RECTANGULAR-TO-QUINCUNX GABOR LATTICE CONVERSION VIA FRACTIONAL FOURIER TRANSFORMATION

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ABSTRACT

Transformations of Gabor lattices are associated with operations on the window functions that arise in Gabor theory. In particular it is shown that transformation from a rectangular to a quincunx lattice can be associated with fractional Fourier transformation. Since a Gaussian function, which plays an important role as a window function in Gabor theory, is an eigenfunction of fractional Fourier transformation, this transformation has a clear advantage over other operations that are used to transform a rectangular lattice into a quincunx lattice.

1. INTRODUCTION

Recently a new sampling lattice – the quincunx lattice – has been introduced [1] as a sampling geometry in the Gabor scheme, which geometry is different from the traditional rectangular sampling geometry [2]. In this paper we will show how results that hold for rectangular sampling (see, for instance, [3, 4]), can be transformed to the quincunx case. In particular we will show that, in order to transform such results, we may apply a fractional Fourier transformation [5] to the window functions, while simultaneously rotating the sampling lattice.

2. GABOR'S SIGNAL EXPANSION

We start with the usual Gabor expansion [2, 3, 4] on a rectangular lattice, in which case a signal $\varphi(t)$ can be expressed as a linear combination of properly shifted and modulated versions of a synthesis window $g(t)$

$$\varphi(t) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_{mk} g(t-mT) e^{jk\Omega t}. \tag{1}$$

The time step T and the frequency step Ω satisfy the relationship $\Omega T \leq 2\pi$. The expansion coefficients a_{mk} follow from sampling the windowed Fourier transform with analysis window $w(t)$, on the rectangular lattice $(\tau = mT, \omega = k\Omega)$:

$$a_{mk} = \int_{-\infty}^{\infty} \varphi(t) w^*(t-mT) e^{-jk\Omega t} dt. \tag{2}$$

The synthesis window $g(t)$ and the analysis window $w(t)$ are related to each other in such a way that their shifted and modulated versions constitute two sets that are biorthogonal:

$$\sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} g(t_1 - mT) w^*(t_2 - mT) e^{jk\Omega(t_1 - t_2)} = \delta(t_1 - t_2). \tag{3}$$

The biorthogonality relation (3) leads immediately to the equivalent but simpler expression

$$\frac{2\pi}{\Omega} \sum_{m=-\infty}^{\infty} g(t-mT) w^*\left(t - \left[m + k\frac{2\pi}{\Omega T}\right] T\right) = \delta_k; \tag{4}$$

note that the factor $2\pi/\Omega T$ represents the degree of oversampling. In this paper we will derive a similar relationship between the synthesis and the analysis window in the case of a quincunx sampling geometry. In particular we will see that the quincunx lattice can be considered as a rotated version of a rectangular lattice over an angle $\alpha = \frac{1}{4}\pi$, which rotation is related to the fractional Fourier transform.

3. FRACTIONAL FOURIER TRANSFORMATION

The fractional Fourier transform $\bar{\varphi}(t)$ of a signal $\varphi(t)$ can be expressed as [5]

$$\begin{aligned}\bar{\varphi}(t_o) &= \sqrt{\frac{\beta}{2\pi \sin\alpha}}\, e^{j\frac{1}{2}(\alpha - \frac{1}{2}\pi)}\, e^{j\frac{1}{2}\beta t_o^2 \cot\alpha} \\ &\times \int_{-\infty}^{\infty} e^{-j\frac{\beta}{\sin\alpha} t_o t_i}\, e^{j\frac{1}{2}\beta t_i^2 \cot\alpha} \varphi(t_i) dt_i.\end{aligned} \tag{5}$$

Note that the fractional Fourier transformation (5) reduces to a normal Fourier transformation for $\alpha = \frac{1}{2}\pi$, and note also that the fractional Fourier transform can be considered as the result of three consecutive elementary operations:

- a pre-multiplication by the quadratic-phase function

$$e^{j\frac{1}{2}\beta t_i^2 \cot\alpha},$$

- a Fourier transformation with the Fourier kernel

$$\sqrt{\frac{\beta}{2\pi\sin\alpha}}\, e^{j\frac{1}{2}(\alpha-\frac{1}{2}\pi)}\, e^{-j\frac{\beta}{\sin\alpha}t_o t_i}, \text{ and}$$

- a post-multiplication by the function

$$e^{j\frac{1}{2}\beta t_o^2 \cot\alpha}.$$

We remark that fractional Fourier transformation – like multiplication by a quadratic-phase function and normal Fourier transformation – is a unitary transformation, which implies that the biorthogonality relation (3) holds also for the transformed window functions $\bar{g}(t)$ and $\bar{w}(t)$.

As an example we might consider the Gaussian function $\exp(-\frac{1}{2}\beta t^2)$, which is adapted to the fractional Fourier transformation (5) by having the same factor β in the exponent. It is not difficult to show that the fractional Fourier transform of such a Gaussian function is identical to the function itself, and that such a function is therefore an eigenfunction of the fractional Fourier transformation.

The relationship between rotation of the sampling lattice and the fractional Fourier transform becomes apparent when we consider the fractional Fourier transform of a shifted and modulated version of a signal $\varphi(t)$. Indeed, if $\bar{\varphi}(t)$ is the fractional Fourier transform of $\varphi(t)$, then the fractional Fourier transform of the shifted and modulated version $\varphi(t-\tau)\exp(j\omega t)$ takes the form

$$\bar{\varphi}(t-\bar{\tau})e^{j\bar{\omega}t}\, e^{j\phi(\tau,\omega)},$$

where the additional phase term $\phi(\tau,\omega)$ reads

$$\phi(\tau,\omega) = \tfrac{1}{2}\frac{\sin\alpha}{\beta}(\beta^2\tau^2\cos\alpha + 2\beta\tau\omega\sin\alpha - \omega^2\cos\alpha) \tag{6}$$

and where the shift $\bar{\tau}$ and the modulation $\bar{\omega}$ of the fractional Fourier transform are related to the shift τ and the modulation ω of the signal itself through the matrix relationship

$$\begin{bmatrix} \beta\bar{\tau} \\ \bar{\omega} \end{bmatrix} = M \begin{bmatrix} \beta\tau \\ \omega \end{bmatrix} \tag{7}$$

with a matrix

$$M = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \tag{8}$$

that corresponds to a rotation. Eqs. (6), (7), and (8) can easily be derived by considering the phase terms and transformation matrices that arise in the case of a multiplication by a quadratic-phase function and in the case of a normal Fourier transformation. Multiplication leads to an additional phase term

$$\phi'(\tau,\omega) = -\tfrac{1}{2}\beta\tau^2\cot\alpha$$

and to a transformation matrix of the form

$$M' = \begin{bmatrix} 1 & 0 \\ \cot\alpha & 1 \end{bmatrix} \tag{9}$$

which corresponds to a shear of the modulation variable, while Fourier transformation leads to the additional phase term

$$\phi''(\tau,\omega) = \omega\tau$$

and to a transformation matrix of the form

$$M'' = \begin{bmatrix} 0 & \sin\alpha \\ -\dfrac{1}{\sin\alpha} & 0 \end{bmatrix} \tag{10}$$

which corresponds to an exchange and an appropriate scaling of the shift and the modulation variables. Adding the phase terms that arise for the three elementary operations that build up a fractional Fourier transformation leads immediately to the phase term (6); and multiplying the matrices for the three elementary operations leads to the matrix (8).

4. QUINCUNX LATTICE

If we substitute from Eqs. (6), (7), and (8) into the simple biorthogonality expression (4), and define $\bar{T} = T\cos\alpha$, $\bar{\Omega} = \Omega/\cos\alpha$, and $r = \beta T\sin(2\alpha)/\Omega$, this expression takes the form

$$\frac{2\pi}{\bar{\Omega}\cos\alpha}\sum_{m=-\infty}^{\infty}\bar{g}(t-m\bar{T})\bar{w}^*\left(t-\left[m+k\frac{2\pi}{\bar{\Omega}\bar{T}}\right]\bar{T}\right)$$

$$\times\, e^{-j\pi mkr} = \delta_k. \tag{11}$$

Note that a phase factor $\exp(-j\pi mkr)$ has entered the expression, and that this factor is real for integer values of r. Furthermore, with $\tau = mT$, $\omega = k\Omega$, $\bar{\tau} = \bar{m}\bar{T}$, and $\bar{\omega} = \bar{k}\bar{\Omega}$, Eqs. (7) and (8) yield

$$\begin{cases} r\bar{m} &= 2k\sin^2\alpha + rm \\ 2\bar{k} &= 2k\cos^2\alpha - rm. \end{cases} \tag{12}$$

For $r = 1$ and $\alpha = \frac{1}{4}\pi$, Eq. (12) reduces to

$$\begin{cases} \bar{m} &= k+m \\ \bar{k} &= \frac{1}{2}(k-m) \end{cases} \tag{13}$$

and we conclude that, while (m,k) represents a rectangular lattice, the $(\bar{m},\bar{k})$ lattice (13) has a quincunx geometry. The biorthogonality expression (11) now takes the form

$$\frac{2\pi}{\frac{1}{2}\bar{\Omega}\sqrt{2}} \sum_{m=-\infty}^{\infty} \bar{g}(t-m\bar{T})\bar{w}^* \left(t-\left[m+k\frac{2\pi}{\bar{\Omega}\bar{T}}\right]\bar{T}\right) \times (-1)^{mk} = \delta_k. \tag{14}$$

Of course, Eq. (14) can be derived directly from the quincunx sampling geometry. But now that we have shown how it can be derived from the rectangular sampling geometry, with the biorthogonality relation (4), we can use all the well-known properties for the rectangular lattice. If, for instance, we want to find an analysis window $w(t)$ that corresponds to a given synthesis window $g(t)$ in the case of quincunx sampling, we can fractionally Fourier transform the given window function $g(t)$, find the corresponding analysis window for the rectangular sampling geometry by means of one of the many methods that have been reported in literature, and perform an inverse fractional Fourier transform to get the resulting analysis window $w(t)$ for the quincunx sampling.

5. DISCUSSION

The transformation matrices (8), (9), and (10) belong to the class of symplectic, 2×2-dimensional matrices – i.e., 2×2-dimensional matrices whose determinant equals 1 – and can be associated with signal transformations that have – like Eq. (5) – a quadratic-phase kernel. And of course, other symplectic matrices than the rotation matrix (8) can be used to transform a rectangular lattice into a quincunx lattice. The matrix (9), for instance, corresponding to a shear of the modulation variable, is able to perform such a transformation, as well. Indeed, if we multiply $g(t)$ and $w(t)$ by $\exp(j\frac{1}{4}\beta t^2)$ to get the transformed window functions $g'(t)$ and $w'(t)$, respectively, and take $\beta = \Omega/T$, we get the biorthogonality expression [cf. Eq. (14)]

$$\frac{2\pi}{\Omega} \sum_{m=-\infty}^{\infty} g'(t-mT)w'^* \left(t-\left[m+k\frac{2\pi}{\Omega T}\right]T\right) \times (-1)^{mk} = \delta_k. \tag{15}$$

The corresponding (m',k') lattice is now defined by

$$\left\{ \begin{array}{rcl} m' &=& m \\ k' &=& k+\frac{1}{2}m \end{array} \right. \tag{16}$$

and has again a quincunx geometry. In the important case of a Gaussian window function $\exp(-\frac{1}{2}\beta t^2)$, however, the fractional Fourier transform has a great advantage over a mere multiplication in that the Gaussian function is an eigenfunction of the fractional Fourier transformation and not of the multiplication.

6. REFERENCES

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