ON THE MEASUREMENT OF WIGNER DISTRIBUTION MOMENTS IN THE FRACTIONAL FOURIER TRANSFORM DOMAIN

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ABSTRACT

It is shown how all global Wigner distribution moments of arbitrary order can be measured as intensity moments in the output plane of an appropriate number of fractional Fourier transform systems (generally anamorphic ones). The minimum number of (anamorphic) fractional power spectra that are needed for the determination of these moments is derived.

1. INTRODUCTION

After the introduction of the Wigner distribution [1] (WD) for the description of coherent and partially coherent optical fields [2], it became an important tool for optical signal/image analysis and beam characterization [3, 4, 5]. The WD completely describes the complex amplitude of a coherent optical field (up to a constant phase factor) or the two-point correlation function of a partially coherent field. Because the WD of a two-dimensional optical field is a function of four variables, it is difficult to analyze. Therefore, the optical field is often represented not by the WD itself, but by its global moments. Beam characterization based on the second-order moments of the WD thus became the basis of an International Organization for Standardization standard. [6]

Some of the WD moments can directly be determined from measurements of the intensity distributions in the image plane or the Fourier plane, but most of the moments cannot be determined in such an easy way. In order to calculate such moments, additional information is required. Since first-order optical systems [7] – also called $ABCD$ systems – produce affine transformations of the WD in phase space, the intensity distributions measured at the output of such systems can provide such additional information. The application of $ABCD$ systems for the measurements of the second-order WD moments have been reported in several publications [8, 9, 10, 11, 12, 13].

It is the aim of this paper to show how all WD moments can be measured as intensity moments only. We therefore consider a particular case of the $ABCD$ system: the anamorphic fractional Fourier transform (FT) system. We show that not only the second-order moments, but all moments of the four-dimensional WD can be obtained from measurements of only intensity distributions in an appropriate number of (anamorphic) fractional FT systems.

2. WIGNER DISTRIBUTION

The Wigner distribution (WD) of a two-dimensional function $f(x, y)$ is defined by

$$W_f(x, u; y, v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x + \tfrac{1}{2}x', y + \tfrac{1}{2}y') \times\ f^*(x - \tfrac{1}{2}x', y - \tfrac{1}{2}y') \exp[-j2\pi(ux' + vy')]\, dx'\, dy'. \tag{1}$$

The WD $W_f(x, u; y, v)$ represents a space function $f(x, y)$ in a combined space/spatial-frequency domain, the so-called phase space, where u is the spatial-frequency variable associated to the space variable x, and v the spatial-frequency variable associated to the space variable y. We remark that the definition of the WD – and all the results of this paper – need not be restricted to coherent light, in which case $f(x, y)$ would represent the complex field amplitude of the light, but can be extended to partially coherent light, in which case the two-point correlation function of the light can be identified with $< f(x+\frac{1}{2}x', y+\frac{1}{2}y') f^*(x-\frac{1}{2}x', y-\frac{1}{2}y') >$.

In this paper we consider the normalized moments of the WD, where the normalization is with respect to the total energy E of the signal:

$$\begin{aligned} E &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(x, u; y, v)\, dx\, du\, dy\, dv \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x, y)|^2\, dx\, dy. \end{aligned} \tag{2}$$

These normalized moments μ_{pqrs} of the WD are thus defined by

$$\mu_{pqrs}E = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(x,u;y,v) \times x^p u^q y^r v^s \, dx\, du\, dy\, dv \quad (p,q,r,s \geq 0) \quad (3)$$

$$= \frac{1}{(4\pi j)^{q+s}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^p y^r \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)^q \left(\frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2}\right)^s f(x_1,y_1) f^*(x_2,y_2)\Big|_{x_1=x_2=x, y_1=y_2=y} dx\, dy.$$

Note that for $q = s = 0$ we have intensity moments, which can easily be measured:

$$\begin{aligned} \mu_{p0r0}E &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(x,u;y,v) \\ &\quad \times x^p y^r \, dx\, du\, dy\, dv \quad (p,r \geq 0) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^p y^r |f(x,y)|^2 \, dx\, dy. \end{aligned} \quad (4)$$

3. FRACTIONAL FOURIER TRANSFORM

The fractional Fourier transform (FT) of a function $f(x, y)$ is defined by [14, 15, 16, 17, 18]

$$F_{\alpha\beta}(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_\alpha(x,u) K_\beta(y,v) f(x,y)\, dx\, dy, \quad (5)$$

where the kernel $K_\alpha(x, u)$ is given by

$$K_\alpha(x,u) = \frac{\exp(j\frac{1}{2}\alpha)}{\sqrt{j \sin\alpha}} \exp\left[j\pi \frac{(x^2+u^2)\cos\alpha - 2ux}{\sin\alpha}\right]. \quad (6)$$

We remark that $F_{0,0}(u, v) = f(u, v)$ represents the function itself, while $F_{\pi/2,\pi/2}(u, v)$ corresponds to the normal two-dimensional FT of the function $f(x, y)$.

The fractional FT can be generated optically by a very simple, anamorphic, coherent-optical set-up, consisting only of two cylindrical lenses, whose focal lengths – in combination with some appropriate sections of free space – are related to the angles α and β.

One of the most important properties of the fractional FT is that it corresponds to a rotation of the WD in phase space:

$$W_{F_{\alpha\beta}}(x,u;y,v) = W_f(x\cos\alpha - u\sin\alpha, x\sin\alpha + u\cos\alpha; y\cos\beta - v\sin\beta, y\sin\beta + v\cos\beta). \quad (7)$$

Moreover, the fractional power spectrum, i.e., the squared modulus $|F_{\alpha\beta}(x, y)|^2$ of the fractional FT, is directly related to the WD through a projection operation:

$$\left|F_{\alpha\beta}(x,y)\right|^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_{F_{\alpha\beta}}(x,u;y,v)\, du\, dv. \quad (8)$$

Note that the fractional power spectrum is related to the intensity distribution in the output plane of an anamorphic fractional FT system and therefore can be easily measured in experiments.

4. MOMENTS IN THE FRACTIONAL DOMAIN

We can as well define normalized moments $\mu_{pqrs}(\alpha, \beta)$ in the fractional domain and relate these to the original moments $\mu_{pqrs} = \mu_{pqrs}(0, 0)$, cf. Eq. (3):

$$\mu_{pqrs}(\alpha,\beta)E = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_{F_{\alpha\beta}}(x,u;y,v)\, x^p u^q y^r v^s \, dx\, du\, dy\, dv \quad (p,q,r,s \geq 0) \quad (9)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(x,u;y,v)(x\cos\alpha + u\sin\alpha)^p \times (-x\sin\alpha + u\cos\alpha)^q (y\cos\beta + v\sin\beta)^r \times (-y\sin\beta + v\cos\beta)^s \, dx\, du\, dy\, dv.$$

The general relationship thus takes the form

$$\mu_{pqrs}(\alpha,\beta) = \sum_{k=0}^{p}\sum_{l=0}^{q}\sum_{m=0}^{r}\sum_{n=0}^{s} (-1)^{l+n} \times \binom{p}{k}\binom{q}{l}\binom{r}{m}\binom{s}{n} \mu_{p-k+l,q-l+k,r-m+n,s-n+m} \times (\cos\alpha)^{p-k+q-l}(\sin\alpha)^{k+l}(\cos\beta)^{r-m+s-n}(\sin\beta)^{m+n}, \quad (10)$$

and for the intensity moments in particular we have

$$\mu_{p0r0}(\alpha,\beta) = \sum_{k=0}^{p}\sum_{m=0}^{r} \binom{p}{k}\binom{r}{m} \mu_{p-k,k,r-m,m} \times \cos^{p-k}\alpha \sin^k\alpha \cos^{r-m}\beta \sin^m\beta. \quad (11)$$

Note that the total energy E, see Eq. (2), is invariant under fractional Fourier transformation.

From the definitions of the normalized moments in the fractional domain, it follows directly that the $x^p u^q$ moments (with $r = s = 0$) are not affected by the fractional Fourier transformation in the y-direction (with angle β), while the $y^r v^s$ moments (with $p = q = 0$) are not affected by the one in the x-direction (with angle α):

$$\begin{aligned} \mu_{pq00}(\alpha,\beta) &= \mu_{pq00}(\alpha,0), \\ \mu_{00rs}(\alpha,\beta) &= \mu_{00rs}(0,\beta). \end{aligned} \quad (12)$$

Moreover, the $x^p y^r$ moments (with $q = s = 0$) can easily be measured as intensity moments again,

$$\mu_{p0r0}(\alpha,\beta)E = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^p y^r |F_{\alpha\beta}(x,y)|^2 \, dx\, dy, \quad (13)$$

and from the relationship

$$\mu_{pqrs}(\alpha,\beta)=(-1)^{p+r}\mu_{qpsr}(\alpha+\tfrac{1}{2}\pi,\beta+\tfrac{1}{2}\pi), \qquad (14)$$

we conclude that the $u^q v^s$ moments (with $p=r=0$) can be measured as intensity moments, as well.

5. RELATIONS BETWEEN MOMENTS IN THE FRACTIONAL DOMAIN

5.1. First-order moments

For the first-order moments, the following two equations are relevant:

$$\begin{aligned}\mu_{1000}(\alpha,\beta) &= \mu_{1000}\cos\alpha+\mu_{0100}\sin\alpha, &(15)\\ \mu_{0010}(\alpha,\beta) &= \mu_{0010}\cos\beta+\mu_{0001}\sin\beta, &(16)\end{aligned}$$

which equations correspond to Eq. (11) with $pqrs=1000$ and $pqrs=0010$, respectively, and the 4 moments μ_{1000}, μ_{0100}, μ_{0010}, and μ_{0001} can be determined by measuring the intensity moments $\mu_{1000}(\alpha,\cdot)$ and $\mu_{0010}(\cdot,\beta)$ in the fractional domain for two values of α and β, for instance for 0 and $\frac{1}{2}\pi$: $\mu_{1000}=\mu_{1000}(0,\cdot)$, $\mu_{0100}=\mu_{1000}(\frac{1}{2}\pi,\cdot)$, $\mu_{0010}=\mu_{0010}(\cdot,0)$, and $\mu_{0001}=\mu_{0010}(\cdot,\frac{1}{2}\pi)$.

5.2. Second-order moments

For the 3+4+3=10 second-order moments, the following equations are relevant:

$$\begin{aligned}\mu_{2000}(\alpha,\beta) &= \mu_{2000}\cos^2\alpha\\ &\quad+2\mu_{1100}\cos\alpha\sin\alpha\\ &\quad+\mu_{0200}\sin^2\alpha, &(17)\\ \mu_{1010}(\alpha,\beta) &= \mu_{1010}\cos\alpha\cos\beta\\ &\quad+\mu_{1001}\cos\alpha\sin\beta\\ &\quad+\mu_{0110}\sin\alpha\cos\beta\\ &\quad+\mu_{0101}\sin\alpha\sin\beta, &(18)\\ \mu_{0020}(\alpha,\beta) &= \mu_{0020}\cos^2\beta\\ &\quad+2\mu_{0011}\cos\beta\sin\beta\\ &\quad+\mu_{0002}\sin^2\beta, &(19)\end{aligned}$$

which equations correspond to Eq. (11) with $pqrs=2000$, $pqrs=1010$, and $pqrs=0020$, respectively. The 3 moments μ_{2000}, μ_{1100}, and μ_{0200} can be determined by measuring the intensity moment $\mu_{2000}(\alpha,\cdot)$ in the fractional domain [19] for three values of α, for instance for 0, $\frac{1}{4}\pi$, and $\frac{1}{2}\pi$: $\mu_{2000}=\mu_{2000}(0,\cdot)$, $\mu_{0200}=\mu_{2000}(\frac{1}{2}\pi,\cdot)$, and then $\mu_{1100}=\mu_{2000}(\frac{1}{4}\pi,\cdot)-\frac{1}{2}(\mu_{2000}+\mu_{0200})$. Likewise, the 3 moments μ_{0020}, μ_{0011}, and μ_{0002} can be determined by measuring the intensity moment $\mu_{0020}(\cdot,\beta)$ for three values of β, for instance for 0, $\frac{1}{4}\pi$, and $\frac{1}{2}\pi$: $\mu_{0020}=\mu_{0020}(\cdot,0)$, $\mu_{0002}=\mu_{0020}(\cdot,\frac{1}{2}\pi)$, and then $\mu_{0011}=\mu_{0020}(\cdot,\frac{1}{4}\pi)-\frac{1}{2}(\mu_{0020}+\mu_{0002})$. The other 4 moments μ_{1010}, μ_{1001}, μ_{0110}, and μ_{0101} follow from measuring the intensity moment $\mu_{1010}(\alpha,\beta)$, for instance as follows: $\mu_{1010}=\mu_{1010}(0,0)$, $\mu_{0101}=\mu_{1010}(\frac{1}{2}\pi,\frac{1}{2}\pi)$, $\mu_{0110}=\mu_{1010}(\frac{1}{2}\pi,0)$, and then $\mu_{1001}=2\mu_{1010}(\frac{1}{4}\pi,\frac{1}{4}\pi)-\mu_{1010}-\mu_{0110}-\mu_{0101}$. We conclude that all 10 second-order moments can be determined from the knowledge of 4 fractional power spectra, where one of them has to be anamorphic (i.e., $\alpha\neq\beta$), for instance $|F_{0,0}(x,y)|^2$, $|F_{\pi/4,\pi/4}(x,y)|^2$, $|F_{\pi/2,\pi/2}(x,y)|^2$, and $|F_{\pi/2,0}(x,y)|^2$.

5.3. Higher-order moments

For higher-order moments we can proceed analogously. For the 4+6+6+4=20 third-order moments, the following equations are relevant:

$$\begin{aligned}\mu_{3000}(\alpha,\beta) &= \mu_{3000}\cos^3\alpha\\ &\quad+3\mu_{2100}\cos^2\alpha\sin\alpha\\ &\quad+3\mu_{1200}\cos\alpha\sin^2\alpha\\ &\quad+\mu_{0300}\sin^3\alpha, &(20)\\ \mu_{2010}(\alpha,\beta) &= \mu_{2010}\cos^2\alpha\cos\beta\\ &\quad+\mu_{2001}\cos^2\alpha\sin\beta\\ &\quad+2\mu_{1110}\cos\alpha\sin\alpha\cos\beta\\ &\quad+2\mu_{1101}\cos\alpha\sin\alpha\sin\beta\\ &\quad+\mu_{0210}\sin^2\alpha\cos\beta\\ &\quad+\mu_{0201}\sin^2\alpha\sin\beta, &(21)\\ \mu_{1020}(\alpha,\beta) &= \mu_{1020}\cos\alpha\cos^2\beta\\ &\quad+2\mu_{1011}\cos\alpha\cos\beta\sin\beta\\ &\quad+\mu_{1002}\cos\alpha\sin^2\beta\\ &\quad+\mu_{0120}\sin\alpha\cos^2\beta\\ &\quad+2\mu_{0111}\sin\alpha\cos\beta\sin\beta\\ &\quad+\mu_{0102}\sin\alpha\sin^2\beta, &(22)\\ \mu_{0020}(\alpha,\beta) &= \mu_{0030}\cos^3\beta\\ &\quad+3\mu_{0021}\cos^2\beta\sin\beta\\ &\quad+3\mu_{0012}\cos\beta\sin^2\beta\\ &\quad+\mu_{0003}\sin^3\beta. &(23)\end{aligned}$$

Note again that these equations correspond to Eq. (11) with $pqrs=3000$, $pqrs=2010$, $pqrs=1020$, and $pqrs=0030$, respectively. The 20 third-order moments can be determined from the knowledge of 6 fractional power spectra, where 2 of them have to be anamorphic.

For the 5+8+9+8+5=35 fourth-order moments, the relevant equations follow from Eq. (11) with $pqrs=4000$, $pqrs=3010$, $pqrs=2020$, $pqrs=1030$, and $pqrs=0040$, respectively. The 35 fourth-order moments can be determined from the knowledge of 9 fractional power spectra, where 4 of them have to be anamorphic.

To find the number of nth-order moments N, and the total number of fractional power spectra t (with a the number of anamorphic ones) that we need to determine these N moments, use can be made of the following triangle, which can easily be extended to higher order:

n	number of nth order moments	N	t	a
0	1	1	1	0
1	$2+2$	4	2	0
2	$3+4+3$	10	4	1
3	$4+6+6+4$	20	6	2
4	$5+8+9+8+5$	35	9	4
5	$6+10+12+12+10+6$	56	12	6
6	$7+12+15+16+15+12+7$	84	16	9
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Note that N (the number of nth-order moments) is equal to the sum of the values in the nth row of the triangle, $N = \frac{1}{6}(n+1)(n+2)(n+3)$; that t (the total number of fractional power spectra) is equal to the highest value that appears in the nth row of the triangle, $t = \frac{1}{4}(n+2)^2$ for $n =$ even, and $t = \frac{1}{4}(n+3)(n+1)$ for $n =$ odd; and that a (the number of anamorphic fractional power spectra) follows from $a = t - n - 1$.

6. CONCLUSIONS

We have shown how all global WD moments of arbitrary order can be measured as intensity moments in the output plane of an appropriate number of fractional FT systems (generally anamorphic ones, i.e., with different angles α and β), and we have derived the minimum number of (anamorphic) fractional power spectra that are needed for the determination of these moments. The results followed directly from the general relationship (11) that expresses the intensity moments in the output plane of an anamorphic fractional FT system in terms of the moments in the input plane and the two angles α and β.

7. REFERENCES

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