mu Y_o, m\frac{\beta_i}{\beta_o}\mu X_o\right) \tag{26}$$

in the output plane. Note that relationship (25) represents the output signal as an interpolated version of the Gabor transform, where the interpolation kernel consists of two sinc-functions, in accord with the rectangular aperture in the intermediate (Fourier) plane.

The technique described in this section to generate the Gabor transform, fully utilizes the two-dimensional nature of the optical system, its parallel processing features, and the large space-bandwidth product possible in optical processing. The technique exhibits a resemblance to folded spectrum techniques, where space-bandwidth products in the order of 300 000 are reported.[14]

6. NON-ORTHOGONAL SAMPLING

The rectangular (or orthogonal) lattice that we considered in the previous sections, where sampling occurred on the lattice points $(\tau = mT, \omega = k\Omega)$, can be obtained by integer combinations of two orthogonal vectors $[T,0]^t$ and $[0,\Omega]^t$, see Fig. 5a, which vectors constitute the lattice generator matrix

$$\begin{bmatrix} T & 0 \\ 0 & \Omega \end{bmatrix}.$$

We now consider a time-frequency lattice that is no longer orthogonal. Such a lattice is obtained by integer combinations of two linearly independent, but no longer orthogonal vectors, which we express in the forms $[aT, c\Omega]^t$ and $[bT, d\Omega]^t$, with a, b, c and d integers, and which constitute the lattice generator matrix

$$\begin{bmatrix} aT & bT \\ c\Omega & d\Omega \end{bmatrix} = \begin{bmatrix} T & 0 \\ 0 & \Omega \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Without loss of generality, we may assume that the integers a and b have no common divisors, and that the same holds for the integers c and d; possible common divisors can be absorbed in T and Ω. Note that we only consider lattices that have samples on the time and frequency axes and that are therefore suitable for a discrete-time approach, as well.

The area of a cell (a parallelogram) in the time-frequency plane, spanned by the two vectors $[aT, c\Omega]^t$ and $[bT, d\Omega]^t$, is equal to the determinant of the lattice generator matrix, which determinant is equal to $\Omega T D$, with $D = |ad - bc|$. To be usable as a proper Gabor sampling lattice, this area should satisfy the condition $D \le 2\pi/\Omega T$.

There are a lot of lattice generator matrices that generate the same lattice. We will use the one that is based on the Hermite normal form, unique for any lattice,

$$\begin{bmatrix} T & 0 \\ R\Omega & D\Omega \end{bmatrix} = \begin{bmatrix} T & 0 \\ 0 & \Omega \end{bmatrix} \begin{bmatrix} 1 & 0 \\ R & D \end{bmatrix},$$

where R and D are relatively prime integers and $0 \le |R| < D$. Sampling then occurs on the lattice points $(\tau = mT, \omega = [mR + nD]\Omega)$, and it is evident that these points of the non-orthogonal lattice form a subset of the points $(\tau = mT, \omega = k\Omega)$ of the orthogonal lattice. To be more specific: the non-orthogonal lattice is formed by those points of the rectangular (orthogonal) lattice for which $k - mR$ is an integer multiple of D. Note that the original rectangular lattice arises for $R = 0$ and $D = 1$, see Fig. 5a, and that a hexagonal lattice occurs for $R = 1$ and $D = 2$, see Fig. 5b.

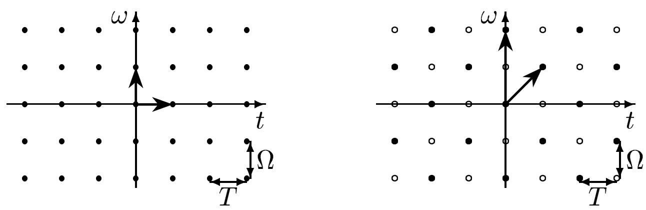


Figure 5. (a) A rectangular lattice with lattice vectors $[T, 0]^t$ and $[0, \Omega]^t$, and thus $R = 0$ and $D = 1$; and (b) a hexagonal lattice with lattice vectors $[T, \Omega]^t$ and $[0, 2\Omega]^t$, and thus $R = 1$ and $D = 2$.

7. GABOR'S SIGNAL EXPANSION ON A NON-ORTHOGONAL LATTICE

If we define the two-dimensional array λ_{mk} as

$$\lambda_{mk} = \sum_{n=-\infty}^{\infty} \delta_{k-mR-nD}, \tag{27}$$

Gabor's signal expansion on a non-orthogonal lattice can be expressed as [cf. Eq. (4)]

$$\varphi(t) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \lambda_{mk} a_{mk} g_{mk}(t) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_{mk}^s g_{mk}(t), \tag{28}$$

while – with a different analysis window $w(t)$, though! – the expansion coefficients a_{mk} are still determined by the Gabor transform (5). Of course, since we only need the limited array $a_{mk}^s = \lambda_{mk} a_{mk}$ – which is, in fact, a properly sampled version of the full array a_{mk} – we need only calculate the coefficients a_{mk} for those values of m and k for which $k - mR$ is an integer multiple of D. We note that the Fourier transform $\bar{a}^s(\xi,\eta)$ of the limited array a^s_{mk} is related to the Fourier transform $\bar{a}(\xi,\eta)$ of the full array a_{mk} via the periodization relation

$$\bar{a}^s(\xi,\eta) = \frac{1}{D}\sum_{n=0}^{D-1} \bar{a}\left(\xi - \frac{n}{D}, \eta - \frac{nR}{D}\right) \tag{29}$$

and thus

$$a^s_r(\xi,\eta) = \frac{1}{D}\sum_{n=0}^{D-1} a_r\left(\xi - \frac{n}{D}, \eta - \frac{nR}{D}\right). \tag{30}$$

In the non-orthogonal case, the biorthogonality condition takes the form [cf. Eq. (6)]

$$\sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \lambda_{mk} g_{mk}(t_1) w^*_{mk}(t_2) = \delta(t_1 - t_2) \tag{31}$$

and leads to the equivalent but simpler expression [cf. Eq. (7)]

$$\frac{2\pi}{D\Omega}\sum_{m=-\infty}^{\infty} g(t-mT)w^*\left(t - \left[mT + n\frac{2\pi}{D\Omega}\right]\right)e^{j2\pi mnR/D} = \delta_n. \tag{32}$$

Note that for $R=0$ and $D=1$, for which we have a rectangular lattice (see Fig. 5a), Eq. (32) reduces to Eq. (7), and that for $R=1$ and $D=2$, for which we have a hexagonal lattice (see Fig. 5b), Eq. (32) takes the form

$$\frac{\pi}{\Omega}\sum_{m=-\infty}^{\infty} g(t-mT)w^*\left(t - \left[mT + n\frac{\pi}{\Omega}\right]\right)(-1)^{mn} = \delta_n. \tag{33}$$

The biorthogonality condition expressed in terms of the Zak transforms of the window functions now takes the form, cf. Eq. (16),

$$\frac{T}{Dq}\sum_{r=0}^{p-1} g_{s_1,r}(\xi,\eta) w^*_{s_2,r}\left(\xi - \frac{n}{D}, \eta - \frac{nR}{D}\right) = \delta_n \delta_{s_1-s_2}, \tag{34}$$

with $s_1, s_2 = 0, 1, \ldots, q-1$ and $n = 0, 1, \ldots, D-1$, and allows an easy determination of the analysis window $w(t)$ for a given synthesis window $g(t)$. For $R=0$ and $D=1$, for instance, relation (34) reduces to Eq. (16), while for $R=1$, $D=2$, $q=1$, and p an even integer – which corresponds to the integer ($\frac{1}{2}p$-times) oversampled hexagonal case – it reduces to

$$\frac{T}{2}\sum_{r=0}^{p-1} g_{0,r}(\xi,\eta) w^*_{0,r-np/2}(\xi,\eta)(-1)^{nr} = \delta_n \quad (n = 0, 1;\ p \text{ even}), \tag{35}$$

from which the Zak transform $\tilde{w}(t,\omega;T)$ and hence the window function $w(t)$ can easily be determined.

Since we have related Gabor's signal expansion on a non-orthogonal lattice to sampling on a denser but orthogonal lattice, followed by restriction to a sub-lattice that corresponds to the non-orthogonal lattice, we can still use all the techniques that are developed for rectangular lattices, in particular the technique of determining Gabor's expansion coefficients via the Zak transform, cf. Eqs. (15) and (17). Hence, the optical setup described in Section 5 can still be applied; the only difference is that the sampling in the output plane is now on the non-orthogonal lattice.

8. FROM NON-ORTHOGONAL TO RECTANGULAR SAMPLING VIA SHEARING

If we eliminate the array λ_{mk} from Gabor's signal expansion (28), we can directly write

$$\varphi(t) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} a_{m,mR+nD}\, g(t-mT) e^{j(mR+nD)\Omega t}. \tag{36}$$

We now write the shifted and modulated version of the synthesis window $g(t)$ in the form

$$g(t-mT)e^{j(mR+nD)\Omega t} = g'_{mn}(t)e^{jR\Omega t^2/2T}e^{jRm^2\Omega T/2},$$

where we have introduced the sheared version $g'(t)$ of $g(t)$:

$$g'(t) = g(t)e^{-jR\Omega t^2/2T}. \tag{37}$$

If we apply the same shear operation to the signal $\varphi(t)$ to get $\varphi'(t) = \varphi(t)e^{-jR\Omega t^2/2T}$ and introduce the sheared array

$$a'_{mn} = a_{m,mR+nD}e^{jRm^2\Omega T/2}, \tag{38}$$

we get Gabor's signal expansion in the original, rectangular form, again:

$$\varphi'(t) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} a'_{mn}g'_{mn}(t). \tag{39}$$

And, after introducing the sheared version $w'(t)$ of the analysis window $w(t)$, the same holds for the Gabor transform:

$$a'_{mn} = \int_{-\infty}^{\infty} \varphi'(t){w'}^*_{mn}(t)dt. \tag{40}$$

We conclude that multiplication of the signal, the window functions, and the expansion coefficients with proper quadratic phase factors, correponds to shearing of the Gabor lattice; this shearing transforms a non-orthogonal lattice into a rectangular one. Consequently, in calculating Gabor's expansion coefficients in the non-orthogonal case, algorithms for the rectangular case can still be used, but the signal $\varphi(t)$ and the window functions $w(t)$ and $g(t)$ have to be pre-multiplied by a quadratic phase factor, cf. Eq. (37). After calculating the Gabor coefficients a'_{mn}, these coefficients have to be post-multiplied by a quadratic phase factor to obtain the actual coefficients a_{mn}, see Eq. (38).

Note that shearing is not the only way to transform a non-orthogonal lattice into an orthogonal one. In some recent papers,[15,16] a fractional Fourier transform combined with a proper scaling was used to do this.

The procedure described above, when applied to discrete-time signals and combined with the Zak transform, leads to fast algorithms for the calculation of Gabor's expansion coefficients by means of a digital computer.[16] It is unclear, however, whether an optical setup based upon this procedure – i.e., pre- and post-multiplication by quadratic-phase factors to shear a non-orthogonal lattice into an orthogonal one – can be found.

9. CONCLUSIONS

Gabor's signal expansion and the Gabor transform on a rectangular lattice have been introduced, along with the Fourier transform of the array of expansion coefficients and the Zak transforms of the signal and the window functions. Based on these Fourier and Zak transforms, the sum-of-products forms for the Gabor expansion and the Gabor transform, which hold in the rationally oversampled case, have been derived. An optical setup for the generation of Gabor's expansion coefficients in the case of integer oversampling has been presented.

We have then studied Gabor's signal expansion and the Gabor transform based on a non-orthogonal sampling geometry. We have done this by considering the non-orthogonal lattice as a sub-lattice of an orthogonal lattice. This procedure allows us to use all the formulas that hold for the orthogonal sampling geometry. In particular we can use the sum-of-products forms that hold in the case of a rationally oversampled rectangular lattice. Moreover, in the case of integer oversampling, the optical setup presented before can still be used.

We finally note that if everything remains to be based on a rectangular sampling geometry, it will be easier to extend the theory of the Gabor scheme to higher-dimensional signals; see, for instance, Ref. 16, where the multi-dimensional case is treated for continuous-time as well as discrete-time signals.

Appendix A. Derivation of Eq. (14)

In Gabor's signal expansion (4) we substitute from the inverse Fourier transform [cf. Eq. (9)]

$$\varphi(t)=\sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\left[\int_0^1\int_0^1 \bar{a}(\xi,\eta)e^{j2\pi(m\eta-k\xi)}d\xi d\eta\right]g(t-mT)e^{jk\Omega t}$$

and rearrange factors

$$\varphi(t)=\int_0^1\int_0^1 \bar{a}(\xi,\eta)\left[\sum_{m=-\infty}^{\infty}g(t-mT)e^{j2\pi m\eta}\right]\left[\sum_{k=-\infty}^{\infty}e^{-jk(2\pi\xi-\Omega t)}\right]d\xi d\eta.$$

We replace the sum of exponentials by a sum of Dirac functions and recognize the Zak transform of the elementary signal $g(t)$ [cf. Eq. (10)]:

$$\varphi(t)=\int_0^1\int_0^1 \bar{a}(\xi,\eta)\tilde{g}\left(t,\eta\frac{2\pi}{T};T\right)\left[\sum_{k=-\infty}^{\infty}\delta\left(\xi-\frac{\Omega}{2\pi}t+k\right)\right]d\xi d\eta.$$

We rearrange factors again and substitute from the periodicity property of $\bar{a}(\xi,\eta)$

$$\varphi(t)=\int_0^1\left[\sum_{k=-\infty}^{\infty}\int_0^1 \bar{a}(\xi+k,\eta)\delta\left(\xi+k-\frac{\Omega}{2\pi}t\right)d\xi\right]\tilde{g}\left(t,\eta\frac{2\pi}{T};T\right)d\eta$$

and we replace the summation over k together with the integral over the finite ξ-interval by an integral over the entire ξ-axis

$$\varphi(t)=\int_0^1\left[\int_{-\infty}^{\infty}\bar{a}(\xi,\eta)\delta\left(\xi-\frac{\Omega}{2\pi}t\right)d\xi\right]\tilde{g}\left(t,\eta\frac{2\pi}{\Omega};T\right)d\eta.$$

Evaluation of the resulting integral over ξ yields the intermediate result

$$\varphi(t)=\int_0^1\bar{a}\left(\frac{\Omega}{2\pi}t,\eta\right)\tilde{g}\left(t,\eta\frac{2\pi}{T};T\right)d\eta.$$

We now write down the definition of the Zak transform [cf. Eq. (10)]

$$\tilde{\varphi}\left(\xi\frac{pT}{q},\eta_o\frac{2\pi}{T};pT\right)=\tilde{\varphi}\left(\xi\frac{pT}{q},\eta_o\frac{2\pi}{T};q\frac{pT}{q}\right)=\sum_{n=-\infty}^{\infty}\varphi\left([\xi+nq]\frac{pT}{q}\right)e^{-jnq(pT/q)\eta_o(2\pi/T)}$$

and substitute from the intermediate result above

$$\tilde{\varphi}\left(\xi\frac{pT}{q},\eta_o\frac{2\pi}{T};pT\right)=\sum_{n=-\infty}^{\infty}\left[\int_0^1\bar{a}\left(\frac{\Omega}{2\pi}[\xi+nq]\frac{pT}{q},\eta\right)\tilde{g}\left([\xi+nq]\frac{pT}{q},\eta\frac{2\pi}{T};T\right)d\eta\right]e^{-jnq(pT/q)\eta_o(2\pi/T)}.$$

We rearrange things, using the relation $2\pi/\Omega=pT/q$,

$$\tilde{\varphi}\left(\xi\frac{pT}{q},\eta_o\frac{2\pi}{T};pT\right)=\sum_{n=-\infty}^{\infty}\left[\int_0^1\bar{a}(\xi+nq,\eta)\tilde{g}\left(\xi\frac{pT}{q}+pnT,\eta\frac{2\pi}{T};T\right)d\eta\right]e^{-j2\pi pn\eta_o}$$

and use the periodicity property of $\bar{a}(\xi,\eta)$ and the quasi-periodicity property of $\tilde{g}(\xi pT/q,y2\pi/T;T)$

$$\tilde{\varphi}\left(\xi\frac{pT}{q},\eta_o\frac{2\pi}{T};pT\right)=\sum_{n=-\infty}^{\infty}\left[\int_0^1\bar{a}(\xi,\eta)\tilde{g}\left(\xi\frac{pT}{q},\eta\frac{2\pi}{T};T\right)e^{jpnT\eta(2\pi/T)}d\eta\right]e^{-j2\pi pn\eta_o}.$$

We rearrange factors

$$\tilde{\varphi}\left(\xi\frac{pT}{q},\eta_o\frac{2\pi}{T};pT\right)=\int_0^1\bar{a}(\xi,\eta)\tilde{g}\left(\xi\frac{pT}{q},\eta\frac{2\pi}{T};T\right)\left[\sum_{n=-\infty}^{\infty}e^{j2\pi np(\eta-\eta_o)}\right]d\eta$$

and replace the sum of exponentials by a sum of Dirac functions

$$\tilde{\varphi}\left(\xi\frac{pT}{q},\eta_o\frac{2\pi}{T};pT\right)=\int_0^1\bar{a}(\xi,\eta)\tilde{g}\left(\xi\frac{pT}{q},\eta\frac{2\pi}{T};T\right)\left[\frac{1}{p}\sum_{n=-\infty}^{\infty}\delta\left(\eta-\eta_o-\frac{n}{p}\right)\right]d\eta.$$

We replace the summation over n by a double summation over r and k through the substitution $n=kp+r$, where r extends over an interval of length p

$$\tilde{\varphi}\left(\xi\frac{pT}{q},\eta_o\frac{2\pi}{T};pT\right)=\int_0^1\bar{a}(\xi,\eta)\tilde{g}\left(\xi\frac{pT}{q},\eta\frac{2\pi}{T};T\right)\left[\frac{1}{p}\sum_{k=-\infty}^{\infty}\sum_{r=0}^{p-1}\delta\left(\eta-\eta_o-k-\frac{r}{p}\right)\right]d\eta$$

and rearrange factors

$$\tilde{\varphi}\left(\xi\frac{pT}{q},\eta_o\frac{2\pi}{T};pT\right)=\frac{1}{p}\sum_{r=0}^{p-1}\left[\sum_{k=-\infty}^{\infty}\int_0^1\bar{a}(\xi,\eta)\tilde{g}\left(\xi\frac{pT}{q},\eta\frac{2\pi}{T};T\right)\delta\left(\eta-k-\eta_o-\frac{r}{p}\right)d\eta\right].$$

We use the periodicity of $\bar{a}(\xi,\eta)$ and $\tilde{g}(\xi pT/q,\eta 2\pi/T;T)$

$$\tilde{\varphi}\left(\xi\frac{pT}{q},\eta_o\frac{2\pi}{T};pT\right)=\frac{1}{p}\sum_{r=0}^{p-1}\left[\sum_{k=-\infty}^{\infty}\int_0^1\bar{a}(\xi,\eta-k)\tilde{g}\left(\xi\frac{pT}{q},[\eta-k]\frac{2\pi}{T};T\right)\delta\left(\eta-k-\eta_o-\frac{r}{p}\right)d\eta\right]$$

and replace the summation over k together with the integral over the finite η-interval by an integral over the entire η-axis

$$\tilde{\varphi}\left(\xi\frac{pT}{q},\eta_o\frac{2\pi}{T};pT\right)=\frac{1}{p}\sum_{r=0}^{p-1}\int_{-\infty}^{\infty}\bar{a}(\xi,\eta)\tilde{g}\left(\xi\frac{pT}{q},\eta\frac{2\pi}{T};T\right)\delta\left(\eta-\eta_o-\frac{r}{p}\right)d\eta.$$

Evaluation of the integral and replacing η_o by η results in

$$\tilde{\varphi}\left(\xi\frac{pT}{q},\eta\frac{2\pi}{T};pT\right)=\frac{1}{p}\sum_{r=0}^{p-1}\bar{a}\left(\xi,\eta+\frac{r}{p}\right)\tilde{g}\left(\xi\frac{pT}{q},\left[\eta+\frac{r}{p}\right]\frac{2\pi}{T};T\right).$$

We finally replace ξ by $\xi+s$ and use the periodicity of the Fourier transform $\bar{a}(\xi,\eta)$, which leads to the required result, Eq. (14):

$$\tilde{\varphi}\left([\xi+s]\frac{pT}{q},\eta\frac{2\pi}{T};pT\right)=\frac{1}{p}\sum_{r=0}^{p-1}\bar{a}\left(\xi,\eta+\frac{r}{p}\right)\tilde{g}\left([\xi+s]\frac{pT}{q},\left[\eta+\frac{r}{p}\right]\frac{2\pi}{T};T\right).$$

Appendix B. Derivation of Eq. (15)

In the Fourier transform (9) of the array a_{mk}, we substitute from the Gabor transform (5)

$$\bar{a}(\xi,\eta)=\sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\varphi(t)w^*(t-mT)e^{-jk\Omega t}dt\right]e^{-j2\pi(m\eta-k\xi)}$$

and rearrange factors

$$\bar{a}(\xi,\eta)=\sum_{m=-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\varphi(t)w^*(t-mT)\left\{\sum_{k=-\infty}^{\infty}e^{-jk(\Omega t-2\pi\xi)}\right\}dt\right]e^{-j2\pi m\eta}.$$

We replace the sum of exponentials by a sum of Dirac functions

$$\bar{a}(\xi,\eta)=\sum_{m=-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\varphi(t)w^*(t-mT)\left\{\frac{2\pi}{\Omega}\sum_{k=-\infty}^{\infty}\delta\left(t-[\xi+k]\frac{2\pi}{\Omega}\right)\right\}dt\right]e^{-j2\pi m\eta}$$

and rearrange factors again

$$\bar{a}(\xi,\eta) = \frac{2\pi}{\Omega}\sum_{m=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(t)w^*(t-mT)\delta\left(t-[\xi+k]\frac{2\pi}{\Omega}\right)dt\right]e^{-j2\pi m\eta}.$$

We evaluate the integral

$$\bar{a}(\xi,\eta) = \frac{2\pi}{\Omega}\sum_{m=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty}\varphi\left([\xi+k]\frac{2\pi}{\Omega}\right)w^*\left([\xi+k]\frac{2\pi}{\Omega}-mT\right)\right]e^{-j2\pi m\eta}$$

and rearrange factors again

$$\bar{a}(\xi,\eta) = \frac{2\pi}{\Omega}\sum_{k=-\infty}^{\infty}\varphi\left(\xi\frac{2\pi}{\Omega}+k\frac{2\pi}{\Omega}\right)e^{-jk(2\pi/\Omega)\eta(2\pi/T)}$$

$$\times\left[\sum_{m=-\infty}^{\infty}w^*\left(\xi\frac{2\pi}{\Omega}-\left[m-k\frac{2\pi}{\Omega T}\right]T\right)e^{-j[m-k(2\pi/\Omega T)]T\eta(2\pi/T)}\right].$$

We replace $2\pi/\Omega T$ by p/q

$$\bar{a}(\xi,\eta) = \frac{2\pi}{\Omega}\sum_{k=-\infty}^{\infty}\varphi\left(\xi\frac{2\pi}{\Omega}+k\frac{2\pi}{\Omega}\right)e^{-jk(2\pi/\Omega)\eta(2\pi/T)}$$

$$\times\left[\sum_{m=-\infty}^{\infty}w\left(\xi\frac{2\pi}{\Omega}-\left[m-k\frac{p}{q}\right]T\right)e^{j[m-k(p/q)]T\eta(2\pi/T)}\right]^*$$

and replace the summation over k by a double summation over s and n through the substitution $k = nq+s$, where s extends over an interval of length q

$$\bar{a}(\xi,\eta) = \frac{2\pi}{\Omega}\sum_{n=-\infty}^{\infty}\sum_{s=0}^{q-1}\varphi\left(\xi\frac{2\pi}{\Omega}+[nq+s]\frac{2\pi}{\Omega}\right)e^{-j(nq+s)(2\pi/\Omega)\eta(2\pi/T)}$$

$$\times\left[\sum_{m=-\infty}^{\infty}w\left(\xi\frac{2\pi}{\Omega}-\left[m-(nq+s)\frac{p}{q}\right]T\right)e^{j[m-(nq+s)(p/q)]T\eta(2\pi/T)}\right]^*.$$

We substitute $m-np$ by $-k$

$$\bar{a}(\xi,\eta) = \frac{2\pi}{\Omega}\sum_{n=-\infty}^{\infty}\sum_{s=0}^{q-1}\varphi\left(\xi\frac{2\pi}{\Omega}+\left[n+\frac{s}{q}\right]q\frac{2\pi}{\Omega}\right)e^{-j[n+(s/q)]q(2\pi/\Omega)\eta(2\pi/T)}$$

$$\times\left[\sum_{k=-\infty}^{\infty}w\left(\xi\frac{2\pi}{\Omega}+\left[k+s\frac{p}{q}\right]T\right)e^{-j[k+s(p/q)]T\eta(2\pi/T)}\right]^*$$

and rearrange factors, while using the relation $pT = q(2\pi/\Omega)$

$$\bar{a}(\xi,\eta) = \frac{2\pi}{\Omega}\sum_{s=0}^{q-1}\sum_{n=-\infty}^{\infty}\varphi\left([\xi+s]\frac{2\pi}{\Omega}+npT\right)e^{-jnpT\eta(2\pi/T)}\left[\sum_{k=-\infty}^{\infty}w\left([\xi+s]\frac{2\pi}{\Omega}+kT\right)e^{-jkT\eta(2\pi/T)}\right]^*.$$

In the last expression we recognize the definitions [cf. Eq. (10)] for the Zak transforms $\tilde{\varphi}(\xi 2\pi/\Omega, \eta 2\pi/T; pT)$ and $\tilde{w}(x2\pi/\Omega, y2\pi/T; T)$ of the signal $\varphi(t)$ and the window function $w(t)$, respectively, and can write

$$\bar{a}(\xi,\eta) = \frac{2\pi}{\Omega}\sum_{s=0}^{q-1}\tilde{\varphi}\left([\xi+s]\frac{2\pi}{\Omega},\eta\frac{2\pi}{T};pT\right)\tilde{w}^*\left([\xi+s]\frac{2\pi}{\Omega},\eta\frac{2\pi}{T};T\right)$$

or, with $2\pi/\Omega = pT/q$,

$$\bar{a}(\xi,\eta) = \frac{pT}{q}\sum_{s=0}^{q-1}\tilde{\varphi}\left([\xi+s]\frac{pT}{q},\eta\frac{2\pi}{T};pT\right)\tilde{w}^*\left([\xi+s]\frac{pT}{q},\eta\frac{2\pi}{T};T\right).$$

We finally replace η by $\eta + r/p$ and use the periodicity of the Zak transform $\tilde{\varphi}(\xi pT/q, \eta 2\pi/T; pT)$, which leads to the required result, Eq. (15):

$$\bar{a}\left(\xi,\eta+\frac{r}{p}\right) = \frac{pT}{q}\sum_{s=0}^{q-1}\tilde{\varphi}\left([\xi+s]\frac{pT}{q},\eta\frac{2\pi}{T};pT\right)\tilde{w}^*\left([\xi+s]\frac{pT}{q},\left[\eta+\frac{r}{p}\right]\frac{2\pi}{T};T\right).$$

REFERENCES

1. D. Gabor, "Theory of communication," *J. Inst. Elec. Eng.* **93 (III)**, pp. 429–457, 1946.
2. E. Wigner, "On the quantum correction for thermodynamic equilibrium," *Phys. Rev.* **40**, pp. 749–759, 1932.
3. W. Mecklenbrauker and F. Hlawatsch, Eds., *The Wigner Distribution – Theory and Applications in Signal Processing*, Springer, Amsterdam, 1997.
4. M. J. Bastiaans, "A sampling theorem for the complex spectrogram and Gabor's expansion of a signal in Gaussian elementary signals," *Opt. Eng.* **20**, pp. 594–598, 1981.
5. M. J. Bastiaans, "Gabor's signal expansion and its relation to sampling of the sliding-window spectrum," in R. J. Marks II, Ed., *Advanced Topics in Shannon Sampling and Interpolation Theory*, pp. 1–35, Springer, New York, 1993.
6. M. J. Bastiaans, *Gabor's expansion and the Zak transform for continuous-time and discrete-time signals: critical sampling and rational oversampling*, Eindhoven University of Technology Research Report 95-E-295, Technische Universiteit Eindhoven, Eindhoven, Netherlands, 1995.
7. M. J. Bastiaans and M. C. W. Geilen, "On the discrete Gabor transform and the discrete Zak transform," *Signal Process.* **49**, pp. 151–166, 1996.
8. H. G. Feichtinger and T. Strohmer, Eds., *Gabor Analysis and Algorithms – Theory and Applications*, Birkhäuser, Berlin, 1998.
9. J. Zak, "Finite translations in solid state physics," *Phys. Rev. Lett.* **19**, pp. 1385–1397, 1967.
10. J. Zak, "The kq-representation in the dynamics of electrons in solids," *Solid State Physics* **27**, pp. 1–62, 1972.
11. A. J. E. M. Janssen, "The Zak transfrom: A signal transform for sampled time-continuous signals," *Philips J. Res.* **43**, pp. 23–69, 1988.
12. M. J. Bastiaans, "Optical generation of Gabor's expansion coefficients for rastered signals," *Opt. Acta* **29**, pp. 1349–1357, 1982.
13. Y. Li and Y. Zhang, "Coherent optical processing of Gabor and wavelet expansions of one- and two-dimensional signals," *Opt. Eng.* **31**, pp. 1865–1885, 1992.
14. D. Casasent, "Optical signal processing," in D. Casasent, Ed., *Optical Data Processing; Applications, Topics in Applied Physics*, vol. 23, Chap. 8, Springer, Berlin, 1978.
15. M. J. Bastiaans and A. J. van Leest, "From the rectangular to the quincunx Gabor lattice via fractional Fourier transformation," *IEEE Signal Process. Lett.* **5**, pp. 203-205, 1998.
16. A. J. van Leest, *Non-separable Gabor schemes: their design and implementation*, PhD thesis, Technische Universiteit Eindhoven, Eindhoven, Netherlands, 2001.