Comment on "The T-class of time-frequency distributions: Time-only kernels with amplitude estimation"

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Abstract

It is shown that the recently introduced T-class of time-frequency distributions is a subclass of the S-method distributions. From the generalization of the S-method distribution by rotating it in the time-frequency plane, a similar generalization of the T-class distribution follows readily. The generalized T-class distribution is then applicable to signals that behave chirp-like, with their instantaneous frequency slowly varying around the slope of the chirp; this slope need no longer be zero, as is the case for the original T-class distribution, but may take an arbitrary value.

Key words: Time-frequency analysis, Chirp-like signals, Instantaneous frequency, Fractional Fourier transform, T-class
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In a recent paper [1], the T-class time-frequency distribution $\rho_x^T(t,f;g)$ of a time signal $x(t)$ has been introduced. In terms of the signal's Fourier transform

$$X(f) = \int_{-\infty}^{\infty} x(t)\,\exp(-\mathrm{j}2\pi f t)\,\mathrm{d}t \tag{1}$$

and the Fourier transform $G(f)$ of the window function $g(t)$ that characterizes the distribution, it can most easily be written as

$$\rho_x^T(t,f;g) = \int_{-\infty}^{\infty} X(f+\nu/2)\,G(-\nu)\,X^*(f-\nu/2)\,\exp(\mathrm{j}2\pi\nu t)\,\mathrm{d}\nu. \tag{2}$$

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The T-class distribution $\rho_x^T(t,f;g)$ is related to the signal's Wigner distribution

$$W_x(t,f) = \int_{-\infty}^{\infty} x(t+\tau/2)\, x^*(t-\tau/2)\, \exp(-\mathrm{j}2\pi f\tau)\, \mathrm{d}\tau \tag{3}$$

through the time-domain convolution

$$\rho_x^T(t,f;g) = \int_{-\infty}^{\infty} g(t_\circ)\, W_x(t-t_\circ,f)\, \mathrm{d}t_\circ \tag{4}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi^T(t_\circ,f_\circ;g)\, W_x(t-t_\circ,f-f_\circ)\, \mathrm{d}t_\circ\, \mathrm{d}f_\circ, \tag{5}$$

where $\Phi^T(t,f;g)$ is the kernel that smoothes the Wigner distribution $W_x(t,f)$:

$$\Phi^T(t,f;g) = g(t)\, \delta(f). \tag{6}$$

Note that for $g(t) = \delta(t)$, i.e., $G(f) = 1$, the T-class distribution $\rho_x^T(t,f;g)$ reduces to the Wigner distribution $W_x(t,f)$. We remark that the smoothing is only in the t direction, and that there is no smoothing at all in the perpendicular f direction. Therefore, this time-frequency distribution is well suited for the description of signals with a slowly varying instantaneous frequency.

We recall that the S-method time-frequency distribution [2] is defined as

$$\rho_x^S(t,f;h,z) = \int_{-\infty}^{\infty} S_x(t,f+\nu/2;h)\, Z(\nu)\, S_x^*(t,f-\nu/2;h)\, \mathrm{d}\nu, \tag{7}$$

where $S_x(t,f;h)$ is the short-time Fourier transform of $x(t)$ with window function $h(t)$,

$$S_x(t,f;h) = \int_{-\infty}^{\infty} x(t+t_\circ)\, h^*(t_\circ)\, \exp(-\mathrm{j}2\pi f t_\circ)\, \mathrm{d}t_\circ. \tag{8}$$

In Eq. (7), we notice in particular the averaging of the product $S_x(t,f+\nu/2;h)\, S_x^*(t,f-\nu/2;h)$ in the frequency direction, with the averaging function $Z(\nu)$. For $Z(\nu) = \delta(\nu)$, the S-method reduces to the well-known spectrogram $|S_x(t,f;h)|^2$, and for $Z(\nu) = 1$, we are led to what is known as the pseudo Wigner distribution function.

It has been shown [3] that the S-method distribution $\rho_x^S(t,f;h,z)$ is related to the Wigner distribution $W_x(t,f)$ through the convolution

$$\rho_x^S(t,f;h,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi^S(t_\circ,f_\circ;h,z)\,W_x(t-t_\circ,f-f_\circ)\,\mathrm{d}t_\circ\,\mathrm{d}f_\circ, \tag{9}$$

where the smoothing kernel $\Phi^S(t,f;h,z)$ takes the form

$$\Phi^S(t,f;h,z) = W_h(-t,-f)\,z(-t). \tag{10}$$

We readily verify that the T-class kernel $\Phi^T(t,f;g)$ is a subclass of the S-method kernel $\Phi^S(t,f;h,z)$, and that $\Phi^S(t,f;h,z)$ reduces to $\Phi^T(t,f;g)$ if $z(-t)=g(t)$ and $W_h(-t,-f)=\delta(f)$, i.e., $h(t)=1$. The T-class distribution $\rho_x^T(t,f;g)$ is thus a special case of the more general S-method distribution $\rho_x^S(t,f;h,z)$.

We recall that the S-method distribution can be generalized to the form [3, Eq. (20)]

$$\begin{aligned}\rho_x^{S_\alpha}(t,f;h,z) = \int_{-\infty}^{\infty} & S_x(t+(\nu/2)\sin\alpha, f+(\nu/2)\cos\alpha;h) \\ & \times Z(\nu)\,\exp(-\mathrm{j}2\pi f\nu\sin\alpha) \\ & \times S_x^*(t-(\nu/2)\sin\alpha, f-(\nu/2)\cos\alpha;h)\,\mathrm{d}\nu,\end{aligned} \tag{11}$$

in which case the smoothing kernel $\Phi^{S_\alpha}(t,f;h,z)$ in the convolution

$$\rho_x^{S_\alpha}(t,f;h,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi^{S_\alpha}(t,f;h,z)\,W_x(t-t_\circ,f-f_\circ)\,\mathrm{d}t_\circ\,\mathrm{d}f_\circ \tag{12}$$

takes the generalized form [cf. Eq. (10)]

$$\Phi^{S_\alpha}(t,f;h,z) = W_h(-t,-f)\,z(-t\cos\alpha+f\sin\alpha). \tag{13}$$

The angle α simply determines the direction in the time-frequency plane in which the averaging with $Z(\nu)$ takes place. Note that for $\alpha=0$ the generalized S-method [see Eq. (11)] reduces to the original S-method, which averages in the frequency direction [see Eq. (7)], while for $\alpha=\pi/2$ we have the version of the S-method [4,5] that averages in the time direction:

$$\rho_x^{S_{\pi/2}}(t,f;h,z) = \int_{-\infty}^{\infty} S_x(t+\nu/2,f;h)\,Z(\nu)\,\exp(-\mathrm{j}2\pi f\nu)$$

$$\times S_x^*(t-\nu/2, f; h)\,\mathrm{d}\nu. \tag{14}$$

In the generalized case, we might want to choose the window $h(t)$ such that $W_h(-t,-f)$ becomes a function of $(t\sin\alpha + f\cos\alpha)$, in which case the kernel $\Phi^{S_\alpha}(t,f;h,z)$ becomes a function of the two perpendicular combinations of variables u and v, defined as

$$\begin{aligned} u &= t\cos\alpha - f\sin\alpha, \\ v &= t\sin\alpha + f\cos\alpha. \end{aligned} \tag{15}$$

In the special case of the T-class distribution, the window $h(t)=1$ would then be replaced by its fractional Fourier transform with fractional angle α, i.e., by the chirp signal

$$h_\alpha(t) = [\exp(\mathrm{j}\alpha/2)/\sqrt{\cos\alpha}\,]\,\exp(-\mathrm{j}\pi t^2\tan\alpha). \tag{16}$$

Note that the fractional Fourier transform $h_\alpha(u)$ of a function $h(t)$ is defined as [6–8]

$$h_\alpha(u) = \frac{\exp(\mathrm{j}\alpha/2)}{\sqrt{\mathrm{j}\sin\alpha}}\int_{-\infty}^{\infty} h(t)\exp\left(\mathrm{j}\pi\frac{t^2\cos\alpha - 2ut + u^2\cos\alpha}{\sin\alpha}\right)\mathrm{d}t \tag{17}$$

with the square root $\sqrt{\mathrm{j}a}$ defined as $\sqrt{\mathrm{j}a} = \sqrt{|a|}\,\exp[\mathrm{sgn}(a)\,\mathrm{j}\pi/4]$. With $h_\alpha(t)$ as in Eq. (16), its Wigner distribution $W_{h_\alpha}(t,f)$ then reads $W_{h_\alpha}(t,f) = \delta(t\sin\alpha + f\cos\alpha) = \delta(v)$ and the T-class kernel takes the generalized form [cf. Eq. (6)]

$$\Phi^{T_\alpha}(t,f;g) = g(t\cos\alpha - f\sin\alpha)\,\delta(t\sin\alpha + f\cos\alpha) = g(u)\,\delta(v). \tag{18}$$

By taking α different from 0, we may choose the direction in the time-frequency plane such that the smoothing is optimal, see for instance Ref. [9], and we are no longer restricted to signals $x(t)$ with a slowly-varying instantaneous frequency. The α-rotated version can be used to treat signals with an arbitrary chirp-like character, as long as the instantaneous frequency is slowly varying around the slope α of the chirp.

We now use the property [8, Section IV] that the short-time Fourier transform $S_x(t,f;h_\alpha)$ of the signal $x(t)$ with a window function $h_\alpha(t)$ that is the α-fractional Fourier transform of $h(t)$, is – apart from a coordinate rotation and a phase multiplication – the same as the short-time Fourier transform

$S_{z_{(-\alpha)}}(t, f; h)$ of the $(-\alpha)$-fractional Fourier transform $x_{-\alpha}(t)$ of the signal $x(t)$ with the original window $h(t)$. In detail we have [3]

$$S_x(t, f; h_\alpha) = S_{x_{(-\alpha)}}(u, v; h) \exp[\mathrm{j}\pi(tf - uv)] \tag{19}$$

with the coordinate rotation (15). So, we can still use the original window $h(t) = 1$ instead of its fractional Fourier transform $h_\alpha(t)$: we only have to use the fractional Fourier transform $x_{(-\alpha)}(t)$ of the signal $x(t)$ instead of the signal itself. This leads to the conclusion that a generalized version of the T-class distribution (i.e. rotated in time-frequency plane, and working with the rotated coordinates u and v instead of t and f) can easily be defined as being based on the fractional Fourier transform of the signal: we only have to change the Fourier transform $X(\nu) = x_{\pi/2}(\nu)$ in the original definition (2) by the fractional Fourier transform $x_{(\pi/2-\alpha)}(\nu)$.

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