

MODIFIED ZAK TRANSFORM FOR THE QUINCUNX-TYPE GABOR LATTICE

Martin J. Bastiaans and Arno J. van Leest

Technische Universiteit Eindhoven, Faculteit Elektrotechniek
 Postbus 513, 5600 MB Eindhoven, Netherlands
 phone: +31 40 2473319 – fax: +31 40 2448375 – e-mail: M.J.Bastiaans@ele.tue.nl

ABSTRACT

Gabor's signal expansion and the Gabor transform are formulated on a quincunx lattice instead of on the traditional rectangular lattice; the representation of the quincunx lattice is based on the rectangular lattice via either a shear operation or a rotation operation. A modified Zak transformation is defined with the help of which Gabor's signal expansion and the Gabor transform can be brought into product forms that are identical to the ones that are well known for the rectangular sampling geometry.

1. INTRODUCTION

Recently a new sampling lattice – the quincunx lattice – has been introduced [1] as a sampling geometry in the Gabor scheme, which geometry is different from the traditional rectangular sampling geometry [2]. In this paper we will show how results that hold for rectangular sampling (see, for instance, [3, 4]) can be transformed to the quincunx case. In particular we will concentrate on the well-known product forms [3] of Gabor's signal expansion and the Gabor transform, in terms of the Fourier transform of the expansion coefficients and the Zak transforms of the signal and the window functions; these product forms hold in the case of critical sampling, to which case we will confine ourselves. We will show that identical product forms can be formulated in the case of a quincunx sampling geometry, as well, but then in terms of a modified version of the Zak transform.

2. GABOR'S SIGNAL EXPANSION AND THE ZAK TRANSFORM

We start with the usual Gabor expansion [2, 3, 4] on a rectangular lattice, in which case a signal $\varphi(t)$ can be expressed as a linear combination of properly shifted and modulated versions of a synthesis window $g(t)$:

$$\varphi(t) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_{mk} g(t - mT) e^{jk\Omega t}. \quad (1)$$

The time step T and the frequency step Ω satisfy the relationship $\Omega T \leq 2\pi$. The expansion coefficients a_{mk} follow

from sampling the windowed Fourier transform with analysis window $w(t)$,

$$\int_{-\infty}^{\infty} \varphi(t) w^*(t - \tau) e^{-j\omega t} dt,$$

on the rectangular lattice ($\tau = mT, \omega = k\Omega$):

$$a_{mk} = \int_{-\infty}^{\infty} \varphi(t) w^*(t - mT) e^{-jk\Omega t} dt. \quad (2)$$

The latter equation is known as the Gabor transform.

The synthesis window $g(t)$ and the analysis window $w(t)$ are related to each other in such a way that their shifted and modulated versions constitute two sets that are biorthogonal:

$$\begin{aligned} \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} g(t_1 - mT) w^*(t_2 - mT) e^{jk\Omega(t_1 - t_2)} \\ = \delta(t_1 - t_2). \end{aligned} \quad (3)$$

The biorthogonality relation (3) leads immediately to the equivalent but simpler expression

$$\frac{2\pi}{\Omega} \sum_{m=-\infty}^{\infty} g(t - mT) w^* \left(t - \left[m + k \frac{2\pi}{\Omega T} \right] T \right) = \delta_k, \quad (4)$$

where δ_k is the Kronecker delta; note that the factor $2\pi/\Omega T$ represents the degree of oversampling.

In this paper we will restrict ourselves to the case of critical sampling, i.e., $\Omega T = 2\pi$; the case of rational oversampling will be the subject of a forthcoming paper. It is well known (see, for instance, [3, 4]) that in the case of critical sampling, Gabor's signal expansion (1) and the Gabor transform (2) can be transformed into product form. We therefore need the Fourier transform $\bar{a}(t/T, \omega/\Omega)$ of the two-dimensional array of Gabor coefficients a_{mk} , defined by

$$\bar{a} \left(\frac{t}{T}, \frac{\omega}{\Omega} \right) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_{mk} e^{-j(m\omega T - k\Omega t)}, \quad (5)$$

and the Zak transforms $\tilde{\varphi}(t, \omega; T)$, $\tilde{g}(t, \omega; T)$, and $\tilde{w}(t, \omega; T)$ of the signal $\varphi(t)$ and the window functions $g(t)$ and $w(t)$, respectively, where the Zak transform

$\tilde{f}(t, \omega; T)$ of a function $f(t)$ is defined as (see, for instance, [3, 4, 5, 6])

$$\tilde{f}(t, \omega; T) = \sum_{n=-\infty}^{\infty} f(t + nT) e^{-jn\omega T}. \quad (6)$$

Upon substituting from the Fourier transform (5) and the Zak transforms [cf. Eq. (6)] into Eqs. (1) and (2), it is not too difficult to show that Gabor's signal expansion (1) can be transformed into the product form

$$\tilde{\varphi}(t, \omega; T) = \bar{a} \left(\frac{t}{T}, \frac{\omega}{\Omega} \right) \tilde{g}(t, \omega; T), \quad (7)$$

while the Gabor transform (2) can be transformed into the product form

$$\bar{a} \left(\frac{t}{T}, \frac{\omega}{\Omega} \right) = T \tilde{\varphi}(t, \omega; T) \tilde{w}(t, \omega; T). \quad (8)$$

It is the aim of this paper to show that product forms can be formulated not only in the case of a rectangular sampling lattice, but also in the case of a quincunx lattice.

3. QUINCUNX SAMPLING GEOMETRY

Instead of formulating Gabor's signal expansion on a rectangular sampling lattice, we will now use a quincunx sampling geometry. In that case Gabor's expansion coefficients follow from sampling the windowed Fourier transform on the quincunx lattice, represented, for instance, by

$$(\tau = mT, \omega = (k - \frac{1}{2}m)\Omega), \quad (9)$$

which representation is based on the rectangular lattice via a shear of the frequency variable k , or by

$$(\tau = (m + k)T, \omega = \frac{1}{2}(k - m)\Omega), \quad (10)$$

which representation is based on the rectangular lattice via a rotation. Based on Eqs. (9) or (10), Gabor's signal expansion on the quincunx lattice can be formulated as [cf. Eq. (1)]

$$\varphi(t) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a'_{mk} g(t - mT) e^{j(k - \frac{1}{2}m)\Omega t} \quad (11)$$

or

$$\varphi(t) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a''_{mk} g(t - (m + k)T) e^{j\frac{1}{2}(k - m)\Omega t}, \quad (12)$$

respectively, while the corresponding Gabor transforms take the form [cf. Eq. (2)]

$$a'_{mk} = \int_{-\infty}^{\infty} \varphi(t) w^*(t - mT) e^{-j(k - \frac{1}{2}m)\Omega t} dt \quad (13)$$

and

$$a''_{mk} = \int_{-\infty}^{\infty} \varphi(t) w^*(t - (m + k)T) e^{-j\frac{1}{2}(k - m)\Omega t} dt, \quad (14)$$

respectively.

4. MODIFIED ZAK TRANSFORM

In order to bring Eq. (13) into product form, we start with the Fourier transform $\bar{a}'(t/T, \omega/\Omega)$ of the array of Gabor coefficients $a'_{mk} \exp[-j\frac{1}{2}\pi m^2]$ [cf. Eq. (5)]:

$$\begin{aligned} \bar{a}' \left(\frac{t}{T}, \frac{\omega}{\Omega} \right) &= \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a'_{mk} e^{-j\frac{1}{2}\pi m^2} \\ &\times e^{-j[m\omega T - (k - \frac{1}{2}m)\Omega t]} \end{aligned} \quad (15)$$

and substitute from the Gabor transform (13). We rearrange factors,

$$\begin{aligned} \bar{a}' \left(\frac{t}{T}, \frac{\omega}{\Omega} \right) &= \int_{-\infty}^{\infty} dt' \sum_{m=-\infty}^{\infty} \varphi(t') w^*(t' - mT) \\ &\times e^{j\frac{1}{2}m\Omega(t' - t)} e^{-j\frac{1}{2}\pi m^2} e^{-jm\omega T} \\ &\times \sum_{k=-\infty}^{\infty} e^{-jk\Omega(t' - t)}, \end{aligned}$$

replace the sum of exponentials by a sum of Dirac functions,

$$\sum_{k=-\infty}^{\infty} e^{-jk\Omega(t' - t)} = T \sum_{k=-\infty}^{\infty} \delta(t' - t - kT),$$

and evaluate the integral,

$$\begin{aligned} \bar{a}' \left(\frac{t}{T}, \frac{\omega}{\Omega} \right) &= T \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \varphi(t + kT) w^*(t + kT - mT) \\ &\times e^{j\pi mk} e^{-j\frac{1}{2}\pi m^2} e^{-jm\omega T}. \end{aligned}$$

We rearrange factors again,

$$\begin{aligned} \bar{a}' \left(\frac{t}{T}, \frac{\omega}{\Omega} \right) &= T \sum_{k=-\infty}^{\infty} \varphi(t + kT) e^{j\frac{1}{2}\pi k^2} e^{-jk\omega T} \\ &\times \sum_{m=-\infty}^{\infty} w^*(t + (k - m)T) \\ &\times e^{-j\frac{1}{2}\pi(k - m)^2} e^{j(k - m)\omega T}, \end{aligned}$$

and recognize a modified version of the Zak transform

$$\tilde{f}'(t, \omega; T) = \sum_{n=-\infty}^{\infty} f(t + nT) e^{j\frac{1}{2}\pi n^2} e^{-jn\omega T} \quad (16)$$

for the signal $\varphi(t)$ and for the analysis window $w(t)$. We conclude that, with this modified version of the Zak transformation, the product form (8) of the Gabor transform that we found in the case of a rectangular sampling geometry holds in the case of a quincunx geometry, as well:

$$\bar{a}'\left(\frac{t}{T}, \frac{\omega}{\Omega}\right) = T\tilde{\varphi}'(t, \omega; T)\tilde{w}'(t, \omega; T). \quad (17)$$

It is not difficult to show that the same modified Zak transform (16) and the same product form (17) arise, if we try to bring Eq. (14) into product form, by starting with the Fourier transform of the array of Gabor coefficients $a''_{mk} \exp[-j\frac{1}{2}\pi(m+k)^2]$ [cf. Eq. (15)]:

$$\begin{aligned} \bar{a}'\left(\frac{t}{T}, \frac{\omega}{\Omega}\right) &= \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a''_{mk} e^{-j\frac{1}{2}\pi(m+k)^2} \\ &\times e^{-j[(m+k)\omega T - \frac{1}{2}(k-m)\Omega t]}. \end{aligned} \quad (18)$$

We remark that, compared to the previously defined Fourier transform (5), the Fourier transforms (15) and (18) contain an additional phase factor $\exp[-j\frac{1}{2}\pi m^2]$ and $\exp[-j\frac{1}{2}\pi(m+k)^2]$, respectively. Likewise, compared to the traditional Zak transform (6), an additional phase factor $\exp[j\frac{1}{2}\pi n^2]$ arises in the definition of the modified Zak transform (16). Moreover, we remark that – like the normal Zak transform – the modified Zak transform $\tilde{f}'(t, \omega; T)$ is periodic in the frequency variable ω with period $\Omega = 2\pi/T$, and quasi-periodic in the time variable t with quasi-period T :

$$\tilde{f}'(t+mT, \omega+k\Omega; T) = \tilde{f}'(t, \omega; T) e^{-j\frac{1}{2}\pi m^2} e^{jm\omega T}. \quad (19)$$

Note, finally, that since n^2 is 4-fold for n even and 4-fold+1 for n odd, the phase factor $\exp(-j\frac{1}{2}\pi n^2)$ equals 1 for n even and $-j$ for n odd; this suggests a separation of the quincunx lattice into two rectangular lattices.

In order to transform the Gabor expansions (11) and (12) into product form, as well, we substitute from these expansions into the modified Zak transform $\tilde{\varphi}'(t, \omega; T)$ [cf. Eq. (16)]. After some straightforward calculations, and using the modified Zak transform $\tilde{g}'(t, \omega; T)$ of the synthesis window $g(t)$ and the Fourier transform $\bar{a}'(t/T, \omega/\Omega)$ of the array of Gabor coefficients $a'_{mk} \exp[-j\frac{1}{2}\pi m^2]$ or $a''_{mk} \exp[-j\frac{1}{2}\pi(m+k)^2]$ again, we arrive at the product form that we already found in the case of rectangular sampling [cf. Eq. (7)]:

$$\tilde{\varphi}'(t, \omega; T) = \bar{a}'\left(\frac{t}{T}, \frac{\omega}{\Omega}\right) \tilde{g}'(t, \omega; T). \quad (20)$$

If we combine the product forms (17) and (20), we get a relation between the (modified) Zak transforms of the analysis and the synthesis window:

$$T\tilde{g}'(t, \omega; T)\tilde{w}'^*(t, \omega; T) = 1. \quad (21)$$

After substituting from the modified Zak transforms $\tilde{g}'(t, \omega; T)$ and $\tilde{w}'(t, \omega; T)$ [cf. Eq. (16)] and performing a Fourier transformation on both sides of this equation – as is usually done for rectangular sampling – it is not too difficult to show that the relationship

$$T \sum_{m=-\infty}^{\infty} g(t+mT)w^*(t+(m+k)T)(-1)^{mk} = \delta_k \quad (22)$$

holds. The latter result is the counterpart of the biorthogonality expression (4), but now for a quincunx sampling geometry.

5. REFERENCES

- [1] P. Prinz, "Calculating the dual Gabor window for general sampling sets," *IEEE Trans. Signal Processing*, vol. 44, no. 8, pp. 2078-2082, 1996.
- [2] D. Gabor, "Theory of communication," *J. Inst. Elec. Eng.*, vol. 93 (III), pp. 429-457, 1946.
- [3] M.J. Bastiaans, "Gabor's signal expansion and its relation to sampling of the sliding-window spectrum," in *Advanced Topics in Shannon Sampling and Interpolation Theory*, R.J. Marks II, ed., Springer, New York, pp. 1-35, 1993.
- [4] H.G. Feichtinger and T. Strohmer, Eds. *Gabor Analysis and Algorithms: Theory and Applications*. Berlin: Birkhäuser, 1998.
- [5] J. Zak, "The kq -representation in the dynamics of electrons in solids," *Solid State Physics*, vol. 27, pp. 1-62, 1972.
- [6] A.J.E.M. Janssen, "The Zak transform: a signal transform for sampled time-continuous signals," *Philips J. Res.*, vol. 43, pp. 23-69, 1988.